COMPLETE REGULARITY AS A SEPARATION AXIOM

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1. Introduction. Although the axiom of complete regularity ought to be a separation axiom, in none of its usual forms does it look like an intrinsic separation axiom. Our aim in this paper is to establish such characterizations of complete regularity which naturally fit in between regularity and normality and which already have proved to be fundamental and useful. This can simply be achieved by replacing the family of all open sets (as used in the formulation of the separation axioms) by some suitable (sub)base of open sets. For the sake of simplicity, we *assume all our spaces to be* T_1 and we shall operate with (sub)bases of *closed* sets (instead of open sets). It is appropriate to define the notion of *a screening*.

Two subsets A and B of a set X are said to be *screened by* the pair (C, D) if $C \cup D = X, A \cap D = \emptyset$ and $C \cap B = \emptyset$. (Consequently, $A \subset C$ and $B \subset D$.)

Then we have the following result.

THEOREM 1. *A space X is completely regular if and only if there is a base $\mathfrak{B}$ for the closed subsets of X such that*

1. (*Base-regularity*) *If $B \in \mathfrak{B}$ and $x \notin B$, then $\{x\}$ and B are screened by a pair from $\mathfrak{B}$;*

2. (*Base-normality*) *Every two disjoint elements of $\mathfrak{B}$ are screened by a pair from $\mathfrak{B}$.*

Two subsets A and B of a space X are said to be *screened* by a finite family $\mathfrak{C}$ if $\mathfrak{C}$ covers X and each element of $\mathfrak{C}$ meets[1] at most one of A and B.

THEOREM 2. *A space X is completely regular if and only if there is a subbase $\mathfrak{S}$ for the closed subsets of X such that*

(1) (*Subbase-regularity*) *If $S \in \mathfrak{S}$ and $x \notin S$, then $\{x\}$ and S are screened by a finite subcollection of $\mathfrak{S}$;*

(2) (*Subbase-normality*) *Every two disjoint elements of $\mathfrak{S}$ are screened by a finite subcollection of $\mathfrak{S}$.*

Observe that in this context *Hausdorff* spaces and regular spaces can be defined as follows:

A space X is *Hausdorff* if and only if it is *(sub)base Hausdorff* relative to *any* (sub)base $\mathfrak{B}$ of closed sets; i.e., every two points of X are screened by (a finite subcollection of $\mathfrak{B}$), a pair of elements of $\mathfrak{B}$;

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[1]Throughout, "C meets A" means $C \cap A \neq \emptyset$.

A space X is *regular* if and only if it is (sub)base *regular* to some *suitable* (sub)base $\mathfrak{B}$ of closed sets (as defined in Theorems 1 and 2 above).

Observe that a discrete space consisting of three points has a base for the closed sets (namely, all those which do not contain a singleton) which is not base-regular.

Complete regularity means (sub)base-regularity and (sub)base-normality relative to some suitable (sub)base of closed sets (Theorems 1 and 2). Observe that a space is normal if and only if it is (sub)base regular and (sub)base normal relative to the base of all closed sets.

As shown in § 3 by Examples 1 and 2, we cannot fall back on Theorem 1 for the proof of Theorem 2. The proof presented here depends on the notion of a *linked system*, which is a generalization of the concept of a filter. A linked system in a space X is a family $\mathfrak{F}$ of subsets such that every two members of $\mathfrak{F}$ have non-void intersection. Given a T_1-space X and a subbase $\mathfrak{S}$ for the closed subsets of X which satisfies the conditions of subbase-regularity and subbase-normality, we consider all linked systems of elements of $\mathfrak{S}$ which are generated by maximal centred systems of elements of $\mathfrak{S}$. Those linked systems which have empty intersection serve as new points and are added to the space X. By choosing a suitable topology for the enlarged space, a Hausdorff compactification X^* of X is obtained (for a formal description, see § 2, Theorem 3). Observe also, that a weight-preserving Hausdorff compactification can be obtained (§ 2, Theorem 4).

Intrinsic characterizations of complete regularity have been discussed by several authors; see, e.g., Smirnov (**6**) and Frink (**3**). The characterization of Frink is related to our characterization in Theorem 1. Frink proved that a T_1-space X is completely regular if and only if there is a base $\mathfrak{B}$ for the closed subsets of X such that

(1) all finite unions and intersections of elements of $\mathfrak{B}$ belong to $\mathfrak{B}$;

(2) $\mathfrak{B}$ satisfies the conditions of base-regularity and base-normality.

Therefore, Theorem 1 shows that the algebraic, actually semi-ring, condition can be dropped.

Theorems 1 and 2 have been known to us for some time (cf. **1**; **2**; and **10**). However, the *methods* (*linked systems*) applied here for the proof of Theorem 2 (and the results of Theorems 3 and 4 below) seem to be fundamental, yield a better insight, and give shorter proofs.

After the completion of this paper, we observed a recent publication by Steiner (**7**). He proved Theorem 1 independently, and the proof is the same as that in (**10**).

2. Theorems for subbases. A family $\mathfrak{S}$ of closed subsets of a space X is a *subbase* for the closed subsets if the family of all finite unions of members of $\mathfrak{S}$ is a base for the closed subsets of X. As is easily seen, two subsets A and B of a space X are screened by a finite family from a subbase $\mathfrak{S}$ if A and B are screened by a pair from the base which is generated by $\mathfrak{S}$ by the taking of all finite unions.

Theorem 2 is proved by obtaining a Hausdorff compactification of the given space.

Suppose that X has a subbase $\mathfrak{S}$ which satisfies the conditions of subbase-regularity and subbase-normality. Let $\xi', \eta', \ldots$ denote maximal centred systems from $\mathfrak{S}$, i.e., $\xi', \eta', \ldots$ have the finite intersection property and are maximal with respect to this property. Let X' be the collection of all maximal centred systems from $\mathfrak{S}$. Define $S' = \{\xi' | \xi' \in X', S \in \xi'\}$ for every $S \in \mathfrak{S}$. X' will be endowed with a topology for which $\mathfrak{S}' = \{S' | S \in \mathfrak{S}\}$ serves as a subbase. X' is a compact T_1-space, which is a compactification of X. The embedding ν of X into X' is defined by the rule

$$\nu(x) = \{S | S \in \mathfrak{S}, x \in S\}.$$

Observe that this compactification is similar to the Wallman compactification (**9**). In order to obtain a *Hausdorff* compactification of X, we proceed as follows. For each $\xi' \in X'$ we define ξ^* as follows:

$$\xi^* = \{T \in \mathfrak{S} | T \cap S \neq \emptyset \text{ for every } S \in \xi'\}.$$

Observe that $\xi' \subset \xi^*$ and that different ξ' might define the same ξ^*. It turns out that for each ξ', the system ξ^* is a *linked* system, i.e. any two members of ξ^* have non-void intersection. Let X^* be the collection of all linked systems obtained in this way. Define $S^* = \{\xi^* | \xi^* \in X^*, S \in \xi^*\}$ for every $S \in \mathfrak{S}$. X^* is endowed with the topology for which $\{S^* | S \in \mathfrak{S}\}$ is a subbase. Since $\nu(x) = \nu(x)^*$ for every $x \in X$, there is a natural embedding μ of X into X^* defined by

$$\mu(x) = \nu(x) = \nu(x)^*.$$

The natural projection π of X' onto X^* is defined by $\pi(\xi') = \xi^*$. We have the following commutative triangle

$$\begin{tikzcd} & X \arrow[dl, "\nu"'] \arrow[dr, "\mu"] & \\ X' \arrow[rr, "\pi"'] & & X^* \end{tikzcd}$$

It will be proved that X^* is a Hausdorff space and that π is a continuous map. It follows that X^* is a Hausdorff compactification of X (X^* is compact, since it is the continuous image of a compact space).

Remark. If $\mathfrak{S}$ is closed under finite unions and finite intersections, then $\xi' = \xi^*$ for all ξ'; whence $X' = X^*$. In this case, the construction resembles the construction for the Čech-Stone compactification (as given in **4**, p. 86).

Actually, if $\mathfrak{S}$ is the family of all zero-sets, then X^* is the Čech-Stone compactification. It is an open problem whether each compactification of X can be obtained as an X^*, starting with a suitably chosen $\mathfrak{S}$. As pointed out

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by Frink (3), the one-point compactification of a locally compact space X can be obtained in this way. Our result can be stated as follows.

THEOREM 3. *Let X be a T_1-space. If X has a subbase $\mathfrak{S}$ for the closed subsets which satisfies the conditions of subbase-regularity and subbase-normality, then X has a Hausdorff compactification X^* for which the family $\mathfrak{S}^* = \{S^* | S \in \mathfrak{S}\}$ is a subbase. Moreover, $\mathfrak{S}^*$ satisfies the conditions of subbase-regularity and subbase-normality.*

Closely related to Theorem 3 is the following theorem.

THEOREM 4. *If $\mathfrak{S}$ is a subbase for the closed subsets of a T_1-space X such that $\mathfrak{S}$ satisfies the conditions of subbase-regularity and subbase-normality, then there is a Hausdorff compactification X^* of X which has the same weight as X.*

Remark. It might be conjectured that in Theorem 3 the closed set S^* equals the closure of S in X^* for each $S \in \mathfrak{S}$. In § 3, Example 4, it is shown that this conjecture is false.

The proof of Theorem 3 is given in Lemmas 1–10 below.

LEMMA 1. *Let $S_i \in \mathfrak{S}$, $i = 1, \ldots, n$, and $\bigcup\{S_i | i = 1, \ldots, n\} = X$. Then $\bigcup\{S_i' | i = 1, \ldots, n\} = X'$ and $\bigcup\{S_i^* | i = 1, \ldots, n\} = X^*$.*

Proof. If $S_i \notin \xi'$, then there are $T_j \in \xi'$, $j = 1, \ldots, m$, such that

$$S_i \cap \{\bigcap\{T_j | j = 1, \ldots, m\}\} = \emptyset.$$

Now, suppose that $\xi' \notin \bigcup\{S_i' | i = 1, \ldots, n\}$. Then $S_i \notin \xi'$ for each $i = 1, \ldots, n$. It follows that some finite subfamily of ξ' has void intersection which is a contradiction. If $\xi^* \notin S^*$, then $S \notin \xi^*$, and consequently $S \notin \xi'$. From the first part of the proof it now follows that $\bigcup\{S_i^* | i = 1, \ldots, n\} = X^*$.

LEMMA 2. *Each $\xi^* \in X^*$ is a linked system; i.e., if $S_1, S_2 \in \xi^*$, then*

$$S_1 \cap S_2 \neq \emptyset.$$

Proof. Suppose that $S_1 \cap S_2 = \emptyset$, $S_1, S_2 \in \xi^*$. Due to the subbase-normality of $\mathfrak{S}$, there are $T_j \in \mathfrak{S}$, $j = 1, \ldots, m$, such that $\{T_j | j = 1, \ldots, m\}$ covers X and each T_j meets at most one of S_1 and S_2. Due to Lemma 1, $\xi' \in T_j'$ for some j. Consequently, $T_j \in \xi'$. If $T_j \cap S_1 = \emptyset$, then $S_1 \notin \xi^*$. If $T_j \cap S_2 = \emptyset$, then $S_2 \notin \xi^*$. Since T_j meets at most one of S_1 and S_2, we have a contradiction.

LEMMA 3. *Let $S_i \in \mathfrak{S}$, $i = 1, \ldots, n$. Then $\bigcap\{S_i | i = 1, \ldots, n\} = \emptyset$ if and only if $\bigcap\{S_i' | i = 1, \ldots, n\} = \emptyset$.*

The proof of Lemma 3 is obvious. As shown in § 3, Example 2, Lemma 3 does not hold for linked systems. For linked systems we have the following lemma.

LEMMA 4. *Let $S_1, S_2 \in \mathfrak{S}$. Then $S_1 \cap S_2 = \emptyset$ if and only if $S_1^* \cap S_2^* = \emptyset$.*

The "only if" part of Lemma 4 follows from Lemma 2. Obviously, the "if" part also holds for finitely many S.

LEMMA 5. *If $\xi^*, \eta^* \in X^*$ and $\xi^* \neq \eta^*$, then there are $T_1 \in \xi^*$ and $T_2 \in \eta^*$ such that $T_1 \cap T_2 = \emptyset$.*

Proof. Suppose that for every $T_1 \in \xi^*$ and $T_2 \in \eta^*$ we have that $T_1 \cap T_2 \neq \emptyset$. Then, if $T \in \xi^*$, T meets every element of η^*. In particular, T meets every element of η'. Consequently, $T \in \eta^*$. Hence $\xi^* \subset \eta^*$. Since similarly we have $\eta^* \subset \xi^*$, we have that $\xi^* = \eta^*$, which is a contradiction.

LEMMA 6. *The subbase $\mathfrak{S}^*$ of X^* satisfies the conditions of subbase-regularity and subbase-normality.*

Proof. Suppose that $\xi^* \notin S^* \in \mathfrak{S}^*$. Then, $S \notin \xi^*$ and $S \notin \xi'$. Hence, there is an element $T \in \xi'$ such that $S \cap T = \emptyset$. Because of the subbase-normality of $\mathfrak{S}$, the sets T and S are screened by a cover $\{S_1, \ldots, S_n\} \subset \mathfrak{S}$. From Lemmas 1 and 4, it follows that T^* and S^* are screened by the cover $\{S_1^*, \ldots, S_n^*\} \subset \mathfrak{S}^*$. Since $T \in \xi' \subset \xi^*$, $\xi^* \in T^*$ and ξ^* and S^* are screened by $\{S_1^*, \ldots, S_n^*\}$. Subbase-normality of $\mathfrak{S}^*$ is proved similarly.

LEMMA 7. *ν and μ are homeomorphisms.*

Proof. First, observe that ν and μ are well-defined. The maximality of $\nu(x)$ follows from the subbase-regularity of $\mathfrak{S}$. From the subbase-regularity, it also follows that $\nu(x) = \nu(x)^* = \mu(x)$. ν is one-to-one, since X is a T_1-space. If $S \in \mathfrak{S}$, then $\nu^{-1}(S') = \{x \mid \nu(x) \in S'\} = \{x \mid x \in X, S \in \nu(x)\} = S$. Hence $\nu(S) = S' \cap \nu(X)$. It follows that ν is a homeomorphism. Similarly, μ is a homeomorphism.

LEMMA 8. *X' is a compact T_1-space.*

Proof. If $\xi', \eta' \in X'$ and $\xi' \neq \eta'$, then by the maximality of ξ' and η' there are $S_1 \in \xi'$ and $S_2 \in \eta'$ such that $S_1 \notin \eta'$ and $S_2 \notin \xi'$. Consequently, $\eta' \in S_2'$, $\eta' \notin S_1'$, $\xi' \in S_1'$, and $\xi' \notin S_2'$.

This proves that X' is a T_1-space. According to Alexander's lemma, in order to prove the compactness of X', it suffices to show that each centred system $\mathfrak{T}'$ from $\mathfrak{S}'$ has non-void intersection. Because of Lemma 3, the family $\{T \mid T' \in \mathfrak{T}'\}$ is a centred system. Let ξ' be a maximal centred system containing this family. Then for each $T' \in \mathfrak{T}'$ we have that $\xi' \in T'$, since $T \in \xi'$. Consequently, $\bigcap \{T' \mid T' \in \mathfrak{T}'\} \neq \emptyset$.

LEMMA 9. *X^* is a Hausdorff space.*

Proof. If $\xi^* \neq \eta^*$, then by Lemma 5 there are disjoint T_1 and T_2 of ξ^* and η^*, respectively. $\xi^* \in T_1$, $\eta^* \in T_2$, and $T_1^* \cap T_2^* = \emptyset$ by Lemma 4. Because of Lemma 6, T_1^* and T_2^* are screened by a cover $\{S_1^*, \ldots, S_n^*\}$ from $\mathfrak{S}^*$. Consequently, ξ^* and η^* are screened by $\{S_1^*, \ldots, S_n^*\}$. The lemma follows.

LEMMA 10. *π is continuous.*

Proof. We show that $\pi^{-1}(S^*)$ is closed for each $S^* \in \mathfrak{S}^*$. In general, $\pi^{-1}(S^*) \neq S'$; see § 3, Example 2. Obviously, $\pi(S') \subset S^*$. We proceed as follows. Suppose that $\xi' \notin \pi^{-1}(S^*)$. Then $\pi(\xi') = \xi^* \notin S^*$. Because of Lemma 6, ξ^* and S^* are screened by a subfamily $\{S_1^*, \ldots, S_n^*\}$ of $\mathfrak{S}^*$. Assume that $S_1^*, \ldots, S_j^*$ do not meet S^* and that $S_{j+1}^*, \ldots, S_n^*$ meet S^*. By Lemmas 1, 3, and 4 we have that $\{S_1', \ldots, S_n'\}$ covers X', $S_1', \ldots, S_j'$ do not meet S', and $S_{j+1}', \ldots, S_n'$ meet S'. Furthermore, $\{S_1', \ldots, S_n'\}$ screens ξ' and S'. It follows that $U = \bigcup \{S_k' | k = 1, \ldots, j\}$ is a neighbourhood of ξ', which, by π, is mapped into $\bigcup \{S_k^* | k = 1, \ldots, j\}$. Hence $U \subset X' \backslash \pi^{-1}(S^*)$. It follows that $X' \backslash \pi^{-1}(S^*)$ is open and $\pi^{-1}(S^*)$ is closed.

Remark. Since X' is compact and X^* is Hausdorff, π is a closed continuous map. Hence, π induces an upper semi-continuous decomposition $\mathfrak{D}$ of X'; see (**5**, p. 99) for definition. ξ' and η' belong to the same element of the decomposition $\mathfrak{D}$ if and only if for each $T \in \xi'$ and $S \in \eta'$ we have that $T \cap S \neq \emptyset$.

If we drop the requirement that $\mathfrak{S}^*$ satisfies the conditions of subbase-regularity and subbase-normality, then a weight-preserving compactification can be obtained. Recall that the weight of X, $w(X)$, is the minimal cardinal number of a subbase for the closed subsets of X. The following potency lemma is well known.

LEMMA 11. *Let X be a space of infinite weight $w(X)$. If $\mathfrak{S}$ is a subbase for the closed subsets of X, then there is a subcollection $\mathfrak{S}_0$ of $\mathfrak{S}$ such that $\mathfrak{S}_0$ is a subbase for the closed subsets of X and the power of $\mathfrak{S}_0$ is $w(X)$.*

The following lemma asserts that $\mathfrak{S}_0$ in Lemma 11 can be chosen in such a way that if $\mathfrak{S}$ satisfies the conditions of subbase-regularity and subbase-normality, then $\mathfrak{S}_0$ will do the same.

LEMMA 12. *Suppose that $w(X)$ is infinite and $\mathfrak{S}$ is a subbase for the closed subsets of X which satisfies the conditions of subbase-regularity and subbase-normality. Then there is a subcollection $\mathfrak{S}_0$ of $\mathfrak{S}$ which is a subbase for the closed subsets of X satisfying the conditions of subbase-regularity and subbase-normality and which has power $w(X)$.*

Proof. Using Lemma 11, first select a subset $\mathfrak{S}_1$ of $\mathfrak{S}$ of power $w(X)$, which is a subbase for the closed subsets of X: $\mathfrak{S}_1 = \{S_\lambda | \lambda \in \Lambda\}$, $|\Lambda| = w(X)$. Then, choose an open base $\mathfrak{O}$ of power $w(X)$: $\mathfrak{O} = \{O_\mu | \mu \in M\}$, $|M| = w(X)$. For each two disjoint elements S_{λ_1} and S_{λ_2} of $\mathfrak{S}_1$, take a fixed cover $\mathfrak{S}_{\lambda_1,\lambda_2}$ from $\mathfrak{S}$ such that S_{λ_1} and S_{λ_2} are screened by $\mathfrak{S}_{\lambda_1,\lambda_2}$. If $S_\lambda \in \mathfrak{S}_1$ and $O_\mu \in \mathfrak{O}$ *can* be screened by a finite cover from $\mathfrak{S}$, then let $\mathfrak{T}_{\lambda,\mu}$ be a fixed cover which screens S_λ and O_μ. Let

$$\mathfrak{S}_2 = \mathfrak{S}_1 \cup [\bigcup\{\mathfrak{S}_{\lambda_1,\lambda_2} | \lambda_1, \lambda_2 \in \Lambda, S_{\lambda_1} \cap S_{\lambda_2} \neq \emptyset\}] \cup$$
$$[\bigcup\{\mathfrak{T}_{\lambda,\mu} | \lambda \in \Lambda, \mu \in M, S_\lambda \text{ and } O_\mu \text{ can be screened by a finite cover from } \mathfrak{S}\}].$$

An easy computation shows that $|\mathfrak{S}_2| = w(X)$. Proceeding by induction on n, we define subsets $\mathfrak{S}_n$ of $\mathfrak{S}$ with power $w(X)$. We shall show that

$$\mathfrak{S}_0 = \bigcup\{\mathfrak{S}_n | \, n = 1, 2, \ldots\}$$

satisfies all properties required. Obviously, $\mathfrak{S}_0$ is a subbase of power $w(X)$, which satisfies the condition of subbase-normality. Suppose that $S \in \mathfrak{S}_0$ and $x \notin S$. Take a cover $\mathfrak{D} \subset \mathfrak{S}$, which screens x and S, and let

$$U = X \backslash \bigcup\{D | \, D \in \mathfrak{D}, D \cap S \neq \emptyset\}.$$

Choose O such that $x \in O \in \mathfrak{D}$ and $O \subset U$. Then, clearly O and S are screened by $\mathfrak{D}$. Since for some n we have that $S \in \mathfrak{S}_n$, O and S are screened by a finite subfamily of $\mathfrak{S}_{n+1}$. Hence, x and S are screened by a finite subfamily of $\mathfrak{S}_0$.

Proof of Theorem 4. From Lemma 12 it follows that there is a subfamily $\mathfrak{S}_0$ of $\mathfrak{S}$ such that $|\mathfrak{S}_0| = w(X)$ and $\mathfrak{S}_0$ satisfies the conditions of subbase-regularity and subbase-normality. Because of Theorem 3, there is a Hausdorff compactification X^* of X for which the family $\{S^* | \, S \in \mathfrak{S}_0\}$ is a subbase. Consequently, $w(X^*) = w(X)$.

3. Examples.

Example 1.[2] This is an example of a space X and a subbase $\mathfrak{S}$ for the closed subsets of X such that

(1) $\mathfrak{S}$ satisfies the conditions of subbase-regularity and subbase-normality;

(2) The family $\mathfrak{S}^{\cup}$ of all finite unions of elements of $\mathfrak{S}$ does not satisfy the condition of base-normality.

X is the unit interval $[0, 1]$ with the usual topology. $\mathfrak{S}$ is the family of all intervals of the form $[0, x]$ and $[x, 1]$ for $x \in [0, 1]$, and all singletons $\{x\}$ for $x \in [0, 1]$. Now, $A = \{\frac{1}{2}\}$ and $B = [0, \frac{1}{4}] \cup [\frac{3}{4}, 1]$ cannot be screened by a pair from $\mathfrak{S}^{\cup}$. Indeed, suppose that there are $C, D \in \mathfrak{S}^{\cup}$ such that $C \cup D = X$, $C \cap B = \emptyset$, and $A \cap D = \emptyset$. It follows that C is a neighbourhood of A which does not meet B. Hence, C contains a subset of the form $[x, 1]$, $x < \frac{1}{2}$, or $[0, x]$, $x > \frac{1}{2}$. In both cases $C \cap B \neq \emptyset$, which is a contradiction.

Example 2. In this example we use the notation of § 2. Let X be an infinite set which is the disjoint union of five infinite sets A, B, C, D, and E. Consider the following base $\mathfrak{S}$ for the *discrete topology* on X. $\mathfrak{S}$ contains all finite sets and all cofinite sets (a set is cofinite if its complement is finite). Moreover, it contains the following sets: $A \cup B \cup C$, $A \cup D \cup E$, and $B \cup D$.

(1) The family $\mathfrak{S}^{\cap}$ of all finite intersections does not satisfy the condition of subbase-normality.

(2) The family $\mathfrak{S}^{\cap\cup}$ of all finite unions and intersections of elements of $\mathfrak{S}$ does not satisfy the condition of subbase-normality.

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[2]This example is due to J. van der Slot (oral communication).

In order to prove (1) and (2), consider $A \in \mathfrak{S}^{\cap}$ and $B \cup D \in \mathfrak{S}$. We show that A and $B \cup D$ cannot be screened by a pair from $\mathfrak{S}^{\cap\cup}$. Suppose that there are $F, G \in \mathfrak{S}^{\cap\cup}$ such that $F \cup G = X$, $A \cap G = \emptyset$, and $F \cap (B \cup D) = \emptyset$. Since F contains A, F is infinite. It follows that F is cofinite, or the intersection of a cofinite set and $(A \cup B \cup C)$, or the intersection of a cofinite set and $(A \cup D \cup E)$. In each case, the intersection of F and $B \cup D$ is infinite, which is a contradiction.

(3) Let $\mathfrak{C}$ denote the collection of all cofinite sets of X. Then the maximal centred systems from $\mathfrak{S}$ with void intersection are:

$$\xi' = \mathfrak{C} \cup \{A \cup B \cup C, A \cup D \cup E\},$$
$$\eta' = \mathfrak{C} \cup \{A \cup B \cup C, B \cup D\},$$
$$\vartheta' = \mathfrak{C} \cup \{A \cup D \cup E, B \cup D\}.$$

The compactification X' is not Hausdorff.

Indeed, observe that a space is Hausdorff if and only if every subbase for the closed subsets has the property that any two distinct points can be screened by a finite subcollection of the subbase. Now, if C is a cofinite set, then C' contains ξ', η', and ϑ'. Therefore, a cover which screens ξ' and η' has to be a subfamily of $\{(A \cup B \cup C)', (A \cup D \cup E)', (B \cup D)'\}$. However, no such subfamily screens ξ' and η'.

(4) ξ', η', and ϑ' generate the same linked system: $\xi^* = \eta^* = \vartheta^*$. Observe that $(A \cup B \cup C) \cap (A \cup D \cup E) \cap (B \cup D) = \emptyset$. It follows that the linked system ξ^* does *not* have the finite intersection property. However,

$$(A \cup B \cup C)^* \cap (A \cup D \cup E)^* \cap (B \cup D)^* = \xi^* \quad \text{(cf. Lemma 4).}$$

(5) $\pi^{-1}((A \cup B \cup C)^*) = (A \cup B \cup C)' \cup \{\vartheta'\} \neq (A \cup B \cup C)'$ (cf. Lemma 10).

Example 3. We give an example of a space X with the following properties:

(1) X is a regular space which is not completely regular;

(2) X has a base $\mathfrak{B}_1$ which satisfies the condition of base-regularity;

(3) X has a base $\mathfrak{B}_2$ which satisfies the condition of base-normality.

Such an example shows that the conditions of (sub)base-regularity and (sub)base-normality *must* deal with the same base. In fact, the space X is the example of Tychonoff (**8**) of a space which is regular but not completely regular. Let Y denote the Tychonoff plank with a corner point removed: $Y = \Omega' \times \omega' \setminus \{(\Omega, \omega)\}$, where Ω' is the set of ordinal numbers not greater than the first uncountable ordinal Ω, and ω' is the set of ordinal numbers not greater than the first infinite ordinal ω, each with the order topology. For each natural number n, let Y_n be a space homeomorphic to Y. Points of Y_n are denoted by (α, β, n), where $\alpha \leqq \Omega, \beta \leqq \omega$. We assume that the Y_n are pairwise disjoint. X_1 denotes the union of the sets Y_n and a point ξ not in the union of the Y_n.

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A topology on X_1 is determined by the following rules: Each Y_n is an open subspace of X_1 and

$$\left\{\left(\bigcup_{n=m}^{\infty} Y_n\right) \cup \{\xi\} \mid m = 1, 2, \ldots\right\}$$

is a neighbourhood base of ξ. Then X is obtained from X_1 by identifying $(\Omega, \beta, 2n)$ with $(\Omega, \beta, 2n + 1)$ for each $\beta \leqq \omega$ and $n = 1, 2, \ldots,$ and $(\alpha, \omega, 2n - 1)$ with $(\alpha, \omega, 2n)$ for each $\alpha < \Omega$ and $n = 1, 2, \ldots$. Let p denote the natural projection of X_1 onto X. The family of all closed subsets satisfies the condition of base-regularity, as is easily seen. $\mathfrak{B}_2$ is defined as follows. Since Y is locally compact, for each point (α, β, n) we can select an open base for the X_1-neighbourhoods whose closures are compact. $\mathfrak{B}_2$ will consist of the images of the following sets under the map p:

(1) The (compact) closures of the selected neighbourhoods;

(2) The complements of the selected neighbourhoods;

(3) $\bigcup_{n=1}^{m} Y_n$, $m = 1, 2, \ldots$.

It is not hard to show that $\mathfrak{B}_2$ satisfies the condition of base-normality. By Theorem 1, $\mathfrak{B}_2$ cannot satisfy the condition of base-regularity. Indeed, it is easily seen that $p(\bigcup_{n=1}^{m} Y_n)$ and $p(\xi)$ cannot be screened by a pair from $\mathfrak{B}_2$.

Example 4. In this example we use the notation of § 2. Let X be a locally compact, non-compact Hausdorff space. Let p and q be two distinct points of X. We exhibit a base $\mathfrak{S}$ of X, which satisfies the conditions of subbase-regularity and subbase-normality, and a member $S \in \mathfrak{S}$ such that S^* is *not* equal to the closure of S in X^* (see Theorem 3). $\mathfrak{S}$ is the family of all finite unions of the family consisting of the following sets:

(1) all compact subsets of X,

(2) all closed subsets of X which contain at least one of the points p and q and the complements of which have a compact closure in X.

It turns out that X^* is the one-point compactification of X. Put $X^* = X \cup \{\infty\}$. Now $S = \{p, q\}$ is a member of $\mathfrak{S}$ on account of (1) and $S^* = \{p, q, \infty\}$. Since S is compact, the closure of S in X^* equals S.

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