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SYNCHRONIZATION OF COUPLED MAP LATTICES

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Abstract In this paper, we address the issue of synchronization of coupled systems, introducing concepts of local and global synchronization for a class of systems that extend the model of coupled map lattices. A criterion for local synchronization is given; numerical experiments are exhibited to illustrate the criteria and also to raise some questions in the end of the text.

Keywords: stochastic kernel; global/local synchronization; coupled map lattice; Hardy-Littlewood maximal operator; Amari neural network; Markov chain

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1. Introduction

Coupled map lattices were introduced many years ago as a model of identical (nonlinear) dynamical systems interacting by means of some coupling (see, e.g., [3, 10, 11] and references therein). Many copies of the same dynamical system $f\colon X \to X$ are distributed over the points of some lattice Σ (that can be finite or infinite), and an interaction is used to couple the maps on different points. Among some phenomena that can be studied in this context, we have the propagation of some signals through the lattice, solitons, certain kinds of phase transitions when invariant measures are considered (see [12]), the entropy of the map and its dependence on the coupling (see [4]), and the so-called synchronization where the orbits of distinct points of the lattice have the same asymptotic behavior (see, e.g., [9]; for an approach using the ideas of extreme value theory, see [7]).

The purpose of this work is to find some general conditions for ensuring synchronization when the usual linear coupling is replaced by a general Markov kernel, extending part of what appears in [9] to less symmetrical situations and also to understand what may happen when those conditions are relaxed. The dynamics on the base is assumed to be C^1−smooth, allowing also the use of the concept of Lyapunov exponent. Some numerical experiments suggest that part of the results about synchronization can remain valid for a Lipschitz map, but the infinitesimal argument used in the C^1-smooth context cannot be reproduced in this more general case.

We also relax some conditions on the lattice Σ, replacing this by a more general set that is assumed to be a Polish space in what we call a coupled map system; a similar idea appeared in [13], where the lattice is replaced by a Cantor set, and a possible time discretization of the Amari neural field equation (see [1, 2]) used in biomathematics leads to a case where the lattice is replaced by the real line. It is also interesting to notice that some classical analytical objects, like the Hardy–Littlewood operator acting on real functions, can be seen as limits of the action of a certain coupled map system where the lattice is replaced, again, by the real line. In these new situations, it is also possible to adapt some of the arguments used to get synchronization.

This paper is organized as follows: in § 2, we fix the notation and the main definitions. In § 3, we have the results about synchronization of C^1-smooth transformations for coupled map lattices and coupled map systems. In § 4, we present the proofs. In § 5, we introduce several classes of coupled map lattices and systems to which our results apply. In § 6, we provide some counter-examples which illustrate the need for the hypothesis of the theorems as well as the limitations in their conclusions. Some problems and future directions, part of them based on numerical experiments, are discussed in § 7.

2. Definitions

A topological space Σ is called a *Polish space* if it is separable, and there exists a metric d on Σ which defines the topology of Σ such that (Σ, d) is complete. Every Polish space is a Radon space. In particular, every Borel measure on Σ is regular (see [15]). From now on Σ will denote a Polish space and $\mathscr{F}$ will be the Borel σ-algebra on Σ.

A *stochastic kernel* on Σ is any function $K\colon \Sigma \times \mathscr{F} \to [0, 1]$ such that

(1) the function $B \mapsto K(x, B)$, from $\mathscr{F}$ to $[0, 1]$, is a probability measure for any $x \in \Sigma$;
(2) the function $x \mapsto K(x, B)$, from Σ to $[0, 1]$, is $\mathscr{F}$-measurable for any $B \in \mathscr{F}$.

The convolution of two kernels K_1 and K_2 on Σ is

$$(K_2 \times K_1)(x, B) = \int_\Sigma K_1(x, \mathrm{d}y)K_2(y, B), \qquad x \in \Sigma, B \in \mathscr{F}.$$

The space of kernels on Σ is a convolution semigroup with identity, where the identity is the kernel $K(x, .) = \delta_x$, and δ_x stands for the Dirac measure supported on x. The iterated kernels are defined recursively, setting $K^1 = K$, and for $n \geq 1$,

$$K^{n+1}(x, B) = \int_\Sigma K(x, \mathrm{d}y)K^n(y, B).$$

A probability measure μ on $(\Sigma, \mathscr{F})$ is called *K-stationary* if for all $B \in \mathscr{F}$,

$$\mu(B) = \int K(x, B)\mu(\mathrm{d}x).$$

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A set $B \in \mathscr{F}$ is said to be K-*invariant* when $K(x,B) = 1$ for all $x \in B$ and $K(x,B) = 0$ for all $x \in X \setminus B$. A K-stationary measure μ is called *ergodic* when there is no K-invariant set $B \in \mathscr{F}$ such that $0 < \mu(B) < 1$. As usual, ergodic measures are the extremal points in the convex set of K-stationary measures.

We say that a stochastic kernel K is *strongly mixing* if it admits a unique stationary probability measure μ with $\mathrm{supp}(\mu) = \Sigma$ and there are constants $C > 0$ and $0 < \rho < 1$ such that for every $\psi \in L^\infty(\Sigma)$, all $x \in \Sigma$ and $n \in \mathbb{N}$,

$$\left| \int_\Sigma \psi(y)\, K^n(x, \mathrm{d}y) - \int_\Sigma \psi(y)\, \mu(\mathrm{d}y) \right| \le C\rho^n \, \|\psi\|_\infty .$$

We say that K satisfies the *Doeblin condition* if there is a positive finite measure ρ on $(\Sigma, \mathscr{F})$ and some $\varepsilon > 0$ such that for all $x \in \Sigma$ and $B \in \mathscr{F}$,

$$K(x,B) \ge 1 - \varepsilon \quad \Rightarrow \quad \rho(B) \ge \varepsilon.$$

Theorem 2.1 *Let K be a stochastic kernel on $(\Sigma, \mathscr{F})$. If K satisfies the Doeblin condition, then there are sets $\Sigma_1, \ldots, \Sigma_m$ in $\mathscr{F}$ and probability measures $\nu_1, \ldots, \nu_m$ on Σ such that for all $i, j = 1, \ldots, m$,*

(1) $\Sigma_i \cap \Sigma_j = \emptyset$ *when* $i \ne j$,
(2) Σ_i *is K-forward invariant, that is,* $K(x, \Sigma_i) = 1$ *for* $x \in \Sigma_i$,
(3) ν_i *is K-stationary and ergodic with* $\nu_i(\Sigma_j) = \delta_{ij}$,
(4) $\lim_{n \to +\infty} K^n(x, \Sigma_1 \cup \cdots \cup \Sigma_m) = 1$, *with geometric uniform speed of convergence, for all* $x \in \Sigma$,
(5) $\nu(\Sigma_1 \cup \cdots \cup \Sigma_m) = 1$, *for every K-stationary probability* ν.

Moreover, for every $1 \le i \le m$, there is an integer $p_i \in \mathbb{N}$ and measurable sets $\Sigma_{i,1}, \ldots, \Sigma_{i,p_i} \in \mathscr{F}$ such that

(1) $\{\Sigma_{i,1}, \ldots, \Sigma_{i,p_i}\}$ *is a partition of* Σ_i,
(2) $K(x, \Sigma_{i,j+1}) = 1$ *for* $x \in \Sigma_{i,j}$ *and* $1 \le j \le p_i$, *with* $\Sigma_{i,p_i+1} = \Sigma_{i,1}$,
(3) *the stochastic kernel K^{p_i} on $\Sigma_{i,j}$ is strongly mixing for all* $1 \le j \le p_i$.

Proof. See [5, section V-5]. □

In the context of Doeblin condition, we call the sets Σ_i, $1 \le i \le m$, the *ergodic components* of K and the integers p_i their *periods*. We call the sets $\Sigma_{i,j}$, $1 \le j \le p_i$, the *mixing subcomponents* of the ergodic component Σ_i. We shall say that an ergodic component is *aperiodic* when its period is equal to 1.

A stochastic kernel K determines an operator $K : L^\infty(\Sigma) \to L^\infty(\Sigma)$ defined by

$$(K\psi)(x) := \int_\Sigma \psi(y)\, K(x, \mathrm{d}y). \tag{1}$$

This operator satisfies the following.

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Proposition 2.2. *For any $\psi \in L^\infty(\Sigma)$,*

(a) $K\mathbb{1} = \mathbb{1}$, *where* $\mathbb{1}$ *denotes the constant function* 1.
(b) $\int_\Sigma K\psi \, \mathrm{d}\mu = \int_\Sigma \psi \, \mathrm{d}\mu$,
(c) $\|K\psi\|_\infty \le \|\psi\|_\infty$.

In particular, if $H_0 = \{\psi \in L^\infty(\Sigma) \colon \int_\Sigma \psi \, \mathrm{d}\mu = 0\}$, *then* $L^\infty(\Sigma) = \mathbb{R}\mathbb{1} \oplus H_0$ *is a* K*-invariant decomposition and* $\rho(K|_{H_0}) \le 1$.

Proof. Since $\int_\Sigma K(x, dy) = 1$, items (a) and (c) follow. Item (b) is a consequence of μ being a K-stationary probability measure. □

If the kernel K on Σ is strongly mixing, we define its *mixing rate* to be the spectral radius of the restriction of K to the invariant subspace H_0 in Proposition 2.2, that is,

$$\tau^*(K) := \rho(K|_{H_0}).$$

In general, if K satisfies the Doeblin condition, using the notation of Theorem 2.1, we define its *escape rate* $\beta^*(K)$ to be the spectral radius of the operator $r_\Lambda \circ K|_{L^\infty(\Lambda)}$, where $\Lambda = \Sigma \setminus (\Sigma_1 \cup \cdots \cup \Sigma_m)$ and $r_\Lambda : L^\infty(\Sigma) \to L^\infty(\Lambda)$ is the restriction operator. Notice that the operator $K_\Lambda : L^\infty(\Lambda) \to L^\infty(\Lambda)$, $K_\Lambda = r_\Lambda \circ K|_{L^\infty(\Lambda)}$ is sub-stochastic and hence has spectral radius less than 1. We also define the *mixing rate of the component* Σ_i as $\tau^*(K, \Sigma_i) := \sqrt[p_i]{\tau^*(K^{p_i}, \Sigma_{i,j})}$, where $\tau^*(K^{p_i}, \Sigma_{i,j})$ denotes the mixing rate of the strongly mixing stochastic kernel obtained by restriction of the kernel K^{p_i} to any of the mixing subcomponents $\Sigma_{i,j}$, $1 \le j \le p_i$.

Let Σ be a discrete space, that we call a *lattice*. Consider a convex and compact set $X \subset \mathbb{R}^d$, a map $f \colon X \to X$, the base dynamical system and a probability kernel $K \colon \Sigma \times \Sigma \to [0, 1]$. If Σ is finite with k elements, then the stochastic kernel K can be identified with a $k \times k$ stochastic matrix. We recall that a *(row) stochastic matrix* on a finite set Σ is any square matrix $A = [a_{ij}] \in \mathbb{R}^{k \times k}$ such that $a_{ij} \ge 0$ for all $i, j \in \Sigma$ and $\sum_{j \in \Sigma} a_{ij} = 1$ for all $i \in \Sigma$. A stochastic matrix A is called *primitive* if there exists $n \ge 1$ for which the power matrix $A^n = [a^n_{ij}]$ has all entries strictly positive, that is, $a^n_{ij} > 0$ for $i, j \in \Sigma$. Notice that if A is primitive, then the kernel K on Σ determined by A is strongly mixing. In particular, Σ is an aperiodic ergodic component. We define the *mixing rate of a primitive matrix* A, denoted by $\tau^*(A)$, as the mixing rate of the strongly mixing kernel defined by A on Σ.

The *coupled map lattice* is the dynamical system $F : X^\Sigma \to X^\Sigma$ defined by

$$F(\varphi) := K(f \circ \varphi),$$

where X^Σ denotes the space of all functions from Σ to X. Since K is a probability kernel, we get that for the constant function $\varphi = c$, the image $K(f \circ \varphi) = Kf(c) = f(c)$. Hence, the set of constant functions is invariant under the dynamics defined by F.

More generally, given a Polish space Σ, consider the space $L^\infty(\Sigma, X)$ of all measurable functions $\varphi : \Sigma \to X$. Notice that since X is compact, all functions in $L^\infty(\Sigma, X)$ are bounded. We endow this space with the uniform convergence topology, which is

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determined by the uniform distance

$$d_\infty(\varphi, \psi) := \|\varphi - \psi\|_\infty .$$

As before, we can define a transformation $F : L^\infty(\Sigma, X) \to L^\infty(\Sigma, X)$ by

$$F(\varphi) := K(f \circ \varphi),$$

which we will refer to as a *coupled map system.*

We say that the coupled map system $F : L^\infty(\Sigma, X) \to L^\infty(\Sigma, X)$ has *global synchronization over a subset* $\Lambda \subset \Sigma$ if for every $\delta > 0$, there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ and for every $\varphi \in L^\infty(\Sigma, X)$ and all $x, y \in \Lambda$, one has

$$\|F^n(\varphi)(x) - F^n(\varphi)(y)\| < \delta.$$

When $\Lambda = \Sigma$, we simply talk about global synchronization of F. Let $\Delta \subset L^\infty(\Sigma, X)$ denote the compact and convex subset of all constant functions. Notice that F has global synchronization if and only if Δ is a global attractor of F on $L^\infty(\Sigma, X)$.

We say that F has *local synchronization over a subset* $\Lambda \subset \Sigma$ if there exists a neighbourhood $\mathscr{U}$ of Δ in $L^\infty(\Sigma, X)$ such that for every $\delta > 0$, there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ and for every $\varphi \in \mathscr{U}$ and all $x, y \in \Lambda$, one has

$$\|F^n(\varphi)(x) - F^n(\varphi)(y)\| < \delta.$$

The map F has local synchronization (over Σ) if and only if Δ is a local attractor of F on $L^\infty(\Sigma, X)$.

Clearly, global synchronization implies local synchronization, but the converse is not true.

3. Main results

Throughout this section, we assume that $X \subset \mathbb{R}^d$ is a compact convex set and $f : X \to X$ is a C^1-smooth map whose Lipschitz constant is denoted by $\mathrm{Lip}(f)$.

We define

$$\ell(f) := \lim_{n\to\infty} \frac{1}{n} \log \mathrm{Lip}(f^n).$$

This limit exists by Fekete's Lemma. As usual, the derivative of the map f at a point $x \in X$ is denoted by $Df_x : \mathbb{R}^d \to \mathbb{R}^d$. The *top Lyapunov exponent* of a periodic point $x = f^n(x)$ with period n is defined to be

$$\lambda(f, x) := \frac{1}{n} \log \|Df^n_x\| .$$

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By the mean value theorem, $\|Df_x\| \leq \mathrm{Lip}(f)$. Hence, for all periodic points $x \in X$ of f, one has

$$\lambda(f,x) \leq \ell(f).$$

We define the *oscillation* semi-norm $\|.\|_{\mathrm{o}}$ of a function $\varphi \in L^\infty(\Sigma, \mathbb{R}^d)$ by

$$\|\varphi\|_{\mathrm{o}} := \sup_{x,y\in\Sigma} \|\varphi(x) - \varphi(y)\|\,.$$

Proposition 3.1. *The oscillation semi-norm satisfies*

(1) $\|g \circ \varphi\|_{\mathrm{o}} \leq \mathrm{Lip}(g)\, \|\varphi\|_{\mathrm{o}}$ *for any Lipschitz function* $g\colon \mathbb{R}^d \to \mathbb{R}^d$, *and*
(2) $\|\varphi\|_{\mathrm{o}} = 0 \Leftrightarrow \varphi$ *is constant.*

Proof. Given any Lipschitz function $g\colon \mathbb{R}^d \to \mathbb{R}^d$,

$$\begin{aligned}\|g \circ \varphi\|_{\mathrm{o}} &= \sup_{x,y\in\Sigma} \|g(\varphi(x)) - g(\varphi(y))\| \\ &\leq \sup_{x,y\in\Sigma} \mathrm{Lip}(g)\, \|\varphi(x) - \varphi(y)\| \\ &\leq \mathrm{Lip}(g)\, \|\varphi\|_{\mathrm{o}}\,,\end{aligned}$$

and, clearly, $\|\varphi\|_{\mathrm{o}} = 0$ if and only if φ is constant. □

We define the norm of a stochastic kernel K, regarded as an operator, by

$$|||K|||_{\mathrm{o}} := \sup_{\|\varphi\|_{\mathrm{o}}\neq 0} \frac{\|K\varphi\|_{\mathrm{o}}}{\|\varphi\|_{\mathrm{o}}}.$$

Theorem 3.2 *If* $|||K|||_{\mathrm{o}}\, \mathrm{Lip}(f) < 1$, *then* F *has global synchronization.*

Proof. Given $\varphi \in L^\infty(\Sigma, X)$,

$$\begin{aligned}\|F(\varphi)\|_{\mathrm{o}} = \|K(f\circ\varphi)\|_{\mathrm{o}} &\leq |||K|||_{\mathrm{o}}\, \|f \circ \varphi\|_{\mathrm{o}} \\ &\leq |||K|||_{\mathrm{o}}\, \mathrm{Lip}(f)\; \|\varphi\|_{\mathrm{o}}\,.\end{aligned}$$

Hence,

$$\|F^n(\varphi)\|_{\mathrm{o}} \leq [|||K|||_{\mathrm{o}}\, \mathrm{Lip}(f)]^n\, \|\varphi\|_{\mathrm{o}}$$

converges to 0 as $n \to +\infty$. This proves that F has global synchronization. □

Theorem 3.3 *Given a finite set* Σ *and a stochastic primitive matrix* A *on* Σ, *if there exists a periodic orbit* $x \in X$ *with Lyapunov exponent* $\lambda = \lambda(f,x)$ *such that*

$$\tau^*(A)\, \mathrm{e}^{\lambda} > 1,$$

then F *does not have local synchronization over* Σ.

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Theorem 3.4 *Given a Polish space* Σ *and a stochastic kernel* K *on* Σ *satisfying the Doeblin condition with a unique aperiodic ergodic component* Σ_0*, if*

$$\max\{\beta^*(K), \tau^*(K, \Sigma_0)\}\, \mathrm{e}^{\ell(f)} < 1,$$

then F *has local synchronization over* Σ.

The next theorem provides a sufficient condition for local synchronization over the mixing subcomponents of an ergodic component.

Theorem 3.5 *Given a Polish space* Σ *and a stochastic kernel* K *on* Σ *satisfying the Doeblin condition with an ergodic component*

$$\Sigma_i = \Sigma_{i,1} \cup \cdots \cup \Sigma_{i,p},$$

where the sets $\Sigma_{i,j}$ *stand for the mixing subcomponents of* Σ_i*, if*

$$\tau^*(K, \Sigma_i)\, \mathrm{e}^{\ell(f)} < 1,$$

then F *has local synchronization over all mixing subcomponents* $\Sigma_{i,j}$ *of* Σ_i.

4. Proofs

Proof of Theorem 3.3. Let $\Delta = \{u \in X^\Sigma\colon u_i = u_j\ \forall\, i, j \in \Sigma\}$. The transformation F has local synchronization over Σ iff Δ is a local attractor whose basin of attraction is a neighbourhood of Δ. To see that F does not have local synchronization over Σ, it is enough to find the F-periodic point $u \in \Delta$ with period p such that the derivative DF^p_u has an eigenvalue $\alpha \in \mathbb{C}$ with $|\alpha| > 1$ associated to some (complex) eigenvector, which does not lie in the complexification of the tangent space $T_u\Delta$.

Consider the periodic point $x = f^p(x)$ whose existence is hypothesized in this theorem, and let $u = (x, x, \ldots, x) \in X^\Sigma$. Then $u \in \Delta$ is a periodic point of F with the same period p. Let $w \in \mathbb{C}^\Sigma$ be an eigenvector of A such that $\sum_{i\in\Sigma} w_i = 0$ and $Aw = \beta w$, where $|\beta| = \tau^*(A)$. Writing $A = [a_{ij}] \in \mathbb{R}^{\Sigma\times\Sigma}$, let $\hat{A} \in (\mathbb{R}^{d\times d})^{\Sigma\times\Sigma}$ be the matrix with entries $\hat{a}_{ij} := a_{ij} I_d$, where I_d is the identity matrix in $\mathbb{R}^{d\times d}$. Let $M_j = Df_{f^j(x)}$ and denote by D_{M_j} the block diagonal matrix

$$D_{M_j} = \begin{bmatrix} M_j & 0 & \cdots & 0 \\ 0 & M_j & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_j \end{bmatrix}.$$

Finally, let $v \in \mathbb{C}^d$ be a non-zero vector such that $Df^p_x\, v = M_{p-1} \cdots M_1\, M_0\, v = \alpha\, v$ with $\alpha \in \mathbb{C}$ and $\frac{1}{p} \log|\alpha| = \lambda(x, f)$. The Jacobian matrix of F at each point $F^j(u)$ is the matrix

$$DF_{F^j(u)} = \hat{A}\, D_{M_j}.$$

Since the matrices $\hat{A}$ and D_{M_j} commute, we have

$$DF^p_u = \hat{A}^p D_M,$$

where $M = M_{p-1}\cdots M_1\, M_0 = Df^p_x$. Now the complex vector $V = (w_i v)_{i\in\Sigma} \in (\mathbb{C}^d)^\Sigma$ satisfies

$$DF^p_u V = \hat{A}^p D_M V = \beta^p \alpha V.$$

Since the vector V does not lie in the complexification, $\Delta_{\mathbb{C}} = \{(w, w, \ldots, w)\colon w \in \mathbb{C}\}$ of the tangent space $T_u\Delta$ and

$$|\beta^p\alpha| = \tau^{*p}\, \mathrm{e}^{p\lambda} = (\tau^*\, \mathrm{e}^{\lambda})^p > 1,$$

the F-invariant set Δ cannot be a local attractor. □

Proof of Theorem 3.4. Assume first that the stochastic kernel K is strongly mixing, and consider the space

$$H_0 := \{\varphi \in L^\infty(\Sigma, \mathbb{R}^d)\colon \int_\Sigma \varphi\, \mathrm{d}\mu = 0\}$$

introduced in Proposition 2.2. Let $\tau^* = \tau^*(K) < 1$ be the spectral radius of $K|_{H_0}$.

In the end, we explain how to proceed in the general case. Taking $\kappa > 0$ with $\tau^*(K)\, \mathrm{e}^{\ell(f)} < \kappa < 1$, there exists an integer $m \in \mathbb{N}$ such that $|||K^m|||_{\mathrm{o}}\, \mathrm{Lip}(f^m) < \kappa^m$.

Because f is C^1-smooth, then so is F. On the diagonal Δ, the derivatives of the linear action of K and f commute. Consequently, for all $\varphi \in \Delta$

$$\left\|DF^n_\varphi\right\| \le \tau^{*n}\, \mathrm{e}^{n\ell} < \kappa^n,$$

with $0 < \kappa < 1$. Hence, there is a neighbourhood $\mathscr{U}$ of Δ where F^n is a contraction. This implies that for all $\varphi \in \mathscr{U}$,

$$\lim_{k\to+\infty} \left\|F^{kn}\varphi\right\|_{\mathrm{o}} = 0.$$

Thus, for all $0 \le j < n$,

$$\lim_{k\to+\infty} \left\|F^{kn+j}\varphi\right\|_{\mathrm{o}} = 0,$$

and therefore

$$\lim_{k\to+\infty} \left\|F^{k}\varphi\right\|_{\mathrm{o}} = 0.$$

The case where K is not strongly mixing but there is only one ergodic acyclic component Σ_0 is the same as the previous one because the operator $K|_{H_0}$ has spectral radius equal to $\max\{\beta^*(K), \tau^*(K, \Sigma_0)\}$. □

Proof of Theorem 3.5. Same argument as before working in the mixing subcomponent. □

5. Coupled map classes

In this section, we introduce several classes of coupled map systems and lattices to which our results apply. In all examples below, $X \subset \mathbb{R}^d$ is any compact convex set and $f : X \to X$ is a C^1−smooth function.

Finite coupled map lattices

Consider a finite set $\Sigma = \{1, \ldots, k\}$ and a stochastic $k \times k$ matrix $A = [a_{ij}]$. The associated coupled map lattice is the transformation $F : X^k \to X^k$ with components

$$F_i(x) = \sum_{j=1}^{k} a_{ij} f(x_j), \quad \text{where } \ x = (x_1, \ldots, x_k).$$

In this setting, Theorem 2.1 is a simple consequence of classical Markov Chain's Theory. Let $\Sigma_1, \ldots, \Sigma_m$ be the ergodic components of the stochastic kernel K defined by A on Σ, and let $\Lambda = \Sigma \setminus (\Sigma_1 \cup \ldots \cup \Sigma_m)$. The escape rate $\beta^*(K)$ and the mixing rate $\tau^*(K, \Sigma_i)$ can be estimated as described below. See [5, section V-2] and also [6, section 5]*. Recall that $\tau^*(K, \Sigma_i) := \sqrt[p_i]{\tau^*(K^{p_i}, \Sigma_{i,j})}$, where $\tau^*(K^{p_i}, \Sigma_{i,j})$ denotes the mixing rate of the strongly mixing stochastic kernel obtained by restriction of the kernel K^{p_i} to any of the subcomponents $\Sigma_{i,j}$. Let $A^{p_i} = [a_{ij}^{p_i}]$. We have that

$$\beta^*(K) = \inf_{n \geq 1} [\beta(K^n)]^{1/n}, \tag{2}$$

where

$$\beta(K) = 1 - \min_{\ell \in \Sigma} \sum_{k \in \Sigma_1 \cup \cdots \cup \Sigma_m} a_{\ell k} = \max_{\ell \in \Sigma} \sum_{k \in \Lambda} a_{\ell k}$$

and

$$\tau^*(K^{p_i}, \Sigma_{i,j}) = \inf_{n \geq 1} [\tau(K^{n p_i}, \Sigma_{i,j})]^{1/n}, \tag{3}$$

* We notice that all results in § 5 of this reference are abstract and hence hold for general measure spaces (X, m). This includes the case where X is finite.

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where

$$\tau(K^{p_i}, \Sigma_{i,j}) = \frac{1}{2} \max_{\ell,\ell' \in \Sigma_{i,j}} \sum_{k \in \Sigma_{i,j}} \left| a^{p_i}_{\ell k} - a^{p_i}_{\ell' k} \right| = 1 - \min_{\ell,\ell' \in \Sigma_{i,j}} \sum_{k \in \Sigma_{i,j}} a^{p_i}_{\ell k} \wedge a^{p_i}_{\ell' k}.$$

This class contains, for example, coupled map lattices with translation-invariant coupling and arbitrary Lipschitz continuous individual map on the periodic lattice $\mathbb{Z}_L := \{s \in \mathbb{Z} \bmod L\}$ $(L > 1)$.

Infinite coupled map lattices

Assume now that Σ is an infinite (countable) lattice, say $\Sigma = \mathbb{N} = \{0, 1, \ldots\}$. Consider an infinite (row) stochastic matrix $A = [a_{ij}]$ such that for some $\alpha > 0$, we have $a_{i0} \geq \alpha$ for all $i \in \mathbb{N}$. This assumptions ensure that the stochastic kernel defined by A satisfies the Doeblin condition.

The associated coupled map lattice is the transformation $F : X^{\mathbb{N}} \to X^{\mathbb{N}}$ with components

$$F_i(x) = \sum_{j \in \mathbb{N}} a_{ij} f(x_j) \quad \text{where} \quad x = (x_j)_{j \in \mathbb{N}}.$$

The previous finite state formulas also hold in this countable case.

Coupled map systems

Assume that Σ is a compact metric space, say $\Sigma = [0, 1]$, and consider a bounded measurable function $k : [0, 1] \times [0, 1] \to [0, +\infty)$ such that

$$\int_0^1 k(x, y)\, \mathrm{d}y = 1 \quad \text{for all} \quad x \in [0, 1].$$

This function determines a kernel K on $[0, 1]$ defined by

$$K(x, B) = \int_B k(x, y)\, \mathrm{d}y \quad \text{for all} \quad B \subset [0, 1].$$

In this setting, the stochastic kernel K satisfies the Doeblin condition.

The associated coupled map system is the transformation $F : L^\infty([0, 1], X) \to L^\infty([0, 1], X)$

$$F(\varphi)(x) = \int_0^1 k(x, y) f(\varphi(y))\, \mathrm{d}y \quad \text{where} \quad \varphi \in L^\infty([0, 1], X).$$

Remark 5.1. It is interesting to mention that the uncoupled map, say, the case where $F(\varphi)(x) = f(\varphi(x))$, corresponds to a dynamical system already considered in [8].

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Remark 5.2. We note that the class of coupled map systems may contain the mean field limit of all-to-all coupled finite systems. In [16], the authors study models of this type, with very similar dynamics to our current setup of coupled map systems.

The previous given formulas for the escape rate and the mixing rate of the ergodic components Σ_i, $1 \leq i \leq m$, extend to stochastic kernels satisfying the Doeblin condition (see [5, section V-5] and also [6, section 5])*. We have that

$$\beta^*(K) = \inf_{n\geq 1} [\beta(K^n)]^{1/n},$$

where

$$\beta(K) = 1 - \inf_{x\in\Sigma} \int_{\Sigma_1\cup\cdots\cup\Sigma_m} k(x,y)\,\mathrm{d}y = \sup_{x\in\Sigma} \int_{\Lambda} k(x,y)\,\mathrm{d}y,$$

and

$$\tau^*(K^{p_i}, \Sigma_{i,j}) = \inf_{n\geq 1} [\tau(K^{np_i}, \Sigma_{i,j})]^{1/n},$$

where

$$\begin{aligned} \tau(K^{p_i}, \Sigma_{i,j}) &= \tfrac{1}{2} \sup\nolimits_{x,z\in\Sigma_{i,j}} \int_{\Sigma_{i,j}} |k^{p_i}(x,y) - k^{p_i}(z,y)|\,\mathrm{d}y \\ &= 1 - \inf_{x,z\in\Sigma_{i,j}} \int_{\Sigma_{i,j}} k^{p_i}(x,y) \wedge k^{p_i}(z,y)\,\mathrm{d}y. \end{aligned}$$

Example 5.3. The Amari neural field equation used in biomathematics is the integro-differential equation

$$\partial_t u(t,x) = -u(t,x) + \int_{\mathbb{R}} \mathcal{K}(x-y) S(u(t,y))\,\mathrm{d}y \qquad t > 0,\ x \in \mathbb{R},$$

where u represents the average neural activity, $\mathcal{K}$ is the connectivity kernel (modelling the interaction between neurons at distinct places) and S is the firing rate function. A first-order time discretization, considering a time step of size 1, consists in replacing the partial derivative by the difference $u(t+1,x) - u(t,x)$; we then get the map

$$u(t+1,x) = F(u(t,x)) = \int_{\mathbb{R}} \mathcal{K}(x-y) S(u(t,y))\,\mathrm{d}y$$

which is an example of a coupled map system where Σ is the real line (for another approximation of the Amari's model, the reader can see [14]).

* All results in § 5 of this reference are abstract and apply to the current setting.

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Example 5.4. Consider an integrable function $\varphi\colon \mathbb{R} \to \mathbb{R}$. The Hardy–Littlewood maximal operator maps φ to the function that is defined for any $x \in \mathbb{R}$ as

$$M\varphi(x) = \sup_{r>0} F_r(\varphi)(x),$$

where, for any given $r > 0$, F_r is given by

$$F_r(\varphi)(x) := \frac{1}{|B(x,r)|} \int_{B(x,r)} |\varphi(y)|\, \mathrm{d}y;$$

in this expression, $\mathrm{d}y$ is the Lebesgue measure on $\mathbb{R}$, $B(x,r)$ is the open interval $(x-r, x+r)$ and $|B|$ denotes its Lebesgue measure. Notice that F_r corresponds to the coupled map system, where $X = \Sigma = \mathbb{R}$, $f(\cdot) = |\cdot|$ and $k(x,y) = \frac{dy}{|B(x,r)|}$ is the normalized Lebesgue measure restricted to the ball $B(x,r)$.

6. Counter-examples

In this section, we provide two counter-examples which illustrate the need for the hypothesis of the theorems as well as the limitations in their conclusions.

In the first example, the hypothesis of Theorem 3.4 is satisfied but the coupled lattice map has no global synchronization.

Example 6.1. Let $X = [0,3]$ and $f : X \to X$ be a C^1-smooth map such that $f(1) = 2/3$, $f(8/5) = 2$, $f(2) = 16/9$ and

$$\frac{3}{2} = |f'(2)| < \mathrm{Lip}(f) = \frac{20}{9} = \mathrm{Lip}\left(f|_{\{1,\frac{8}{5}\}}\right).$$

An example (see Figure 1) is given by

$$f(t) := \begin{cases} \frac{8\,t^3}{9} - \frac{2\,t^2}{9} & \text{for} \quad 0 \le t < 1 \\ \frac{20(t-1)}{9} + \frac{2}{3} & \text{for} \quad 1 \le t < \frac{8}{5} \\ \frac{275\,t^3}{24} - \frac{2395\,t^2}{36} + \frac{1144\,t}{9} - 78 & \text{for} \quad \frac{8}{5} \le t < 2 \\ \frac{t^3}{18} + \frac{t^2}{3} - \frac{7\,t}{2} + 7 & \text{for} \quad 2 \le t \le 3 \end{cases}.$$

Indeed, given $(t_0, x_0, v_0), (t_1, x_1, v_1) \in \mathbb{R}^3$ with $t_0 < t_1$, there exists a unique cubic polynomial $p(t)$ that interpolates these tuples in the sense that

$$p(t_0) = x_0,\ p'(t_0) = v_0,\ p(t_1) = x_1 \text{ and } \ p'(t_1) = v_1.$$

The above function is the unique piecewise cubic polynomial of class C^1 that interpolates the sequence of tuples

$$(0,0,0),\ \left(1, \frac{2}{3}, \frac{20}{9}\right),\ \left(\frac{8}{5}, 2, \frac{20}{9}\right),\ \left(2, \frac{16}{9}, -\frac{3}{2}\right),\ (3,1,0).$$

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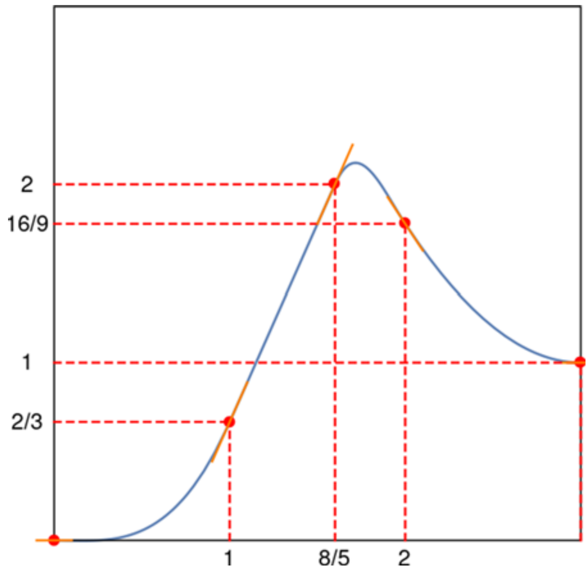

Figure 1. The map f of Example 6.1.

One can easily check that, in each branch, the maximum absolute value of $f'(t)$ is less than or equal to 20/9.

Consider the stochastic matrix

$$A = \begin{bmatrix} \frac{7}{10} & \frac{3}{10} & 0 \\ 0 & \frac{7}{10} & \frac{3}{10} \\ 1 & 0 & 0 \end{bmatrix}$$

and the coupled lattice map $F : [0,3]^3 \to [0,3]^3$ defined by

$$F(x,y,z) = \left(\frac{7}{10} f(x) + \frac{3}{10} f(y), \frac{7}{10} f(y) + \frac{3}{10} f(z), f(x) \right).$$

The point $p = (8/5, 1, 2)$ is a fixed point of F. Because this fixed point is off the diagonal $\Delta = \{(x,x,x) \colon x \in [0,3]\}$, the map F cannot have global synchronization.

The matrix A has eigenvalues 1 and $\frac{1}{10}(2 \pm \sqrt{5}\, i)$ (with absolute value 3/10). The stochastic kernel K determined by A has a unique aperiodic ergodic component $\Sigma_0 = \Sigma$. Hence, $\tau^*(K, \Sigma) = 3/10$ and $\beta^*(K) = 0$. Consequently,

$$\max\{\beta^*(K), \tau^*(K,\Sigma)\} \operatorname{Lip}(f) \leq \frac{3}{10} \cdot \frac{20}{9} = \frac{2}{3} < 1.$$

The second example illustrates a transition from local synchronization over Σ to local synchronization over an ergodic component.

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Example 6.2. Consider the following family of stochastic matrices

$$A_t := \begin{bmatrix} \frac{1-t}{2} + \frac{4t}{9} & \frac{1-t}{2} + \frac{5t}{9} & 0 \\ \frac{1-t}{2} + \frac{t}{6} & \frac{1-t}{2} + \frac{5t}{6} & 0 \\ 0 & \frac{7(1-t)}{18} + \frac{17t}{18} & \frac{11(1-t)}{18} + \frac{t}{18} \end{bmatrix}$$

with $0 \leq t \leq 1$, whose associated family of kernels K_t has a unique aperiodic ergodic component $\Sigma_0 = \{1, 2\}$. Applying the formulas (2)–(3), we get $\beta^*(K_t) = \frac{11-10\,t}{18}$ and $\tau^*(K_t, \Sigma_0) = \frac{5t}{18}$.

Let $f : [0, 1] \to [0, 1]$ be the map $f(x) := 4x(1-x)$, which has Lipschitz constant $\mathrm{Lip}(f) = 4$. We have $\mathrm{e}^{\ell(f)} = 4$ because 0 is a fixed point with derivative $f'(0) = 4$.

In this example, the threshold conditions of Theorems 3.4 and 3.5 are, respectively,

$$(\mathscr{R}_1) \quad \max\{\beta^*(K_t), \tau^*(K_t, \Sigma_0)\}\, \mathrm{e}^{\ell(f)} < 1 \quad \Leftrightarrow \quad 0.65 < t < 0.9;$$

$$(\mathscr{R}_2) \quad \tau^*(K_t, \Sigma_0)\, \mathrm{e}^{\ell(f)} < 1 \quad \Leftrightarrow \quad t < 0.9.$$

An analog of the threshold condition in Theorem 3.3 with Lyapunov exponent $\lambda(f) = \log 2$ w.r.t. Lebesgue measure is

$$(\mathscr{R}_3) \quad \max\{\beta^*(K_t), \tau^*(K_t, \Sigma_0)\}\, e^{\lambda(f)} < 1 \quad \Leftrightarrow \quad 0.2 < t.$$

See Figure 2.

Using Theorems 3.4 and 3.5,

(1) in region $(\mathscr{R}_1)$, (local) synchronization over Σ occurs;
(2) in region $(\mathscr{R}_2)$, (local) synchronization over Σ_0 occurs.

For $t < 0.65 = 13/20$, an easy calculation shows that $p_t = (0, 0, \frac{20t-13}{20t-22})$ is a fixed point of the coupled map F_t off the diagonal. This shows that there is no global synchronization for F_t on this range.

We run several numerical experiments, randomly choosing an initial condition in $[0, 1]^3$, computing $n = 200$ iterates and then plotting the last 100 iterates. These experiments indicate that (see Figures 3 and 4):

(3) in region $(\mathscr{R}_1)$, global synchronization over Σ occurs;
(4) in region $(\mathscr{R}_3)$, local synchronization over Σ occurs;
(5) outside region $(\mathscr{R}_3)$, synchronization over Σ never occurs.

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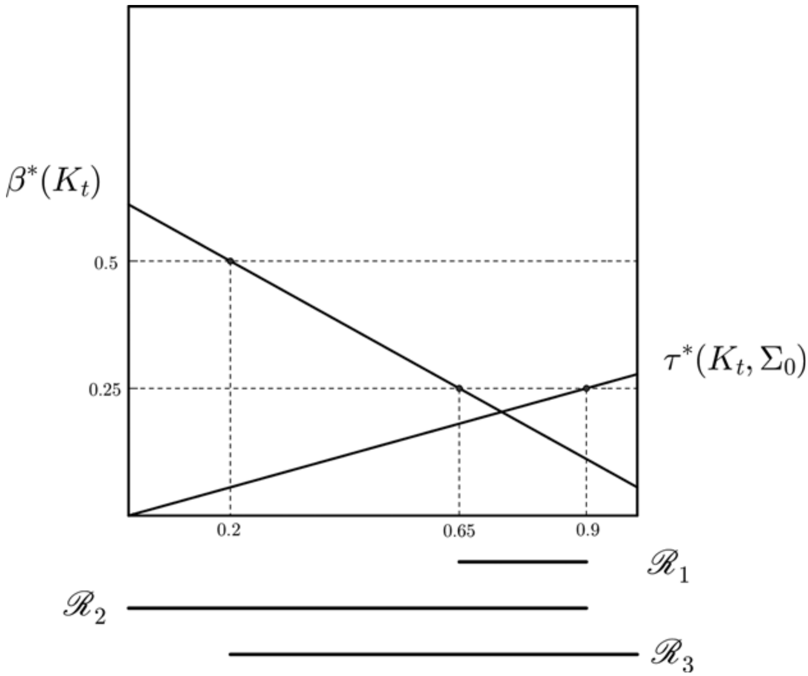

Figure 2. Threshold lines.

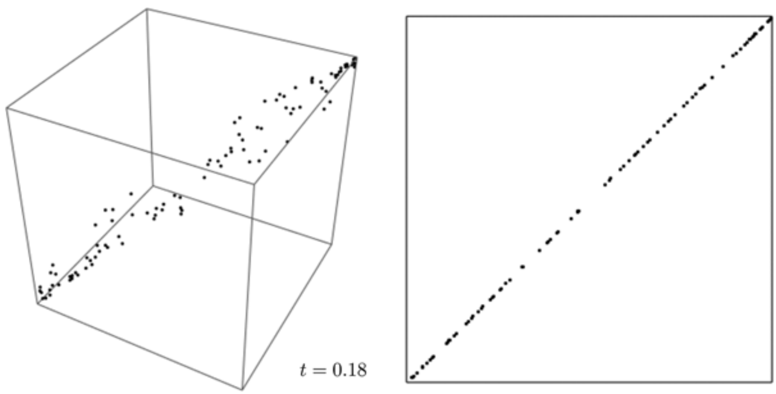

Figure 3. In Example 6.2, synchronization occurs, only over Σ_0, for the values $0 \leq t < 0.2$. The right plot represents the projection of the left one onto the plane $\mathbb{R}^{\Sigma_0}$, where $\Sigma_0 = \{1, 2\}$.

7. Conclusions

All examples that we have analyzed indicate that Theorems 3.4 and 3.5 should hold for any Lipschitz map $f : X \to X$. We note that the infinitesimal argument used in the C^1-smooth context cannot be reproduced in the Lipschitz case.

This motivates the following problem:

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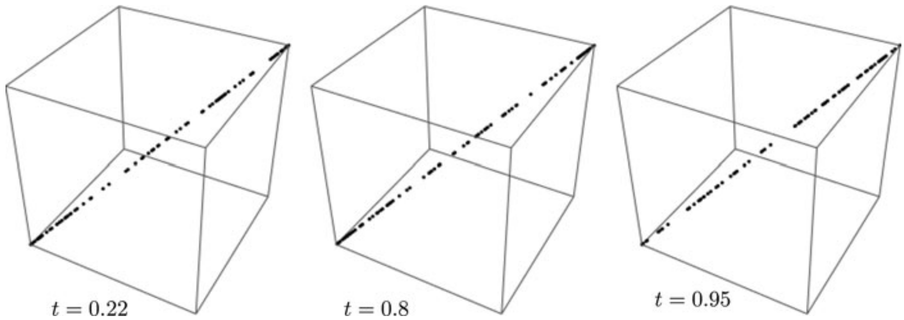

Figure 4. In Example 6.2, synchronization over Σ occurs for the values $0.2 < t \leq 1$.

Problem 1. *Prove Theorems 3.4 and 3.5 for Lipschitz maps $f : X \to X$, or at least for piecewise smooth maps.*

The two (Lipschitz) examples below provide numerical evidence that there is no local synchronization whenever the main hypothesis of Theorems 3.4 and 3.5 fail.

Example 7.1. Consider the following family of stochastic matrices

$$A_t := \begin{bmatrix} \frac{1-t}{2} + \frac{4t}{5} & \frac{1-t}{2} + \frac{t}{5} & 0 \\ \frac{1-t}{2} + \frac{t}{5} & \frac{1-t}{2} + \frac{4t}{5} & 0 \\ 0 & \frac{1-t}{4} + \frac{2t}{3} & \frac{3(1-t)}{4} + \frac{t}{3} \end{bmatrix}$$

with $0 \leq t \leq 1$, whose associated family of kernels K_t has a unique aperiodic ergodic component $\Sigma_0 = \{1, 2\}$. Applying the formulas (2)–(3), we have that $\beta^*(K_t) = \frac{9-5t}{12}$ and $\tau^*(K_t, \Sigma_0) = \frac{3t}{5}$.

Let $f : [0, 1] \to [0, 1]$ be the piecewise linear map

$$f(x) := \begin{cases} 2\,x & \text{if} \quad 0 \leq x \leq \frac{1}{2} \\ 2 - 2\,x & \text{if} \quad \frac{1}{2} < x \leq 1 \end{cases}$$

which has Lipschitz constant $\mathrm{Lip}(f) = 2$. A simple calculation shows that $\mathrm{e}^{\ell(f)} = 2$.

In this example, the threshold condition

$$\max\{\beta^*(K_t), \tau^*(K_t, \Sigma_0)\} < \frac{1}{2}$$

of Theorem 3.4 is equivalent to $0.6 = \frac{3}{5} < t < \frac{5}{6} \sim 0.833$ (see Figure 5). Experiments we run indicate that in this example, local synchronization never occurs whenever the threshold condition fails, that is, $t < \frac{3}{5}$ or $t > \frac{5}{6}$ (see Figure 6).

In the simulations, we randomly chose initial conditions near the diagonal Δ in $[0, 1]^3$ and computed $n = 200$ iterates, plotting the last 100 iterates.

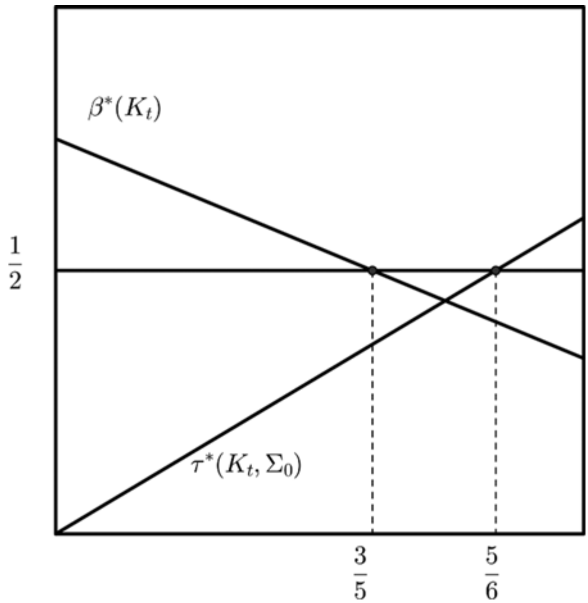

Figure 5. Threshold condition: $\max\{\beta^*(K_t), \tau^*(K_t, \Sigma_0)\} < \frac{1}{2}$.

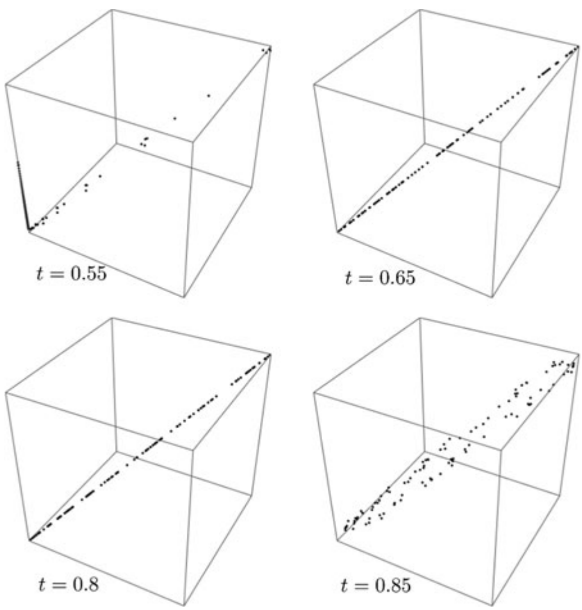

Figure 6. In Example 7.1 local synchronization only occurs for the values $0.6 = \frac{3}{5} < t < \frac{5}{6} \sim 0.833$.

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Example 7.2. Consider the following family of stochastic matrices

$$A_t := \begin{bmatrix} 0 & 0 & \frac{1-t}{2} + \frac{7t}{8} & \frac{1-t}{2} + \frac{t}{8} \\ 0 & 0 & \frac{1-t}{2} + \frac{t}{4} & \frac{1-t}{2} + \frac{3t}{4} \\ \frac{1-t}{2} + \frac{3t}{4} & \frac{1-t}{2} + \frac{t}{4} & 0 & 0 \\ \frac{1-t}{2} + \frac{t}{8} & \frac{1-t}{2} + \frac{7t}{8} & 0 & 0 \end{bmatrix}$$

with $0 \leq t \leq 1$. Each associated kernel K_t has a unique ergodic component $\Sigma_1 = \Sigma = \{1, 2, 3, 4\}$ with period 2. The mixing subcomponents are $\Sigma_{1,1} = \{1, 2\}$ and $\Sigma_{1,2} = \{3, 4\}$. Since the eigenvalues of A_t are 1, -1, $\frac{5\,t}{8}$ and $-\frac{5\,t}{8}$, we have that $\tau^*(K_t, \Sigma_1) = \frac{5\,t}{8}$.

Let $f : [0, 1] \to [0, 1]$ be the piecewise linear map

$$f(x) := \begin{cases} 2\,x & \text{if} \quad 0 \leq x \leq \frac{1}{2} \\ 2 - 2\,x & \text{if} \quad \frac{1}{2} < x \leq 1 \end{cases},$$

which has Lipschitz constant $\mathrm{Lip}(f) = 2$. A simple calculation shows that $\mathrm{e}^{\ell(f)} = 2$.

In this example, the threshold condition

$$\tau^*(K_t, \Sigma_1) < \frac{1}{2}$$

of Theorem 3.5 is equivalent to $t < 0.8$. Experiments we run have shown that in this example, local synchronization over the mixing subcomponents never occurs whenever the threshold condition fails, that is, $t > 0.8$.

In the simulations, we randomly chose initial conditions near the diagonal Δ in $[0, 1]^4$ and computed $n = 200$ iterates, plotting the last 100 iterates. Figures 7 and 8 illustrate the projections of these orbits in $[0, 1]^4$ to all six coordinate 2-planes.

Example 6.2 (in the smooth setting) and Examples 7.1 and 7.2 (in the Lipschitz case) suggest a weaker concept of synchronization.

Let $f : X \to X$ be a piecewise C^1-diffeomorphism preserving a probability measure μ on the Borel σ-algebra of X. Consider on $L^\infty(\Sigma, X) \subset X^\Sigma$ the product measure μ^Σ. We say that a coupled map system has *almost sure global synchronization* over $\Lambda \subset \Sigma$ if for μ^Σ almost every $\varphi \in L^\infty(\Sigma, X)$, given $\delta > 0$, there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ and all $x, y \in \Lambda$, one has

$$\|F^n(\varphi)(x) - F^n(\varphi)(y)\| < \delta.$$

In Example 6.2, the simulations indicate that almost sure global synchronization over $\Sigma = \{1, 2, 3\}$ occurs for $t > 0.2$, in spite of the fact that the existence of a fixed point off the diagonal implies no global synchronization for $t < 0.65$.

This motivates the following problems:

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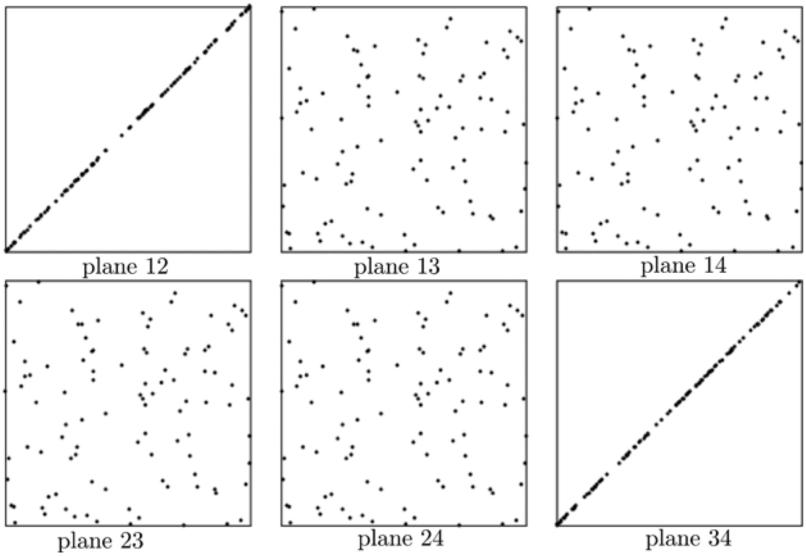

Figure 7. In Example 7.2, local synchronization occurs over the mixing subcomponents for $t = 0.78$.

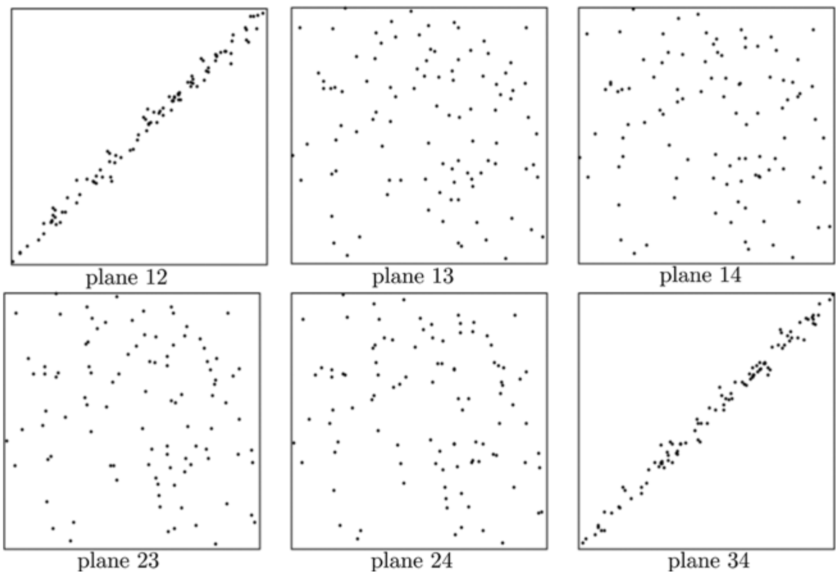

Figure 8. In Example 7.2, local synchronization does not occur over the mixing subcomponents for $t = 0.82$.

Problem 2. *Under the assumptions of Theorem 3.4, denoting by* $\lambda(f) = \lambda(f, \mu)$ *the top Lyapunov exponent of* f *w.r.t.* μ*, if*

$$\max\{\beta^*(K), \tau^*(K, \Sigma_0)\} \, \mathrm{e}^{\lambda(f)} < 1,$$

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can one prove that almost sure global synchronization (over Σ) occurs?

Problem 3. *Under the assumptions of Theorem 3.5, denoting by $\lambda(f) = \lambda(f, \mu)$ the top Lyapunov exponent of f w.r.t. μ, if*

$$\tau^*(K, \Sigma_i)\, e^{\ell(f)} < 1,$$

can one prove that almost sure global synchronization over all mixing subcomponents occur?

In other words, we wonder if almost sure global synchronization occurs when we replace, in the threshold conditions of Theorems 3.4 and 3.5, the Lipschitz term $\ell(f)$ by the Lyapunov exponent $\lambda(f)$.

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Competing Interests. The authors declare none.

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