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A NOTE ON LOCALLY QUASI-UNIFORM SPACES

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ABSTRACT. Locally quasi-uniform spaces are studied, and it is shown that a topological space (X, t) admits exactly one compatible locally quasi-uniform structure if and only if t is finite.

1. **Introduction.** Topological spaces with a unique compatible uniform structure have been characterized by R. Doss [2]. In [3], P. Fletcher initiated the study of spaces with a unique compatible quasi-uniform structure, and he conjectured that (X, t) admits exactly one compatible quasi-uniform structure if and only if t is finite. C. Barnhill and P. Fletcher [1] showed that if t is finite, then (X, t) is uniquely quasi-uniformizable. In [6] and [7], W. Lindgren gave examples where (X, t) is uniquely quasi-uniformizable with t infinite, and showed that the conjecture holds for R_1 spaces. The concept of locally quasi-uniform spaces was defined for T_1 spaces in [5], and it was shown that (X, t) admits a local quasi-uniformity with a countable base if and only if it is a γ space if and only if it is a Nagata first countable space.

A general introduction to quasi-uniform spaces may be found in [8].

2. **Locally quasi-uniform spaces.**

DEFINITION 1. Let X be a non-empty set and let $\mathcal{U}$ be a filter on $X \times X$ such that:

(i) $\Delta \subseteq U$ for every $U \in \mathcal{U}$, where $\Delta = \{(x, x) : x \in X\}$.

(ii) For each $x \in X$ and $U \in \mathcal{U}$, there exists $V(x, U) = V \in \mathcal{U}$ such that $(V \circ V)[x] \subseteq U[x]$. Then $\mathcal{U}$ is called a *locally quasi-uniform* structure for X.

$\mathcal{U}$ gives a topology

$$t_{\mathcal{U}} = \{A \subseteq X: \text{for every } x \in A \text{ there exists } U \in \mathcal{U} \text{ such that } U[x] \subseteq A\}.$$

It is clear that every quasi-uniform structure is a locally quasi-uniform structure. If we use a term without defining it, we are using the quasi-uniform space definition. We say that $(X, \mathcal{U})$ is *strongly complete* if every Cauchy filter converges.

LEMMA 1. *Let (X, t) be a topological space and let*

$$\mathcal{B} = \{U : U \supseteq \Delta \quad \textit{and for every} \quad x \in X, U[x] \in t\}.$$

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Then $\mathcal{B}$ is a base for a locally quasi-uniform structure $\mathcal{FL}$, and $\mathcal{FL}$ is the finest compatible structure.

Proof. If $V \in \mathcal{B}$ and $x \in X$, put

$$U = [(X - V[x]) \times X] \cup (V[x] \times V[x]).$$

If $y \in X$, $U[y] = X$ or $V[x]$. Thus $U \in \mathcal{B}$. Also, $(U \circ U)[x] = V[x]$. Hence $\mathcal{B}$ is a base for a structure $\mathcal{FL}$. If $\mathcal{U}$ is a compatible structure and $U \in \mathcal{U}$, $U \supseteq W$, the interior of U in $t^{-1} \times t$. $W[x] \in t$ for every $x \in X$ gives $W \in \mathcal{FL}$. Thus $\mathcal{U} \subseteq \mathcal{FL}$.

DEFINITION 2. [4] A locally quasi-uniform space $(X, \mathcal{U})$ *has the Lebesgue property* provided that if $\mathcal{C}$ is a $t_{\mathcal{U}}$-open cover of X, then there is a $U \in \mathcal{U}$ such that $\{U[x] : x \in X\}$ refines $\mathcal{C}$.

THEOREM 1. *Let $(X, \mathcal{U})$ be a locally quasi-uniform space.*

(1) *If $(X, \mathcal{U})$ has the Lebesgue property, then $(X, \mathcal{U})$ is strongly complete.*

(2) *$\mathcal{FL}$ is a compatible strongly complete locally quasi-uniform structure.*

(3) *$\mathcal{U}$ is pre-compact if and only if every ultrafilter on X is $\mathcal{U}$-Cauchy.*

(4) *If $t_{\mathcal{U}}$ is compact, every Cauchy filter converges.*

(5) *$(X, t_{\mathcal{U}})$ is compact if and only if $\mathcal{U}$ is strongly complete and pre-compact.*

(6) follows from (5) and the fact that the Pervin structure is pre-compact.

(7) Suppose $\mathcal{FL}$ is pre-compact. By (2), $\mathcal{FL}$ is a compatible strongly complete structure. By (5), $t_{\mathcal{U}} = t_{\mathcal{FL}}$ is compact.

Proof. For (1), we note that the proof in [4] for quasi-uniform spaces carries over.

For (2), we show that $\mathcal{FL}$ has the Lebesgue property and apply (1). If $\mathcal{C}$ is an open cover and $x \in X$, then there exists $C_x \in \mathcal{C}$ such that $x \in C_x$. Let

$$U = \bigcup \{\{x\} \times C_x : x \in X\}.$$

$U \in \mathcal{FL}$ and $\{U[x] : x \in X\}$ refines $\mathcal{C}$.

For (3) and (4), we note that the standard quasi-uniform space arguments hold.

(5) follows from (3) and (4).

(6) follows from (5) and the fact that the Pervin structure is pre-compact.

(7) Suppose $\mathcal{FL}$ is pre-compact. By (2), $\mathcal{FL}$ is a compatible strongly complete structure. By (5), $t_{\mathcal{U}} = t_{\mathcal{FL}}$ is compact.

LEMMA 2. *Suppose (X, t) has a finite compatible locally quasi-uniform structure $\mathcal{U}$. Then $\mathcal{FL} = \mathcal{U}$, and therefore t has only one compatible locally quasi-uniform structure.*

Proof. $U = \bigcap \{V : V \in \mathcal{U}\} \in \mathcal{U}$ gives $\mathcal{U} = \{V : V \supseteq U\}$. Clearly, $U[x]$ is the smallest open set containing x. Also, $\mathcal{U} \subseteq \mathcal{FL}$ by Lemna 1. If $A \in \mathcal{FL}$, $A \supseteq V$

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where $V[x]\in t$ for every $x\in X$. Thus $V[x]\supseteq U[x]$ for every $x\in X$, and therefore $A\supseteq V\supseteq U$ gives $A\in\mathcal{U}$ or $\mathcal{F}\mathcal{L}\subseteq\mathcal{U}$.

COROLLARY. *If t is finite, (X, t) has only one compatible locally quasi-uniform structure.*

Proof. If t is finite the Pervin structure is a finite compatible quasi-uniform structure.

DEFINITION 3. If $\{G_n\}$ is a sequence of open sets and $G_1\subset G_2\subset G_3\subset \ldots$, it is called an *ascending sequence* of open sets.

LEMMA 3. *Let (X, t) be a topological space with t infinite. There exists an ascending infinite sequence of open sets or there exists a descending infinite sequence of open sets.*

Proof. Suppose t contains no ascending infinite sequence of open sets. Then $t-\{X\}$ has the same property. Thus every A in $t-\{X\}$ is contained in a maximal ascending chain in $t-\{X\}$. Let

$$\mathcal{M}=\{M:M \quad \text{is a maximal ascending chain in } t-\{X\}\}.$$

If $M\in\mathcal{M}$, let $V_M=\bigcup\{V:V\in M\}$. Then $V_M\in t-\{X\}$. If $V_{M_1}\neq V_{M_2}$, $V_{M_1}\cup V_{M_2}=X$. For otherwise, M_1 would not be maximal in $t-\{X\}$. Let $\mathcal{V}$ denote the set of distinct V_M, $M\in\mathcal{M}$.

Case 1. $\mathcal{V}$ is infinite. We show that $\mathcal{V}$ has the finite intersection property. If $\Phi=\cap_{i=1}^n V_{M_i}$, choose $V_M\neq V_{M_i}$, $1\le i\le n$. Then $V_M=V_M\cup\Phi= V_M\cup(\bigcap_{i=1}^n V_{M_i})=\bigcap_{i=1}^n(V_M\cup V_{M_i})=\bigcap_{i=1}^n X=X$, a contradiction. We show that $\bigcap_{i=1}^{n-1} V_{M_i}\neq\bigcap_{i=1}^n V_{M_i}$. If equality holds, $\bigcap_{i=1}^{n-1} V_{M_i}\subseteq V_{M_n}$. Thus $V_{M_n}= V_{M_n}\cup(\bigcap_{i=1}^{n-1} V_{M_i})=X$, a contradiction. Since $\mathcal{V}$ is infinite, put $X_n=\bigcap_{i=1}^n V_{M_i}$ and $\{X_n\}$ is a descending infinite sequence of open sets.

Case 2. $\mathcal{V}$ is finite. Since $t-\{X\}$ is infinite and each $V\in t-\{X\}$ is contained in V_M for some $V_M\in\mathcal{V}$, there exists $V_M\in\mathcal{V}$ such that an infinite number of members of $t-\{X\}$ are contained in V_M. Put $V_M=X_1$. Let

$$t_1=\{V:V\in t-\{X\} \quad \text{and} \quad V\subseteq X_1\}.$$

t_1 is an infinite topology for X_1. Repeat the argument just given for (X_1, t_1) and obtain a topological space (X_2, t_2) such that: (1) t_2 is infinite, (2) $X_2\subset X_1\subset X$, and (3) $X_2\in t-\{X_1\}\subset t-\{X\}$. Using induction we obtain a descending infinite sequence of open sets.

THEOREM 2. *A topological space (X, t) is uniquely locally quasi-uniformizable if and only if t is finite.*

Proof. The Corollary gives one half of the theorem. Suppose t is infinite.

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Case 1. t has a descending infinite sequence of open sets. We obtain a sequence $\{a_n\}$ of distinct points and a sequence $\{X_n\}$ of distinct open sets such that $a_n \in X_n$ and $a_n \notin X_m$ for $m > n$. If (X, t) is uniquely locally quasi-uniformizable, $\mathscr{FL}$ is the Pervin structure, and therefore $\mathscr{FL}$ is totally bounded. Let

$$U = \left[\bigcup_{i=1}^{n} (\{a_i\} \times X_i)\right] \cup \left[\left(X - \bigcup_{i=1}^{\infty} \{a_i\}\right) \times X\right].$$

$U \in \mathscr{FL}$ so there exists $A_1, \ldots, A_n$ such that $\bigcup_{i=1}^{n} A_i = X$ and $A_i \times A_i \subseteq U$. There exists j, $1 \leq j \leq n$, such that A_j contains infinitely many elements of $\{a_n\}$. Choose $m > n$ such that $a_n, a_m \in A_j$. We have $(a_m, a_n) \in A_j \times A_j \subseteq U$ or $a_n \in U[a_m] = X_m$, a contradiction.

Case 2. t has an ascending infinite sequence of open sets. We obtain a sequence $\{a_n\}$ of distinct points and a sequence $\{X_n\}$ of distinct open sets such that $a_n \in X_n$ and $a_n \notin X_m$ for $m < n$. Now use the argument in Case 1.

Remark. After looking at Lemma 3, one might wonder when (X, t) has an ascending infinite sequence of open sets. In [6], it is shown that every subset of (X, t) is compact if and only if every ascending open sequence is finite.

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