NORMAL COMPLEMENTS IN FINITE SOLVABLE GROUPS

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(Received 8 August 1972)

Communicated by G. E. Wall

1. Introduction

A well known theorem ([1] page 432) in the study of finite groups states that if P is a Sylow p-subgroup of the finite group G, and if P_0 is a normal subgroup of P such that whenever two elements, σ and τ, of P are conjugate in G, the cosets σP_0 and τP_0 are conjugate in P/P_0, then there is a normal subgroup K of G such that $G = KP$ and $K \cap P = P_0$. In this note we will extend this result to allow P to be any Hall subgroup if G is solvable. More precisely, following theorem will be the proved.

THEOREM *Let G be a finite solvable group, let H be a Hall subgroup of G and let H_0 be a normal subgroup of H such that whenever σ and τ are elements of H with σ conjugate to τ in G, σH_0 and τH_0 are conjugate in H/H_0. Then there exists a normal subgroup K of G with $G = KH$ and $K \cap H = H_0$.*

When H_0 is the trivial subgroup we get the following special case.

COROLLARY 1. *If G is a finite solvable group and H is a Hall subgroup of G such that for $\sigma, \tau \in H$, σ is conjugate to τ in G if, and only if, σ is conjugate to τ in H, then H has a normal complement in G.*

The following result is a direct consequence of Corollary 1.

COROLLARY 2. *If H is a finite solvable group and σ is an automorphism of H of order a prime p which preserves conjugacy classes in H, then $p \mid |H|$.*

The converse of the theorem is, of course, also true and easily seen under more general hypotheses. If H is any subgroup of G, if $K \lhd G$, $G = KH$, and $H_0 = K \cap H$, then σH_0 and τH_0 are H/H_0-conjugate whenever σ and τ are G-conjugate.

Corollary 1 and a fortiori the theorem are false if the assumption of solvability is omitted. This is easily seen by examination of the symmetric group on $p-1$ letters, S_{p-1}, as a subgroup of S_p, where p is any prime greater than 3.

https://doi.org/10.1017/S1446788700022837 Published online by Cambridge University Press

Any two elements of S_{p-1} that are conjugate in S_p are already conjugate in S_{p-1}, but S_p has no normal p-subgroups. These are the only counterexamples known to the author.

The notation in the paper is standard. The expression $\sigma \sim_G \tau$ means that there is an element ρ in G such that $\rho^{-1}\sigma\rho = \tau$. $O_p(G)$ is the maximal normal p-subgroup of G, and $O_{p,p'}(G)$ is the inverse image in G of the maximal p'-subgroup of $G/O_p(G)$. The author would like to thank Professor E. C. Dade for stimulating interest in the question.

2. Proof of the Theorem

The proof of the theorem will be carried out by induction on the order of G. If G is a minimal counterexample, then $[G:H]$ is a power of a prime q. For by looking at a Sylow system for G that contains a Sylow system for H, one can find Sylow subgroups $Q_1, \cdots, Q_k$ such that Q_iH is a subgroup of G properly containing H for all i. If H normalizes Q_iH_0 for each i, then $Q_1 \cdots Q_kH_0$ is the required normal subgroup and G is not a counterexample. Hence there must be a Sylow q_i-subgroup Q_i such that H does not normalize Q_iH_0 but Q_iH is a subgroup. Then Q_iH is a smaller counterexample, since the conjugation properly is inherited by all subgroups of G containing H. From now on, we assume that $[G:H] = q^a$ and Q is a Sylow q-subgroup of G.

Suppose that $Q_1 = O_q(G) \neq 1$. We shall show that $\bar{G} = G/Q_1$, $H = \bar{H}Q_1/Q_1$, $\bar{H}_0 = H_0Q_1/Q_1$ satisfy the hypotheses. Let $\eta_1, \eta_2 \in H$, $\bar{\eta}_i = \eta_iQ_1$, $i = 1,2$. Then $\bar{\eta}_1 \sim_{\bar{G}} \bar{\eta}_2$ implies that

$$\eta_1 \sim_G \eta_2\sigma \ \sigma \in Q_1 .$$

Now H is a Sylow π-subgroup of HQ_1. Hence $\eta_2\sigma$ is HQ_1-conjugate to an element η_3 of H. Since Q_1 acts trivially on $\bar{H}$, $\eta_2 \sim_H \eta_3$. Combining these facts we see that $\eta_1 \sim_G \eta_2$. By our assumptions on G, $\eta_1H_0 \sim_{H/H_0} \eta_2H_0$ and hence

$$\bar{\eta}_1H_0 \sim_{\bar{H}_0/\bar{H}_1} \bar{\eta}_2\bar{H}_0 .$$

Thus in $\bar{G}$, there exists a normal subgroup $\bar{K}$ such that $\bar{K} \cap \bar{H} = \bar{H}_0$, $\bar{G} = \bar{K}\bar{H}$. The inverse image K satisfies $K \cap H = H_0$, $G = KH$, contradicting our choice of G. Thus from now on we may assume that $O_q(G) = 1$, and any minimal normal subgroup of G lies in H.

Let H_1 be any normal subgroup of G contained in H. Let

$$\eta_1, \eta_2 \in H, \ \bar{\eta}_i = \eta_iH_1, \ i = 1,2,$$

and assume $\bar{\eta}_1 \sim_{G/H_1} \bar{\eta}_2$. Then $\eta_1 \sim_G \eta_2\eta_3$ where $\eta_3 \in H_1$. By assumption

$$\eta_1H_0 \sim_{H/H_0} \eta_2\eta_3H_0 \text{ and } \eta_1H_1H_0 \sim_{H/H_0H_1} \eta_2H_0H_1 .$$

Therefore G/H_1, H/H_1, and H_0H_1/H_1 satisfy the hypotheses of the theorem

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and H/H_0H_1 has a normal complement in G/H_1, i.e., $QH_0H_1 \lhd G$ for any normal subgroup H_1 of G contained in H.

Suppose now that H_1 is chosen to be a minimal normal subgroup of G. Then $H_1 \cap H_0 \lhd G$. For H normalizes both H_1 and H_0 and G normalizes H_1. If $\eta \in H_1 \cap H_0$ and $\sigma \in G$, then $\eta^\sigma = \eta_1 \in H_1$. Then $\eta H_0 \sim_{H/H_0} \eta_1 H_0$ by assumption, but $\eta H_0 = H_0$. Thus $\eta_1 \in H_0$, and G normalizes $H_1 \cap H_0$. Since H_1 is minimal, $H_1 \cap H_0 = 1$ or H_1. But if $H_1 \subseteq H_0$, then from the preceding paragraph QH_0 is the required normal complement. Hence we may assume that $H_0 \cap H_1 = 1$ and that $[H_0, H_1] = 1$.

Since $H_1 \lhd G$, both Q and H act as automorphisms on H_1. H decomposes H_1 into H-conjugacy classes $C_1, C_2, \cdots, C_k$. Q must preserve these conjugacy classes. For if $\eta \in C_i$ and $\sigma \in Q$, then $\eta^\sigma = \eta_1 \in H_1$. But $\eta H_0 \sim_{H/H_0} \eta_1 H_0$ implies that $\eta \sim_H \eta_1$ since $H_0 \cap H_1 = 1$ and H_0 centralizes H_1. Therefore $\eta^\sigma \in C_i$. Since the Q-orbits of each C_i have length a power of q and their sum must be a divisor of $|H|$, in each C_i there must be an element centralized by Q. Hence $C_{H_1}(Q) \neq 1$. But

$$C_{H_1}(Q) = z(H_0H_1Q) \cap H_1$$

which is normal in G since $z(H_0H_1Q)$ is characteristic in $H_0H_1Q \lhd G$. By the minimality of H_1, we must have $C_{H_1}(Q) = H_1$. Suppose now that $O_q(G/H_1) = 1$. Then G/H_1 must have a minimal normal subgroup H_2H_1/H_1 contained in H_0H_1/H_1 since

$$QH_0H_1/H_1 \lhd G/H_1.$$

The H_2 is an elementary abelian r-group for some prime r, and H_1 is an elementary abelian p-group for a prime p. If $p \neq r$ then H_2 is characteristic in H_2H_1 and hence normal in G. But the argument in the preceding paragraph rules out the possibility of any subgroups of H_0 being normal in G. On the other hand, if $p = r$, then Q, a q-group with $q \neq p$, must normalize a complement to H_1 in H_1H_2 which we may assume to be H_2 since any complement normalized by Q must lie in H_0. But if $\eta \in H$, $H_2^\eta \subseteq H_0$ and $H_0 \cap H_1H_2 = H_2$. Therefore H normalizes H_2 and again we have a normal subgroup of G contained in H_0. Thus we conclude that $O_q(G/H_1) \neq 1$. But $O_q(G/H_1) = Q_1H_1/H_1$ with Q_1 characteristic in $Q_1H_1 \lhd G$. This contradict $O_q(G) = 1$, and we have completed the proof.

Reference

[1] B. Huppert, *Endliche Gruppen* (Springer-Verlag, 1967).

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