THE LOGARITHMIC SKEW-NORMAL DISTRIBUTIONS ARE MOMENT-INDETERMINATE

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Abstract

We study the class of *logarithmic skew-normal (LSN) distributions*. They have heavy tails; however, all their moments of positive integer orders are finite. We are interested in the problem of moments for such distributions. We show that the LSN distributions are all nonunique (moment-indeterminate). Moreover, we explicitly describe Stieltjes classes for some LSN distributions; they are families of infinitely many distributions, which are different but have the same moment sequence as a fixed LSN distribution.

Keywords: Logarithmic skew-normal distribution; heavy tail; finite moments; problem of moments; nonuniqueness; Stieltjes class

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1. Introduction

The normal and lognormal distributions occupy a central position in both the theory and applications of probability and statistics. However, when modelling complex random phenomena in applied areas, there was a real need of more flexible distributions which were beyond the normal; see, e.g. O'Hagan and Leonard (1976), Azzalini (1985), or Schmoyeri *et al.* (1996). This is why the class of skew-normal distributions was introduced in Azzalini (1985) and intensively studied by many authors.

In this paper we focus on the class of logarithmic skew-normal distributions and investigate their moment determinacy. It is worth mentioning that these distributions have been used recently in Chai and Bailey (2008) to analyze the continuous (coronary artery calcification) data in a two-part stochastic model.

In Section 2 we present basic properties of logarithmic skew-normal distributions, including a probabilistic representation. In Section 3 we study the problem of moments for this class of distributions, showing that they are all moment-indeterminate. This extends the classical result for moment-indeterminacy of the lognormal distribution given in Stieltjes (1894) and developed in Heyde (1963). In Section 4 we present explicit Stieltjes classes for some logarithmic skew-normal distributions. Finally, some possible extensions of the main results are briefly discussed in Section 5.

2. Logarithmic skew-normal distributions: basic properties

We use ϕ and Φ to denote the standard normal density function and the standard normal distribution function, respectively. For any constant $\lambda \in \mathbb{R} = (-\infty, \infty)$, consider the random variable X_λ obeying a *skew-normal distribution* (denoted by SN(λ)), i.e. $X_\lambda \sim$ SN(λ). The density function f_λ of X_λ is $f_\lambda(x) = 2\phi(x)\Phi(\lambda x)$, $x \in \mathbb{R}$. To see that f_λ is a bona fide density function, we have, by the symmetry property of Φ, $\int_{-\infty}^{\infty} f_\lambda(x)\, dx = 2\int_0^{\infty} \phi(x)\, dx = 1$.

Definition. We say that the positive random variable Y_λ has a *logarithmic skew-normal distribution*, and write $Y_\lambda \sim$ LSN(λ), if its logarithmic transform $\ln Y_\lambda \sim$ SN(λ). The density function of Y_λ, denoted by g_λ, is given by:

$$g_\lambda(y) = \frac{2}{y}\phi(\ln y)\Phi(\lambda \ln y), \qquad y > 0. \tag{1}$$

The skew parameter λ regulates the shape of the distribution. If $\lambda = 0$, LSN(λ) reduces to the standard lognormal distribution LN(0, 1).

To derive some basic properties of LSN distributions, we need the following result from Arnold and Lin (2004).

Lemma 1. (Arnold and Lin (2004, Lemma 2, p. 598).) *The standard normal distribution function Φ obeys the property $-\ln \Phi(-x) = O(x^2)$ as $x \to \infty$. More precisely,*

$$\lim_{x \to \infty} \frac{-\ln \Phi(-x)}{x^2} = \frac{1}{2}.$$

The next statement shows that the LSN distributions have heavy tails.

Proposition 1. *For any $\lambda \in \mathbb{R}$, the random variable $Y_\lambda \sim$ LSN(λ) does not have a moment generating function. Equivalently, $\mathrm{E}[e^{tY_\lambda}] = \infty$ for each $t > 0$.*

Proof. Using the expression for the density g_λ, (1), we obtain

$$\mathrm{E}[e^{tY_\lambda}] = \int_0^{\infty} e^{ty} g_\lambda(y)\, dy = \int_0^{\infty} \frac{2}{y} e^{ty}\phi(\ln y)\Phi(\lambda \ln y)\, dy = \int_0^{\infty} h_\lambda(y)\, dy.$$

Here $t > 0$ is fixed and $h_\lambda(y) := (2/y)e^{ty}\phi(\ln y)\Phi(\lambda \ln y) > 0$ for all $y > 0$. We want to show that $J_\lambda := \int_0^{\infty} h_\lambda(y)\, dy = \infty$ for any $\lambda \in \mathbb{R}$.

Indeed, for $\lambda \geq 0$, we have $\liminf_{y \to \infty} \Phi(\lambda \ln y) \geq \frac{1}{2}$. This implies that $h_\lambda(y) \to \infty$ as $y \to \infty$; hence, in this case $J_\lambda = \infty$.

Then, for $\lambda < 0$, we have, by Lemma 1, the asymptotic equivalence

$$\ln \Phi(\lambda \ln y) \simeq -\frac{1}{2}(\lambda \ln y)^2 \quad \text{as } y \to \infty,$$

which implies that

$$\ln h_\lambda(y) \simeq \frac{1}{2}\ln\left(\frac{2}{\pi}\right) - \ln y + ty - \frac{1}{2}(\lambda^2 + 1)(\ln y)^2 \to \infty \quad \text{as } y \to \infty.$$

Hence, also in this case, $J_\lambda = \infty$. This completes the proof.

In general, the existence of the moment generating function implies that all moments of positive integer orders are finite. However, there are distributions that do not have moment generating functions, but still have finite moments of any positive order; see, e.g. Stoyanov (1997, Sections 8 and 11). This is exactly the case for LSN distributions.

Proposition 2. *For any $\lambda \in \mathbb{R}$, the random variable $Y_\lambda \sim \mathrm{LSN}(\lambda)$ has finite moments of all positive integer orders. Equivalently, $\mathrm{E}[Y_\lambda^k] < \infty$ for any positive integer k.*

In the next proposition we give an explicit formula for the moment $\mathrm{E}[Y_\lambda^t] = \mathrm{E}[\mathrm{e}^{tX_\lambda}]$ for any $t \in \mathbb{R}$, not just for positive integers k; see, e.g. Azzalini (1985, p. 174).

Proposition 3. *Let $Y_\lambda \sim \mathrm{LSN}(\lambda)$ for some $\lambda \in \mathbb{R}$. Then, for any real t, the t-order moment $\mathrm{E}[Y_\lambda^t]$ is given by*

$$m_t(\lambda) := \mathrm{E}[Y_\lambda^t] = \mathrm{E}[\mathrm{e}^{tX_\lambda}] = 2\mathrm{e}^{t^2/2}\Phi(\delta t), \quad where \quad \delta = \frac{\lambda}{\sqrt{1 + \lambda^2}}.$$

Proof. The proof follows directly by checking that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}m_t(\lambda) = 2\mathrm{e}^{t^2/2}\phi(\delta t)\frac{\mathrm{d}}{\mathrm{d}\lambda}(\delta t).$$

We next give a probabilistic representation for an LSN random variable.

Proposition 4. *Any LSN random variable is distributionally equivalent to a product of two independent random variables, one which is lognormal and one which is log-half-normal. Specifically, for any $\lambda \in \mathbb{R}$, the random variable $Y_\lambda \sim \mathrm{LSN}(\lambda)$ has the following representation:*

$$Y_\lambda \stackrel{\mathrm{D}}{=} \mathrm{e}^{\delta|Z_1|} \exp[\sqrt{1 - \delta^2}Z_2], \quad where \quad \delta = \frac{\lambda}{\sqrt{1 + \lambda^2}},$$

Z_1 and Z_2 are two independent standard normal random variables, and '$\stackrel{\mathrm{D}}{=}$' denotes equality in distribution.

The proof of Proposition 4 is based on the probabilistic representation of any skew-normal random variable as a linear combination of two independent random variables: one standard normal and one standard half-normal (see Henze (1986)).

From the above we know that $Y_\lambda \sim \mathrm{LSN}(\lambda)$ has all finite moments; hence, we can turn to one of the main questions discussed in this work: is the distribution of Y_λ uniquely determined by its moments? The answer is presented in the next section.

3. The problem of moments for LSN distributions

Suppose that a random variable ξ with distribution function F on the real line has finite moments, $\{\mathrm{E}[\xi^k]\}$, for all positive integer orders k. If F is uniquely determined by the moment sequence $\{\mathrm{E}[\xi^k]\}_{k=1}^{\infty}$, we say that ξ, and also F, is unique, or moment-determinate (M-determinate). Otherwise, ξ, and also F, is M-indeterminate.

In general, if ξ has a moment generating function, not only are all moments finite, but F is also unique, or M-determinate. Note, however, that there are heavy-tailed distributions, i.e. without moment generating functions, such that all their moments are finite and the distributions are M-determinate; see, e.g. Stoyanov (1997, Section 11).

Since, by Proposition 3, we explicitly know the moments of $Y_\lambda \sim \mathrm{LSN}(\lambda)$, an obvious idea is to try to use the well-known Carleman criterion. Recall that if F is a distribution on the interval $(0, \infty)$ with finite moments $m_k = \int_0^\infty x^k \, \mathrm{d}F(x)$, $k = 1, 2, \ldots$, and we calculate the *Carleman quantity*

$$\mathrm{C}[\{m_k\}] := \sum_{k=1}^{\infty}(m_k)^{-1/(2k)},$$

then the condition $C = \infty$ is sufficient for F to be M-determinate. In other words, F is the only distribution with the moment sequence $\{m_k\}$. There are, however, M-determinate distributions for which the Carleman quantity C is finite; see, e.g. Stoyanov (1997, Section 11). In all cases, if F is M-indeterminate then $C < \infty$ necessarily.

In our case $m_k(\lambda) = \mathrm{E}[Y_\lambda^k] = 2\mathrm{e}^{k^2/2}\Phi(k\delta)$, $k = 1, 2, \ldots$, so we need to analyze the Carleman quantity $C = C(\lambda)$, which depends on λ through $\delta = \lambda/\sqrt{1 + \lambda^2}$. We have

$$C(\lambda) := C[\{m_k(\lambda)\}] = \sum_{k=1}^\infty (m_k(\lambda))^{-1/(2k)} = \sum_{k=1}^\infty (2\mathrm{e}^{k^2/2}\Phi(k\delta))^{-1/(2k)} = \sum_{k=1}^\infty c_k \mathrm{e}^{-k/4},$$

where $c_k = (2\Phi(k\delta))^{-1/(2k)}$. Since

$$\lim_{k\to\infty} c_k = 1 \quad \text{for } \lambda \geq 0 \quad \text{and} \quad \lim_{k\to\infty} \frac{\ln c_k}{k\delta^2/4} = 1 \quad \text{for } \lambda < 0,$$

it immediately follows that $C(\lambda) < \infty$ for any $\lambda \in \mathbb{R}$. Hence, the Carleman criterion does not hold, although it perhaps suggests that Y_λ is M-indeterminate. This is the statement of Theorem 1, below, and its proof is based on a different idea.

It is well known that the standard lognormal distribution is M-indeterminate. This result, in a slightly different form, is due to Stieltjes (1894). Heyde (1963) extended the result to general lognormal distributions and his analysis was given in modern probabilistic/statistical terms. See also Stoyanov (1997, Section 11) and Stoyanov (2000). As mentioned before, the random variable $Y_\lambda \sim \mathrm{LSN}(\lambda)$ with $\lambda = 0$ has a standard lognormal distribution, $\mathrm{LN}(0, 1)$. It is one of our aims in this paper to show that, for any $\lambda \in \mathbb{R}$, the LSN distributions, $\mathrm{LSN}(\lambda)$, share the same moment indeterminacy property as that of the standard lognormal distribution.

Theorem 1. *For any $\lambda \in \mathbb{R}$, the random variable Y_λ is M-indeterminate.*

Proof. We use the Krein criterion (see, e.g. Akhiezer (1965, p. 87), Slud (1993), Lin (1997), Stoyanov (2000), or Pakes *et al.* (2001)). So we need to calculate the *Krein quantity,* $K[g_\lambda]$, where

$$K[g_\lambda] := \int_0^\infty \frac{-\ln g_\lambda(y^2)}{1 + y^2}\, \mathrm{d}y.$$

For the M-indeterminacy of Y_λ, it suffices to show that $K[g_\lambda] < \infty$ for $\lambda \in \mathbb{R}$. By (1),

$$K[g_\lambda] = I_1 + I_2 + I_3 + I_4,$$

where

$$I_1 = \int_0^\infty \frac{-\ln 2}{1 + y^2}\, \mathrm{d}y, \qquad\qquad I_2 = \int_0^\infty \frac{2\ln y}{1 + y^2}\, \mathrm{d}y,$$

$$I_3 = \int_0^\infty \frac{-\ln \phi(2\ln y)}{1 + y^2}\, \mathrm{d}y, \qquad I_4 = \int_0^\infty \frac{-\ln \Phi(2\lambda \ln y)}{1 + y^2}\, \mathrm{d}y.$$

Integrals I_1–I_4 can easily be seen to be finite by checking, by means of Lemma 1, that, for a finite positive constant c,

$$|\ln \phi(2\ln y)| + |\ln \Phi(2\lambda \ln y)| \leq c(1 + (\ln y)^2), \qquad y > 0,$$

and that, for $r = 0, 1, 2$, we have

$$\int_0^\infty \frac{|\ln y|^r}{1 + y^2}\, \mathrm{d}y < \infty.$$

Theorem 1 exhibits the moment indeterminacy of $Y_\lambda \sim \text{LSN}(\lambda)$ for any fixed λ. It is, therefore, interesting to clarify what happens if $\lambda \to +\infty$ or $\lambda \to -\infty$.

Theorem 2. *Let Z be a standard normal random variable, $Z \sim N(0, 1)$, and define $\eta = \mathrm{e}^{|Z|}$, usually called the log-half-normal random variable. Then, denoting by $\overset{D}{\to}$ convergence in distribution, the following properties hold.*

(a) $Y_\lambda \overset{D}{\to} \eta$ *as* $\lambda \to +\infty$ *and* $Y_\lambda \overset{D}{\to} 1/\eta$ *as* $\lambda \to -\infty$.

(b) *The left limit $1/\eta$ is M-determinate, while the right limit η is M-indeterminate.*

(c) *For $t > 0$, $m_t(\lambda) = \mathrm{E}[Y_\lambda^t]$ is an increasing function of $\lambda \in \mathbb{R}$, while $m_t(\lambda)$ is decreasing in λ for $t < 0$.*

(d) *For any $k = 1, 2, \ldots,$ we have*

$$\lim_{\lambda \to +\infty} m_k(\lambda) = 2\mathrm{e}^{k^2/2}\Phi(k) = \mathrm{E}[\mathrm{e}^{k|Z|}] = \mathrm{E}[\eta^k],$$

$$\lim_{\lambda \to -\infty} m_k(\lambda) = 2\mathrm{e}^{k^2/2}\Phi(-k) = \mathrm{E}[\mathrm{e}^{-k|Z|}] = \mathrm{E}\left[\left(\frac{1}{\eta}\right)^k\right].$$

Proof. Claim (a) follows immediately from the probabilistic representation of Y_λ given in Proposition 4. We can also use the definitions of Z, X_λ, and Y_λ, and apply Slutsky's theorem to conclude first that $X_\lambda \overset{D}{\to} |Z|$ as $\lambda \to +\infty$ and $X_\lambda \overset{D}{\to} -|Z|$ as $\lambda \to -\infty$, and then that $Y_\lambda \overset{D}{\to} \eta$ as $\lambda \to +\infty$ and $Y_\lambda \overset{D}{\to} 1/\eta$ as $\lambda \to -\infty$. For claim (b), we first note that the random variable $1/\eta = \mathrm{e}^{-|Z|}$ is bounded; hence, it is M-determinate. On the other hand, the Krein quantity for η is finite, which also tells us that η is M-indeterminate. Finally, the proofs of claims (c) and (d) are based on elementary properties of the normal distribution function and are thus omitted.

4. Stieltjes classes for LSN distributions

We start with a distribution function $F = \text{LSN}(\lambda)$, $\lambda \geq 0$, and we let f denote the density function, instead of g_λ as in (1).

In general, a *Stieltjes class* for an M-indeterminate distribution, e.g. F, is a parameterized family of different distributions all having the same moments as F; see Stoyanov (2004). In the absolutely continuous case, the Stieltjes class, denoted by S, is defined in terms of f and another function p, which is called a *perturbation function*:

$$S = S(f, p) = \{f_\varepsilon \colon f_\varepsilon(x) = f(x)[1 + \varepsilon p(x)], \ x \in \mathbb{R}, \ \varepsilon \in [-1, 1]\}.$$

Here p is a measurable function with $|p(x)| \leq 1$ for all $x \in \mathbb{R}$. The density f and the perturbation p are such that the product function $v(x) := f(x)p(x)$, $x \in \mathbb{R}$, has vanishing 'moments' in the sense that $\int_{-\infty}^{\infty} x^k v(x) \, \mathrm{d}x = 0$ for any $k = 0, 1, 2, \ldots$. To construct a Stieltjes class for the LSN(λ) distribution with $\lambda \geq 0$, we need some preparation. First, consider the function $\tilde{p} = (\tilde{p}(x), \ x \in \mathbb{R})$, defined as follows:

$$\tilde{p}(x) = \begin{cases} \left(\dfrac{x}{x-1}\right)^{1+\ln x(x-1)/2} \dfrac{\sin[\pi \ln(x-1)]}{\Phi(\lambda \ln x)} = \dfrac{\ell(x-1)}{\ell(x)} \dfrac{\sin[\pi \ln(x-1)]}{\Phi(\lambda \ln x)} & \text{if } x > 1, \\ 0 & \text{if } x \leq 1. \end{cases}$$

$$(2)$$

Here ℓ is the lognormal $LN(0, 1)$ density:

$$\ell(x) = \frac{1}{\sqrt{2\pi}} \frac{1}{x} \exp\left[-\frac{1}{2}(\ln x)^2\right], \quad x > 0; \qquad \ell(x) = 0, \quad x \le 0.$$

This kind of function with $\lambda = 0$ has been considered in Stoyanov and Tolmatz (2005). Here we need a property of $\tilde{p}$.

Lemma 2. *For any $\lambda \ge 0$, the function $\tilde{p}$ is bounded, i.e. there is a positive constant $\tilde{c}$ such that $|\tilde{p}(x)| \le \tilde{c} < \infty$ for all $x \in \mathbb{R}$.*

Proof. To show the validity of the statement, we need the following three facts.

Fact 1. The function $\tilde{p}$ is continuous on the interval $(1, \infty)$.

Fact 2. At the boundary points 1 and ∞, we have

$$\lim_{x\to 1^+} \frac{\ell(x-1)}{\ell(x)} = 0, \qquad \lim_{x\to\infty} \frac{\ell(x-1)}{\ell(x)} = 1.$$

Fact 3. For any $\lambda \ge 0$ and $x > 1$, we have $\frac{1}{2} \le \Phi(\lambda \ln x) \le 1$.

The statement of the lemma now easily follows from facts 1–3.

Theorem 3. *Suppose that X is a random variable and that $X \sim LSN(\lambda)$ for some $\lambda \ge 0$. Denote by F the distribution function of X, and let f be its density function (see (1)):*

$$f(x) = f(x \mid \lambda) = \frac{2}{x} \phi(\ln x) \Phi(\lambda \ln x), \qquad x > 0.$$

Furthermore, define the function $p = (p(x), x \in \mathbb{R})$ as follows:

$$p(x) = p(x \mid \lambda) = \frac{1}{\tilde{c}} \tilde{p}(x), \qquad x \in \mathbb{R},$$

where $\tilde{p}$ is given by (2) and $\tilde{c}$ is the constant in Lemma 2. Then the family of functions

$$S = S(f, p) = \{f_\varepsilon \colon f_\varepsilon(x) = f(x)[1 + \varepsilon p(x)], \ x \in \mathbb{R}, \ \varepsilon \in [-1, 1]\}$$

is a Stieltjes class for the M-indeterminate distribution F.

Proof. The crucial point is to show that the product function $v := fp$ has vanishing moments, i.e. that $\int_0^\infty x^k f(x) p(x)\, dx = 0$ for any $k = 0, 1, 2, \ldots$. We use the function

$$v_0(x) = \frac{1}{x} \exp\left[-\frac{1}{2}(\ln x)^2\right] \sin(\pi \ln x), \qquad x > 0,$$

and the property that $\int_0^\infty x^k v_0(x)\, dx = 0$ for any $k = 0, 1, 2, \ldots$. Details can be found in Heyde (1963) or Gradshteyn and Ryzhik (2000, Formula 4.133); see also Stoyanov (1997, Section 11). In other words, $v_0(x)$, $x > 0$, is a function with vanishing moments. Then the shifted function $\tilde{v}(x) := v_0(x-1)$, $x > 1$, also has vanishing moments. Finally, we conclude that the product function $v(x) = f(x) p(x) = c\tilde{v}(x)$, $x > 1$, has vanishing moments. This completes the proof.

Remark. We can use the same M-indeterminate distribution F and its density f in combination with other perturbations p to construct new Stieltjes classes, which are different from those mentioned above. Results from Stoyanov (2004) and Stoyanov and Tolmatz (2005) can easily be adapted to LSN distributions. Moreover, for any Stieltjes class, we can introduce the so-called *index of dissimilarity*, which is a measure of how much the distribution functions in this class differ. Details can be found in Stoyanov (2004).

5. Possible extensions

1. Knowing that any random variable $Y_\lambda \sim \text{LSN}(\lambda)$ is M-indeterminate, we can easily show that some nonlinear transformations of Y_λ, called Box–Cox transformations, are also M-indeterminate. In particular, for any nonzero real r, the power $(Y_\lambda)^r$ is M-indeterminate. Also, the Stieltjes class in Theorem 3 can be adapted to this case with $r\lambda > 0$ and r^2 integer.

2. The skew-normal distribution, $\text{SN}(\lambda)$, is in fact a conditional distribution in the hidden truncation model described below.

   Let (Z_1, Z_2) be a bivariate normally distributed random vector with mean $(0, 0)$, variances $\text{var}(Z_1) = \text{var}(Z_2) = 1$, and correlation coefficient $\rho \in (-1, 1)$. Then the conditional distribution of Z_1, given $Z_2 > 0$ (usually ignored and, thus, called a hidden truncation at the mean), is $\text{SN}(\lambda(\rho))$, where $\lambda(\rho) = \rho/\sqrt{1 - \rho^2}$ (see, e.g. Azzalini and Dalla Valle (1996)). In general, for any $a \in \mathbb{R}$, the conditional distribution of Z_1, given $Z_2 > a$ (a hidden truncation at a), has density function

   $$f_{a,\lambda}(x) = [\Phi(-a)]^{-1}\phi(x)\Phi\left(\frac{-a}{\sqrt{1 - \rho^2}} + \lambda(\rho)x\right), \qquad x \in \mathbb{R} \tag{3}$$

   (see, e.g. Arnold and Beaver (2000)). Let $Y_{a,\lambda}$ be a positive random variable such that $\ln Y_{a,\lambda}$ has the density function $f_{a,\lambda}$ in (3). Then $Y_{a,\lambda}$ is M-indeterminate. This will be an extension of our Theorem 1. The proof follows the same steps and the details are thus omitted. The Stieltjes class in Theorem 3 can be adapted to this case with $\rho > 0$.

3. Suppose that in the original definition of the skew-normal density $2\phi(x)\Phi(\lambda x)$, $x \in \mathbb{R}$, we replace the normal density ϕ by an arbitrary symmetric density $g(x)$, $x \in \mathbb{R}$, and the standard normal distribution function Φ by an arbitrary symmetric distribution function $H(x)$, $x \in \mathbb{R}$. Both symmetries are with respect to 0. Then, for $\lambda \in \mathbb{R}$, we define the function $f(x \mid \lambda) = 2g(x)H(\lambda x)$, $x \in \mathbb{R}$, which is a density. The distribution F with density f is said to be a *skew-symmetric distribution*. A positive random variable Y is said to obey a *logarithmic skew-symmetric distribution* if its logarithmic transform $\ln Y$ has a skew-symmetric distribution. It is possible to find conditions on the density g and the distribution H such that Y is M-indeterminate. This also extends our Theorem 1. Details will be reported elsewhere.

4. We suggest that the class of LSN distributions is linked to stochastic models involving distributions which are more flexible than the normal and lognormal distributions. Besides Chai and Bailey's (2008) paper cited in the introduction, other relevant references include Azzalini *et al.* (2003), Williamson and Gaston (2005), and Yamazaki and Lueck (1990).

Acknowledgements

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