BULL. AUSTRAL. MATH. SOC. 13B30, 13D30
VOL. 38 (1988) [373–375]

CONTRACTED PRIMES OF THE COMPLETE RING OF QUOTIENTS

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The generic closure of the set of primes contracted from the complete ring of quotients of a reduced commutative ring is shown to be just the set of those primes not containing a finitely generated dense ideal. It is also the smallest generically closed, quasi-compact set containing the minimal primes.

In the study of the complete ring of quotients $Q(R)$ of a reduced commutative ring R, knowledge of the set

$$G = \{m \cap R | m \in \operatorname{Spec} Q(R)\}$$

of contracted primes is useful. For example [**3**, Theorem 4.3], $Q(R)$ is flat if and only if $G = \min R$, the set of minimal primes of R. In this note we characterise the generic closure of G. Here, $Q(R)$ can be defined as

$$\varinjlim \operatorname{Hom}(I, R)$$

with direct limit taken over all dense ideals I of R, or as

$$Q(R) = \{x \in E(R) | Ix \subseteq R \text{ for some dense ideal } I \subseteq R\}.$$

$E(R)$ is the injective envelope of R, I dense means I has zero annihilator in R (see [**7**] for general considerations). If $H \subseteq \operatorname{Spec} R$, its generic closure is

$$\bigwedge(H) = \{p \in \operatorname{Spec} R | p \subseteq q \in H\},$$

and H is generically closed if $H = \bigwedge(H)$. We use only the Zariski topology on the prime spectrum $\operatorname{Spec} R$. Let G_f be the set of primes of R not containing a finitely generated dense ideal.

Received 15 January, 1987

THEOREM. *For a reduced ring* R, $\bigwedge(G) = G_f$ *and is the smallest generically closed, quasi-compact set of primes containing* $\min R$.

PROOF: First we note that, in a reduced ring, a finitely generated ideal is dense if and only if it is not contained in any minimal prime (since a minimal prime can contain a finitely generated ideal or its annihilator, but not both). Secondly, $Q(R)$ is a von Neumann regular ring since R is reduced [**4**, Proposition 2.4.1].

Clearly, G_f is generically closed. It is also quasi-compact since the set of primes not containing a finitely generated ideal is a "patch", hence so too is an arbitary intersection since patches form the closed sets of a topology ([**2**] or [**1**, Exercise 3.27, 3.28]). To show that G_f contains the set G of contracted primes, we suppose $p \notin G_f$. Choose a finitely generated dense ideal I containing p. Then $IQ(R)$ is dense in $Q(R)$ since $Q(R)$ is an essential extension of R. As the only finitely generated (hence generated by an idempotent) dense ideal in a regular ring is the ring itself, it follows that p is not a contracted prime, $G \subseteq G_f$, hence $\bigwedge(G) \subseteq G_f$.

For the reverse inclusion, we begin by proving $\bigwedge(G)$ contains all the minimal primes. To this end, suppose $p \in \min R$, but that $pQ(R) = Q(R)$. Write $1 = \sum r_i x_i$, for some $r_i \in p$ and $x_i \in Q(R)$. Now there exist dense ideals I_i of R such that $I_i x_i \subseteq R$ for each i. Let I be the product of the I_i, also a dense ideal. We have

$$I = \sum r_i I x_i \subseteq \sum r_i R.$$

This last ideal is a finitely generated dense ideal contained in the minimal prime p, a contradiction to the remark at the begining of the proof. Thus $pQ(R) \neq Q(R)$, that is, $p \in \bigwedge(G)$ and $\bigwedge(G) \supseteq \min R$. Now let C be any subset of Spec R that is quasi-compact, generically closed, and contains $\min R$. We claim that $C \supseteq G_f$. If $p \notin C$, cover C by open sets

$$D(r_\alpha) = \{q \in \operatorname{Spec} R | r_\alpha \notin q\}$$

for $r_\alpha \in p$.

Since C is assumed quasi-compact, finitely many will do, say $D(r_i)$ for $i = 1, \ldots, n$. Then the finitely generated ideal $\sum r_i R$ is not contained in any prime in C, hence not contained in any minimal prime, thus is dense by the remark at the begining of the proof. Therefore, $p \notin G_f$ which establishes our claim.

G is quasi-compact since it is the continuous image of the compact space Spec $Q(R)$ under the spec map [**1**, Exercise 1.17v, 1.21i]. It is clear that $\bigwedge(G)$ is also quasi-compact and we have shown that $\bigwedge(G) \supseteq \min R$, so that we may choose $C = \bigwedge(G)$ in the preceding paragraph. Thus $\bigwedge(G) \supseteq G_f$, the required reverse inclusion. Consideration of the above defined C establishes the remainder of the Theorem. ∎

We may easily obtain the following well-known result:

COROLLARY. *If R is reduced, then $\min R$ is compact $\Leftrightarrow$ $G = \min R$ $\Leftrightarrow$ $Q(R)$ is R-flat $\Leftrightarrow$ each non-minimal prime contains a finitely generated dense ideal.*

PROOF: The paranthetical remark at the begining of the Theorem shows that $\min R$ is Hausdorff, establishing the first equivalence. For the second equivalence, if $G = \min R$ then, to check flatness locally on G [**6**, Item 3.J], we use that R_p is a field for each $p \in \min R$. Conversely, if $Q(R)$ is flat, then $R \subseteq Q(R)$ satisfies going down [**6**, Item 5.D] and we know $\min R \subseteq \bigwedge(G)$, hence $G = \min R$. ∎

CONJECTURES

1. G is generically closed.
2. G_f is an affine subset of $\operatorname{Spec} R$ in the sense of Lazard [**5**, p. 112].

From torsion theory, this would imply the conjecture

3. $M(R) = M_f(R)$, where $M(R)$ is the maximal flat epimorphic extension of R (see [**5**] or [**7**, Chapter XI, Section 4]), and $M_f(R)$ is the ring of quotients with respect to the filter consisting of those ideals that contain a finitely generated dense ideal.

REFERENCES

[1] M.F. Atiyah and I.G. Macdonald, *Introduction to Commutative Algebra* (Addison-Wesley Publ. Co., 1969).
[2] M. Hochster, 'Prime ideal structure in commutative rings', *Trans. Amer. Math. Soc.* **142** (1969), 43–60.
[3] James A. Hukaba, *Commutative Rings with Zero Divisors* (Marcel-Dekker). (to appear) .
[4] Joachim Lambek, *Lectures on Rings and Modules* (Blaisdell Publ. Co, 1969).
[5] Daniel Lazard, 'Autour de la platitude', *Bull. Soc. Math. France* **97** (1969), 81–128.
[6] Hideyuki Matsumura, *Commutative Algebra, 2nd Ed.* (Benjamin/Cummings Publ. Co., 1980).
[7] Bo Stenström, *Rings of Quotients, Grundlehren Math. Wiss. Band 217* (Springer-Verlag, Berlin, Heidelberg, New York, 1975).

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