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Tachikawa's second conjecture, derived recollements, and gendo-symmetric algebras

Hongxing Chen, Ming Fang and Changchang Xi

In memory of Professor Hiroyuki Tachikawa (1930–2022)

Abstract

Tachikawa's second conjecture for symmetric algebras is shown to be equivalent to indecomposable symmetric algebras not having any nontrivial stratifying ideals. The conjecture is also shown to be equivalent to the supremum of stratified ratios being less than 1, when taken over all indecomposable symmetric algebras. An explicit construction provides a series of counterexamples to Tachikawa's second conjecture from each (potentially existing) gendo-symmetric algebra that is a counterexample to Nakayama's conjecture. The results are based on establishing recollements of derived categories and on constructing new series of algebras.

Contents

1. Introduction

In this section we first recall the Nakayama conjecture and Tachikawa's second conjecture, and then give an introductory description of our main results on Tachikawa's second conjecture for symmetric algebras, on constructions of mirror-reflective algebras, and on derived recollements and homological properties of these constructed algebras.

1.1 Homological conjectures and stratifying ideals

In the representation theory of algebras, the long-standing and not yet solved Nakayama conjecture says that a finite-dimensional algebra over a field with infinite dominant dimension

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is self-injective [Nak58]. This is one of the main homological conjectures in representation theory. It is equivalent to the combination of the following two conjectures by Tachikawa [Tac73, pp. 115–116].

(TC1) Let Λ be a finite-dimensional algebra over a field k and $D := \mathrm{Hom}_k(-,k)$. If $\mathrm{Ext}^n_\Lambda(D(\Lambda),\Lambda) = 0$ for all $n \geq 1$, then Λ is a self-injective algebra.

(TC2) Let Λ be a finite-dimensional self-injective algebra over a field k and M a finitely generated Λ-module. Then M is projective if it is self-orthogonal, that is, $\mathrm{Ext}^n_\Lambda(M,M) = 0$ for all $n \geq 1$.

In this paper we deal with (TC2) for symmetric algebras and show that (TC2) is closely related to stratifications of derived categories of algebras. Recall from [CPS96] that an ideal AeA of an Artin algebra A generated by an idempotent element $e \in A$ is called a *stratifying ideal* in A if $Ae \otimes_{eAe} eA \simeq AeA$ and $\mathrm{Tor}_i^{eAe}(Ae, eA) = 0$ for all $i > 0$. In this case, the canonical surjection $\lambda : A \to A/AeA$ is a homological ring epimorphism, that is, the induced derived restriction functor from the derived category $\mathscr{D}(A/AeA)$ of A/AeA to the derived category $\mathscr{D}(A)$ of A is fully faithful, and therefore one has a recollement $(\mathscr{D}(A/AeA), \mathscr{D}(A), \mathscr{D}(eAe))$ of unbounded derived categories of algebras. Such a recollement of derived categories of algebras is called a *standard* recollement. Stratifying ideals are also termed *strong idempotent ideals* in [APT92] and *homological ideals* in [dlPX06]. Special examples of stratifying ideals are heredity ideals which play an important role in the study of quasi-hereditary algebras introduced in [CPS88]. A *heredity ideal* of an algebra A is an ideal I such that I is idempotent (that is, $I^2 = I$), ${}_AI$ is projective as an A-module and $\mathrm{End}_A({}_AI)$ is semisimple.

An algebra Λ is said to be *derived simple* if its derived module category $\mathscr{D}(\Lambda)$ admits no nontrivial recollements of derived module categories of algebras. Examples of derived simple algebras include local algebras, blocks of group algebras and some indecomposable algebras with two simple modules. One should not confuse the notion of derived simple algebras with that of $\mathscr{D}^{\mathrm{b}}$(mod)-*derived simple* algebras in the sense that the bounded derived categories (of finitely generated modules) do not admit any nontrivial recollements of bounded derived categories of any algebras (see [LY12]). Derived simple algebras are $\mathscr{D}^{\mathrm{b}}$(mod)-derived simple, but the converse is not true in general. By [LY12, Theorem 3.2], each indecomposable symmetric algebra is $\mathscr{D}^{\mathrm{b}}$(mod)-derived simple.

Our first main result reads as follows.

THEOREM 1.1. *Let k be a field.*

(I) *The following statements are equivalent.*
   (1) *Tachikawa's second conjecture holds for all symmetric k-algebras.*
   (2) *No indecomposable symmetric k-algebra has a stratifying ideal apart from itself and* 0.

(II) *If each indecomposable symmetric k-algebra is derived simple, then Tachikawa's second conjecture holds for all symmetric k-algebras.*

If an algebra A has a nontrivial stratifying ideal generated by an idempotent element e, then there is a nontrivial recollement $(\mathscr{D}(A/AeA), \mathscr{D}(A), \mathscr{D}(eAe))$. Thus, (II) follows from (I) immediately.

That (1) implies (2) follows from the following elementary observation. Assume that (TC2) holds for all symmetric algebras over k. Let S be an indecomposable symmetric k-algebra and I a stratifying ideal of S. Then $0 = \mathrm{Ext}^i_{S/I}(S/I, S/I) \simeq \mathrm{Ext}^i_S(S/I, S/I)$ for all $i \geq 1$. This means that ${}_SS/I$ is self-orthogonal. Then the S-module S/I is projective by (1), and therefore ${}_SS \simeq I \oplus S/I$. It follows from $I^2 = I$ that $\mathrm{Hom}_S(I, S/I) = 0$. Since S is symmetric and

${}_SI$ is projective, $\operatorname{Hom}_S(S/I, I) \simeq D\operatorname{Hom}_S(I, S/I) = 0$. Consequently, $S \simeq \operatorname{End}_S(I) \oplus \operatorname{End}_S(S/I)$ as algebras. Since S is indecomposable, either $\operatorname{End}_S(I) = 0$ or $\operatorname{End}_S(S/I) = 0$. In other words, $I = 0$ or $I = S$. This implies that S has no stratifying ideal apart from itself and 0. So (1) implies (2).

Thus, the crucial part of Theorem 1.1(I) is to prove that (2) implies (1). Our proof is based on the new ideas and techniques to be discussed in the next subsection.

1.2 Derived recollements of gendo-symmetric algebras

In 1968, Müller investigated dominant dimensions of algebras and proved the following result in [Mul68]:

Let Λ be a finite-dimensional k-algebra over a field k and M a finitely generated Λ-module. Then, for a nonnegative integer n, the dominant dimension of $\operatorname{End}_\Lambda(\Lambda \oplus D(\Lambda) \oplus M)$ is at least $n+2$ if and only if $\operatorname{Ext}^j_\Lambda(D(\Lambda) \oplus M, \Lambda \oplus M) = 0$ for all $1 \leq j \leq n$.

Thus, (TC2) holds for a self-injective algebra Λ if and only if the Nakayama conjecture holds for the endomorphism algebras $\operatorname{End}_\Lambda(\Lambda \oplus M)$ for all finitely generated Λ-modules M. This suggests considering the algebras A of the form $\operatorname{End}_\Lambda(\Lambda \oplus M)$ with Λ a self-injective algebra and M an arbitrary finitely generated Λ-module. Such algebras are called *Morita algebras* [KY13]. In the case that Λ is symmetric, they are called *gendo-symmetric algebras* and studied in [FK11, CX16a, FK16]. In [CX22], self-orthogonal generators over a self-injective Artin algebra have been discussed systematically from the viewpoint of recollements of (relative) stable module categories. In particular, it is shown that the Nakayama conjecture holds true for Gorenstein–Morita algebras [CX22, Corollary 1.4].

To prove that (2) implies (1) in Theorem 1.1, we assume that there is a gendo-symmetric algebra which is a counterexample to Nakayama's conjecture. Then we have to find a nontrivial stratifying recollement, or a nontrivial stratifying ideal in some algebra related to the counterexample. This is based on an inductive construction of a series of new algebras. Roughly speaking, starting with a gendo-symmetric algebra A and an idempotent element e of A such that the A-module Ae is faithful and projective-injective, we construct four families of algebras inductively: R_n, S_n, A_n and B_n for $n \geq 1$ (see Section 5.3 for details). They are called the nth mirror-reflective, reduced mirror-reflective, gendo-symmetric and reduced gendo-symmetric algebras of (A, e), respectively. These algebras are connected by derived recollements, as is shown in the next result. Here, $\mathscr{D}^-(A)$ and $\mathscr{D}(A)$ denote the bounded-above and unbounded derived categories of A, respectively, and domdim(A) stands for the dominant dimension of A.

Theorem 1.2. *Let (A, e) be a gendo-symmetric algebra and n a positive integer. Then the following statements hold.*

(1) *There exist recollements of bounded-above derived categories of algebras induced by stratifying ideals:*

$$\begin{tikzcd} \mathscr{D}^-(A_n) \arrow[r] & \mathscr{D}^-(A_{n+1}) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}^-(A_n) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathscr{D}^-(B_0) \arrow[r] & \mathscr{D}^-(B_{n+1}) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}^-(B_n) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

with $B_0 := (1-e)A(1-e)$.

(2) *Let $R_0 = S_0 := eAe$. If $\operatorname{domdim}(A) = \infty$, then there exist recollements of unbounded derived categories of algebras induced by stratifying ideals:*

$$\begin{tikzcd} \mathscr{D}(A_n) \arrow[r] & \mathscr{D}(R_n) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}(R_{n-1}) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathscr{D}(B_0) \arrow[r] & \mathscr{D}(S_n) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}(S_{n-1}). \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

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Thus, the dominant dimension of a gendo-symmetric algebra A being infinite means that A is a potential counterexample to Nakayama's conjecture. It is a counterexample if and only if the second recollement in Theorem 1.2(2) becomes nontrivial for some n (or equivalently, for all n). In this case, the algebra $B_0 \neq 0$. Hence, if (TC2) for symmetric algebras fails, that is, Nakayama's conjecture for gendo-symmetric algebras fails, then there are arbitrarily long nontrivial stratifying chains or recollements. This explicit construction produces a series of counterexamples provided there is at least one counterexample.

Motivated by Theorem 1.2(2), we introduce the stratified dimension of an algebra. This measures how many steps an algebra can be stratified by its nontrivial stratifying ideals (see Definition 4.7), or equivalently, the derived category of the algebra can be stratified by nontrivial standard recollements of derived module categories. We also define the *stratified ratio* of an algebra to be the ratio of its stratified dimension to the number of isomorphism classes of simple modules (see Definition 4.10). Note that the iteration procedure of constructing A_n, B_n, R_n and S_n gives standard recollements of derived module categories. The connection between (TC2) and stratified dimensions (ratios) of algebras reads as follows.

Theorem 1.3. *Tachikawa's second conjecture holds for all symmetric algebras over a field k if and only if the supremum of stratified ratios of all indecomposable symmetric algebras over k is less than* 1.

1.3 Mirror-reflective algebras and their homological properties

We now briefly outline the construction of mirror-reflective algebras and their homological properties. The first step of the construction is given in a general context.

Let A be an associative algebra over a commutative ring k, e an idempotent element of A, and $\Lambda := eAe$. For $\lambda \in Z(\Lambda)$, the center of the algebra Λ, we introduce an associative algebra $R(A, e, \lambda)$, called the *mirror-reflective algebra* of A at level (e, λ), which has the underlying k-module $A \oplus Ae \otimes_\Lambda eA$, such that $Ae \otimes_\Lambda eA$ is an ideal in $R(A, e, \lambda)$ (see Section 3.1 for details). The terminology 'mirror-reflective' can be justified by Example 3.10 in Section 3.2. Moreover, the k-submodule of $R(A, e, \lambda)$,

$$S(A, e, \lambda) := (1 - e)A(1 - e) \oplus Ae \otimes_\Lambda eA,$$

is closed under the multiplication of $R(A, e, \lambda)$. This is a possibly nonunitary algebra. It is called the *reduced mirror-reflective algebra* of A at level (e, λ). It has fewer simple modules than $R(A, e, \lambda)$ does, that is, the number of simple modules is reduced. The specializations of $R(A, e, \lambda)$ and $S(A, e, \lambda)$ at $\lambda = e$ are called *the mirror-reflective algebra and reduced mirror-reflective algebra of A at e*, denoted by $R(A, e)$ and $S(A, e)$, respectively. Moreover, $S(A, e) = e_0R(A, e)e_0$ for an idempotent element e_0 in $R(A, e)$.

Clearly, each A-module is an $R(A, e)$-module via the canonical surjective homomorphism $R(A, e) \to A$ of algebras. Conversely, each $R(A, e)$-module restricts to an A-module via the canonical inclusion from A into $R(A, e)$. Remark that each module over $(1 - e)A(1 - e)$ can also be regarded as a module over $S(A, e)$. So we have two *basic constructions* associated with (A, e):

$$\mathcal{A}(A, e) := \mathrm{End}_{R(A,e)}(R(A, e) \oplus A(1 - e)), \quad \mathcal{B}(A, e) := \mathrm{End}_{S(A,e)}(S(A, e) \oplus (1 - e)A(1 - e)).$$

Now assume that A is a gendo-symmetric algebra over a field and e is an idempotent element of A such that Ae is a faithful, projective-injective A-module. In this case, we write (A, e) for the gendo-symmetric algebra A. If e' is another idempotent element of A such that Ae' is a faithful, projective-injective A-module, then $R(A, e) \simeq R(A, e')$ as algebras (see Lemma 3.6(1)).

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Hence, up to isomorphism of algebras, we can write $R(A)$ for $R(A,e)$ without referring to e, and call it the *mirror-reflective algebra* of the gendo-symmetric algebra A.

An Artin algebra B is called an *n-Auslander algebra* ($n \geq 0$) if $\mathrm{gldim}(B) \leq n+1 \leq \mathrm{domdim}(B)$; or an *$n$-minimal Auslander–Gorenstein algebra* if $\mathrm{idim}(_BB) \leq n+1 \leq \mathrm{domdim}(B)$ (see [Aus71, Iya07, CK16, IS18]), where $\mathrm{gldim}(B)$, $\mathrm{domdim}(B)$ and $\mathrm{idim}(_BB)$ denote the global, dominant and left injective dimensions of the algebra B, respectively. Clearly, n-Auslander algebras are exactly n-minimal Auslander–Gorenstein algebras of finite global dimension (see § 2).

Theorem 1.4. *Let (A,e) be a gendo-symmetric algebra. Then:*

(1) *$R(A,e,\lambda)$ is a symmetric algebra for λ in the center of eAe.*
(2) $\min\{\mathrm{domdim}(\mathcal{A}(A,e)), \mathrm{domdim}(\mathcal{B}(A,e))\} \geq \mathrm{domdim}(A)+2$.
(3) *Let n be a positive integer. If A is an n-Auslander (respectively, n-minimal Auslander–Gorenstein) algebra, then $\mathcal{A}(A,e)$ is a $(2n+3)$-Auslander (respectively, $(2n+3)$-minimal Auslander–Gorenstein) algebra.*

Theorem 1.4(1) not only implies that R_n and S_n are symmetric algebras and that A_n and B_n are gendo-symmetric algebras, but also lays a basis for the inductive construction of the series of algebras A_n, B_n, R_n and S_n in Theorem 1.2, while Theorem 1.4(2) says that A_n and B_n have higher homological dimensions: $\mathrm{domdim}(A_{n+1}) \geq \mathrm{domdim}(A_n)+2$ and $\mathrm{domdim}(B_{n+1}) \geq \mathrm{domdim}(B_n)+2$. Thus, $2n \leq \mathrm{domdim}(A)+2(n-1) \leq \min\{\mathrm{domdim}(A_n), \mathrm{domdim}(B_n)\}$. For the finitistic dimensions and algebraic K-groups of these algebras, we refer to Corollary 5.10.

1.4 Outline of the paper

The paper is structured as follows. In § 2 we recall the definitions of dominant dimensions, gendo-symmetric algebras, higher Auslander and Auslander–Gorenstein algebras. In § 3 we introduce (reduced) mirror-reflective algebras by reflecting a left (or right) ideal generated by an idempotent element. Further, we describe explicitly the mirror-reflective algebras by quivers with relations for algebras themselves presented by quivers with relations. This description explains visually the terminology of mirror-reflective algebras. In § 4 we recall the definitions of recollements and stratifying ideals (or strong idempotent ideals in other terminology). Also, we present the definitions of stratified dimensions and ratios of algebras (see Definitions 4.7 and 4.10, respectively). We then construct derived recollements from mirror-reflective algebras. In § 5 we first show Theorems 1.4 and 1.2. This relies on the fact that mirror-reflective algebras of gendo-symmetric algebras at any levels are symmetric (see Proposition 5.2). By iteration of forming (reduced) mirror-reflective algebras from a gendo-symmetric algebra, a series of recollements of derived module categories is established. This not only gives proofs of Theorems 1.1 and 1.3, but also establishes a precise relation between the numbers of simple modules over different mirror-reflective algebras (see Corollary 5.10(2)–(3)). Moreover, this construction of mirror-reflective algebras provides a new method to produce a series of n-minimal Auslander–Gorenstein algebras.

2. Dominant dimensions and gendo-symmetric algebras

Let k be a commutative ring. All algebras considered are associative k-algebras with identity.

Let A be a k-algebra. We denote by A-Mod the category of all left A-modules, and by A-mod the full subcategory of A-Mod consisting of finitely generated A-modules. The *global dimension* of A, denoted by $\mathrm{gldim}(A)$, is defined to be the supremum of projective dimensions of all A-modules. The *finitistic dimension* of A, denoted by $\mathrm{findim}(A)$, is defined to be the supremum of projective

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dimensions of those A-modules which have finite projective resolutions by finitely generated projective modules. The *projective and injective dimensions* of an A-module M are denoted by $\mathrm{pdim}({}_AM)$ and $\mathrm{idim}({}_AM)$, respectively. If $f : X \to Y$ and $g : Y \to Z$ are homomorphisms of A-modules, we write fg for the composition of f with g, and $(x)f$ for the image of $x \in X$ under f.

For an additive category $\mathcal{C}$, let $\mathscr{C}(\mathcal{C})$ denote the category of all complexes over $\mathcal{C}$ with chain maps, and $\mathscr{K}(\mathcal{C})$ the homotopy category of $\mathscr{C}(\mathcal{C})$. We denote by $\mathscr{C}^{\mathrm{b}}(\mathcal{C})$ and $\mathscr{K}^{\mathrm{b}}(\mathcal{C})$ the full subcategories of $\mathscr{C}(\mathcal{C})$ and $\mathscr{K}(\mathcal{C})$, respectively, consisting of bounded complexes over $\mathcal{C}$. When $\mathcal{C}$ is abelian, the *(unbounded) derived category* of $\mathcal{C}$ is denoted by $\mathscr{D}(\mathcal{C})$, which is the localization of $\mathscr{K}(\mathcal{C})$ at all quasi-isomorphisms. The full subcategory of $\mathscr{D}(\mathcal{C})$ consisting of bounded-above complexes over $\mathcal{C}$ is denoted by $\mathscr{D}^-(\mathcal{C})$. As usual, we simply write $\mathscr{K}(A)$ for $\mathscr{K}(A\text{-Mod})$, $\mathscr{D}(A)$ for $\mathscr{D}(A\text{-Mod})$, and $\mathscr{D}^-(A)$ for $\mathscr{D}^-(A\text{-Mod})$. Also, we identify A-Mod with the full subcategory of $\mathscr{D}(A)$ consisting of all stalk complexes in degree 0.

For an Artin algebra, we denote by $\#(A)$ the number of isomorphism classes of simple A-modules, and by D the usual duality of an Artin algebra.

Now let A be a finite-dimensional algebra over a field k.

DEFINITION 2.1. The *dominant dimension* of an algebra A, denoted by $\mathrm{domdim}(A)$, is the maximal natural number n or ∞ such that the first n terms $I_0, I_1, \dots, I_{n-1}$ in a minimal injective resolution $0 \to {}_AA \to I_0 \to I_1 \to \cdots \to I_i \to \cdots$ of A are projective.

A module $M \in A$-mod is called a *generator* if ${}_AA \in \mathrm{add}(M)$; a *cogenerator* if $D(A_A) \in \mathrm{add}(M)$; or a *generator-cogenerator* if it is both a generator and a cogenerator. By [Mul68, Lemma 3], if ${}_AM$ is a generator-cogenerator, then $\mathrm{domdim}(\mathrm{End}_A(M)) = \sup\{n \in \mathbb{N} \mid \mathrm{Ext}^i_A(M, M) = 0, 1 \le i \le n\} + 2$.

Algebras of the form $\mathrm{End}_A(A \oplus M)$ with A an algebra and M an A-module have the double centralizer property and have been studied for a long time. Following [FK16], such an algebra is called a *gendo-symmetric* algebra if A is a symmetric algebra. If A is symmetric, then so is eAe for $e = e^2 \in A$.

LEMMA 2.2 [FK11, Theorem 3.2]. *The following statements are equivalent for an algebra A over a field.*

(1) *A is a gendo-symmetric algebra.*
(2) *$\mathrm{domdim}(A) \ge 2$ and $D(Ae) \simeq eA$ as eAe-A-bimodules, where $e \in A$ is an idempotent element such that Ae is a faithful projective-injective A-module.*
(3) *$\mathrm{Hom}_A(D(A), A) \simeq A$ as A-A-bimodules.*
(4) *$D(A) \otimes_A D(A) \simeq D(A)$ as A-A-bimodules.*

In the rest of the paper, we write (A, e) for a gendo-symmetric algebra with e an idempotent element in A such that Ae is a faithful projective-injective A-module. The category $\mathrm{add}(Ae)$ coincides with the full subcategory of A-mod consisting of projective-injective A-modules.

An algebra A is called an *Auslander algebra* if $\mathrm{gldim}(A) \le 2 \le \mathrm{domdim}(A)$. This is equivalent to saying that A is the endomorphism algebra of an additive generator of a representation-finite algebra over a field (see [Aus71]). A generalization of Auslander algebras is the so-called n-Auslander algebras. Let n be a positive integer. Following [Aus71, Iya07, IS18], A is called an *n-Auslander algebra* if $\mathrm{gldim}(A) \le n+1 \le \mathrm{domdim}(A)$; or an *$n$-minimal Auslander–Gorenstein algebra* if $\mathrm{idim}({}_AA) \le n+1 \le \mathrm{domdim}(A)$. Clearly, n-Auslander algebras are n-minimal

Auslander–Gorenstein, while n-minimal Auslander–Gorenstein algebras of finite global dimension are n-Auslander. Moreover, these algebras can be characterized in terms of left or right perpendicular categories. For $M \in A$-mod and $m \in \mathbb{N}$, we define

$$^{\perp_m}M := \{X \in A\text{-mod} \mid \mathrm{Ext}^i_A(X, M) = 0,\ 1 \le i \le m\},$$
$$M^{\perp_m} := \{X \in A\text{-mod} \mid \mathrm{Ext}^i_A(M, X) = 0,\ 1 \le i \le m\ \}.$$

An A-module N is said to be *maximal* $(n-1)$*-orthogonal* or n*-cluster tilting* if $\mathrm{add}({}_AN) = {}^{\perp_{n-1}}N = N^{\perp_{n-1}}$. A generator-cogenerator $M \in A$-mod is said to be $(n-1)$*-ortho-symmetric* or n*-precluster tilting* if $\mathrm{add}({}_AM) \subseteq {}^{\perp_{n-1}}M = M^{\perp_{n-1}}$. The algebra A is n-Auslander if and only if there are an algebra Λ and a maximal $(n-1)$-orthogonal Λ-module ${}_\Lambda X$ such that $A = \mathrm{End}_\Lambda(X)$ by [Iya07, Proposition 2.4.1], and is n-minimal Auslander–Gorenstein if and only if there are an algebra Λ and an $(n-1)$-ortho-symmetric generator-cogenerator ${}_\Lambda X$ such that $A = \mathrm{End}_\Lambda(X)$ by [IS18, Theorem 4.5] or [CK16, Corollary 3.18]. Moreover, by [IS18, Proposition 4.1], if A is n-minimal Auslander–Gorenstein, then either A is self-injective or $\mathrm{idim}({}_AA) = n+1 = \mathrm{domdim}(A)$. In the latter case, $\mathrm{idim}(A_A) = n+1 = \mathrm{domdim}(A)$, and therefore A is $(n+1)$-Gorenstein.

An A-module M is said to be m*-rigid* if $\mathrm{Ext}^i_A(M, M) = 0$ for all $1 \le i \le m$. Over symmetric algebras, ortho-symmetric modules have been characterized as follows.

Lemma 2.3 [CK16, Corollary 5.4]. *Let A be a symmetric algebra and N a basic A-module without any nonzero projective direct summands. For a natural number m, the A-module $A \oplus N$ is m-ortho-symmetric if and only if N is m-rigid and $\Omega^{m+2}_A(N) \cong N$.*

3. Mirror-reflective algebras

In this section we introduce (reduced) mirror-reflective algebras and describe them explicitly by quivers with relations.

3.1 Definition of mirror-reflective algebras

Throughout this subsection, assume that A is an algebra over a commutative ring k. Let M be an A-A-bimodule and $\alpha : {}_AM \otimes_A M \to M$ be a homomorphism of A-A-bimodules, such that the associative law holds:

$$((x \otimes y)\alpha \otimes z)\alpha = (x \otimes (y \otimes z)\alpha)\alpha \quad \text{for } x, y, z \in M. \qquad (\heartsuit)$$

We define a multiplication on the underlying abelian group $A \oplus M$ by setting

$$(a, m) \cdot (b, n) := (ab, an + mb + (m \otimes n)\alpha) \quad \text{for } a, b \in A,\ m, n \in M.$$

Then $A \oplus M$ becomes an associative algebra with the identity $(1, 0)$, denoted by $R(A, M, \alpha)$. In the following, we identify A with $(A, 0)$, and M with $(0, M)$ in $R(A, M, \alpha)$. Thus, A is a subalgebra of $R(A, M, \alpha)$ with the same identity, and M is an ideal of $R(A, M, \alpha)$ such that $R(A, M, \alpha)/M \simeq A$.

We now consider a special case of the above construction. Let $e = e^2 \in A$, $\Lambda := eAe$ and $Z(\Lambda)$ be the center of Λ. For $\lambda \in Z(\Lambda)$, let ω_λ be the composition of the natural maps:

$$(Ae \otimes_\Lambda eA) \otimes_A (Ae \otimes_\Lambda eA) \xrightarrow{\simeq} Ae \otimes_\Lambda (eA \otimes_A Ae) \otimes_\Lambda eA \xrightarrow{\simeq} Ae \otimes_\Lambda \Lambda \otimes_\Lambda eA$$
$$\xrightarrow{\mathrm{Id}\otimes(\cdot\lambda)\otimes\mathrm{Id}} Ae \otimes_\Lambda \Lambda \otimes_\Lambda eA \to Ae \otimes_\Lambda eA,$$

where $(\cdot\lambda) : \Lambda \to \Lambda$ is the multiplication map by λ. Then ω_λ satisfies the associative law ($\heartsuit$).

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Let $R(A,e,\lambda) := R(A, Ae\otimes_\Lambda eA, \omega_\lambda)$. Then the elements of $R(A,e,\lambda)$ are of the form

$$a+\sum_{i=1}^{n} a_ie\otimes eb_i \quad \text{for } a, a_i, b_i \in A,\ 1\leq i\leq n\in\mathbb{N}.$$

Multiplication, denoted by $*$, is explicitly given by

$$(a+be\otimes ec)*(a'+b'e\otimes ec') := aa' + (ab'e\otimes ec' + be\otimes eca' + becb'e\otimes \lambda ec')$$

for $a, b, c, a', b', c' \in A$, and can be extended linearly to elements of general form. In particular,

$$(ae\otimes eb)*(a'e\otimes eb') = aeba'e\lambda\otimes eb' = ae\otimes\lambda eba'eb'. \qquad (\diamondsuit)$$

Now consider the k-submodule $S(A,e,\lambda) := (1-e)A(1-e)\oplus Ae\otimes_\Lambda eA$ of $R(A,e,\lambda)$. It can be checked that $S(A,e,\lambda)$ is closed under the multiplication of $R(A,e,\lambda)$. In general, $S(A,e,\lambda)$ may not have an identity. However, $S(A,e,e)$ has the identity $e_0 := (1-e)+e\otimes e$.

DEFINITION 3.1. The algebra $R(A,e,\lambda)$ defined above is called the *mirror-reflective algebra* of A at level (e,λ). The algebra $S(A,e,\lambda)$ is called the *reduced mirror-reflective algebra* of A at level (e,λ).

The algebra $R(A,e,e)$ is then called the *mirror-reflective algebra* of A at e, denoted by $R(A,e)$. The algebra $S(A,e,e)$ is called the *reduced mirror-reflective algebra* of A at e, denoted by $S(A,e)$.

Compared with $R(A,e)$, $S(A,e)$ has fewer simple modules. So it is termed the reduced mirror-reflective algebra.

Example 3.2. Let A be an algebra over a field k presented by the quiver with a relation:

$$\begin{tikzcd} 1\bullet \arrow[r, bend left, "\alpha"] & \bullet 2 \arrow[l, bend left, "\beta"] \end{tikzcd}, \qquad \alpha\beta=0.$$

The composition $\alpha\beta$ of arrows α and β means that α comes first and then β follows. If k is of characteristic 2, then A is just the Schur algebra $S(2,2)$. Let e be the idempotent of A corresponding to the vertex 2. Then $R(A,e)$ is isomorphic to the algebra presented by the following quiver with relations:

$$\begin{tikzcd} \overline{2}\bullet \arrow[r, bend right, "\overline{\beta}"'] & 1\bullet \arrow[l, bend right, "\overline{\alpha}"'] \arrow[r, bend left, "\alpha"] & \bullet 2 \arrow[l, bend left, "\beta"] \end{tikzcd}, \qquad \alpha\beta+\overline{\alpha}\overline{\beta}=\beta\overline{\alpha}=\overline{\beta}\alpha=0.$$

The algebra $S(A,e)$ is isomorphic to the algebra presented by the quiver with relations:

$$\begin{tikzcd} \overline{2}\bullet \arrow[r, bend right, "\overline{\beta}"'] & \bullet 1 \arrow[l, bend right, "\overline{\alpha}"'] \end{tikzcd}, \qquad \overline{\alpha}\overline{\beta}\overline{\alpha}=\overline{\beta}\overline{\alpha}\overline{\beta}=0.$$

A general description of mirror-reflective algebras presented by quivers with relations will be given in Section 3.2.

The following lemma is obvious.

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Lemma 3.3.

(1) *$R(A, e, \lambda)/(Ae \otimes_\Lambda eA) \simeq A$ as algebras.*
(2) *If $\mu \in Z(\Lambda)$ is an invertible element, then $R(A, e, \lambda) \simeq R(A, e, \lambda\mu)$ as algebras.*

For simplicity, let $R := R(A, e)$, $S := S(A, e)$ and $\bar{e} := e \otimes e \in R$. Then $\bar{e} = \bar{e}^2$, $e\bar{e} = \bar{e} = \bar{e}e$, and $\{\bar{e}, e - \bar{e}, 1 - e\}$ is a set of pairwise orthogonal idempotent elements in R. We define

$$\pi_1 : R \longrightarrow A, \quad a + \sum_{i=1}^{n} a_i\bar{e}b_i \mapsto a, \quad \text{and} \quad \pi_2 : R \longrightarrow A, \quad a + \sum_{i=1}^{n} a_i\bar{e}b_i \mapsto a + \sum_{i=1}^{n} a_i e b_i$$

for $a, a_i, b_i \in A$ and $1 \leq i \leq n$. Then π_1 and π_2 are surjective homomorphisms of algebras. Let

$$I := \mathrm{Ker}(\pi_1), \quad J := \mathrm{Ker}(\pi_2) \quad \text{and} \quad e_0 := (1 - e) + \bar{e} \in R.$$

Lemma 3.4.

(1) *$I = R\bar{e}R$, $J = R(e - \bar{e})R$, $IJ = 0 = JI$, $I + J = ReR$ and $S = e_0Re_0$.*
(2) *As an A-A-bimodule, ${}_AR_A$ has two decompositions: $R = A \oplus I = A \oplus J$.*
(3) *The map $\phi : R \to R$, defined by $a + \sum_{i=1}^{n} a_i\bar{e}b_i \mapsto a + \sum_{i=1}^{n} a_i(e - \bar{e})b_i$, is an automorphism of algebras with $\phi^2 = \mathrm{Id}_R$, such that $\pi_2 = \phi\pi_1$, and the restriction of ϕ to I induces an isomorphism $I \to J$ of A-A-bimodules.*
(4) *Both π_1 and π_2 induce surjective homomorphisms of algebras*

$$\pi_1' : S \longrightarrow (1 - e)A(1 - e) \quad \textit{and} \quad \pi_2' : S \longrightarrow A,$$

respectively. Moreover, $\mathrm{Ker}(\pi_1') = I$ and $\mathrm{Ker}(\pi_2') = (1 - e)J(1 - e) = J \cap S$.

Proof. (1) Clearly, $I = Ae \otimes_\Lambda eA = A\bar{e}A = R\bar{e}R$. Since $(e - \bar{e})\pi_2 = 0$, we have $e - \bar{e} \in \mathrm{Ker}(\pi_2) = J$ and $R(e - \bar{e})R \subseteq J$. Conversely, if $r := a + \sum_{i=1}^{n} a_i\bar{e}b_i \in J$, then $a + \sum_{i=1}^{n} a_i e b_i = (r)\pi_2 = 0$, that is, $a = -\sum_{i=1}^{n} a_i e b_i$. Consequently, $r = -\sum_{i=1}^{n} a_i e b_i + \sum_{i=1}^{n} a_i\bar{e}b_i = -\sum_{i=1}^{n} a_i(e - \bar{e})b_i \in R(e - \bar{e})R$. Thus, $J = R(e - \bar{e})R = A(e - \bar{e})A$. Note that $I + J = R\bar{e}R + R(e - \bar{e})R = ReR$. For any $x, y, x', y' \in A$, since $(x\bar{e}y)(x'(e - \bar{e})y') = x\bar{e}yx'ey' - x\bar{e}yx'ey' = 0$, we have $IJ = 0$. Similarly, $(x'(e - \bar{e})y')(x\bar{e}y) = 0$, and therefore $JI = 0$. Since I is an ideal of R and $IJ = JI = 0$, it follows that $S = e_0Re_0$.

(2) R contains A as a subalgebra with the same identity, and the composition of the inclusion $A \subseteq R$ with π_i, for $i = 1, 2$, is the identity map of A. Thus, (2) follows.

(3) By (2), $I \simeq R/A \simeq J$ as A-A-bimodules. More precisely, the isomorphism from I to J is given by

$$\varphi' : \; I \longrightarrow J, \quad \sum_{i=1}^{n} a_i\bar{e}b_i \mapsto \sum_{i=1}^{n} a_i(e - \bar{e})b_i.$$

Further, the map $\phi : R = A \oplus I \to R = A \oplus J$ is induced from φ', and therefore is a well-defined isomorphism of A-A-bimodules. Moreover, ϕ preserves the multiplication of R and $\phi^2 = \mathrm{Id}_R$. Thus, ϕ is an automorphism of algebras. The equality $\pi_2 = \phi\pi_1$ follows from the definitions of π_1, π_2 and ϕ.

(4) By the left and right multiplications by e_0 of π_1 and π_2, we then get (4) by (1). $\square$

The *annihilator* of an R-module M is defined as $\mathrm{Ann}_R(M) := \{r \in R \mid rM = 0\}$. It is an ideal of R.

Lemma 3.5.

(1) *If the right A-module eA_A is faithful, then $J = \mathrm{Ann}_{R^{\mathrm{op}}}(I)$. Dually, if ${}_AAe$ is faithful, then $J = \mathrm{Ann}_R(I)$.*

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(2) *π_2 induces isomorphisms of abelian groups,*

$$R\bar{e} \xrightarrow{\simeq} Ae, \quad \bar{e}R \xrightarrow{\simeq} eA \quad \text{and} \quad \bar{e}R\bar{e} \xrightarrow{\simeq} eAe,$$

while the map $\pi_2' : S \to A$ in Lemma 3.4(4) induces isomorphisms of abelian groups,

$$S\bar{e} \xrightarrow{\simeq} Ae, \quad \bar{e}S \xrightarrow{\simeq} eA \quad \text{and} \quad \bar{e}S\bar{e} \xrightarrow{\simeq} eAe.$$

(3) *π_1 induces isomorphisms of abelian groups:*

$$R(e-\bar{e}) \xrightarrow{\simeq} Ae, \quad (e-\bar{e})R \xrightarrow{\simeq} eA \quad \text{and} \quad (e-\bar{e})R(e-\bar{e}) \xrightarrow{\simeq} eAe.$$

Proof. (1) Clearly, $J \subseteq \mathrm{Ann}_{R^{\mathrm{op}}}(I)$. This is due to $IJ = 0$ by Lemma 3.4(1). We show $J \supseteq \mathrm{Ann}_{R^{\mathrm{op}}}(I)$. In fact, since $J = \mathrm{Ker}(\pi_2)$, it suffices to prove that $(x)\pi_2 = 0$ for $x \in \mathrm{Ann}_{R^{\mathrm{op}}}(I)$. Let $y := (x)\pi_2$. It follows from $Ix = 0$ that $0 = (Ix)\pi_2 = (I)\pi_2 y = AeAy$. This implies $eAy = 0$. Since eA_A is faithful, we must have $y = 0$, and therefore $x \in \mathrm{Ker}(\pi_2) = J$. Thus, $J = \mathrm{Ann}_{R^{\mathrm{op}}}(I)$. We show the second identity similarly.

(2) Due to $(\bar{e})\pi_2 = e$, the restriction $f_2 : R\bar{e} \to Ae$ of π_2 to $R\bar{e}$ is surjective. As $\mathrm{Ker}(f_2) = R\bar{e} \cap J \subseteq JI = 0$ by Lemma 3.4(1), f_2 is an isomorphism. Dually, the restriction $\bar{e}R \to eA$ of π_2 to $\bar{e}R$ is also an isomorphism. Consequently, π_2 induces an isomorphism of algebras from $\bar{e}R\bar{e}$ to eAe.

Since $IJ = JI = 0$ by Lemma 3.4(1), we have $S\bar{e} = R\bar{e}$ and $\bar{e}S = \bar{e}R$. Clearly, $\bar{e}S\bar{e} = \bar{e}R\bar{e}$. Thus, the second statement in (2) holds.

(3) This follows from (2) and Lemma 3.4(3)–(4). $\square$

Consequently, Lemmas 3.4(1) and 3.5(2) imply that $\#(R) = \#(A) + \#(eAe)$.

To discuss the decomposition of R as an algebra and to lift algebra homomorphisms, we show the following result. For a homomorphism $\alpha : A \to \Gamma$ of algebras, denote by $\mathrm{Hom}_{\alpha\text{-Alg}}(R, \Gamma)$ the set of all algebra homomorphisms $\beta : R \to \Gamma$ such that the restriction of β to A coincides with α.

Lemma 3.6.

(1) *If $u = u^2 \in A$ such that $\mathrm{add}({}_AAu) = \mathrm{add}({}_AAe)$, then $R \simeq R(A, u, u)$ as algebras.*
(2) *If ${}_AAe$ is a generator, then $R \simeq A \times A$ as algebras.*
(3) *Let $\alpha : A \to \Gamma$ be a homomorphism of algebras and define $f := (e)\alpha$. Then there is a bijection*

$$\mathrm{Hom}_{\alpha\text{-Alg}}(R, \Gamma) \xrightarrow{\simeq} \{x \in f\Gamma f \mid x^2 = x, (c)\alpha\, x = x(c)\alpha \text{ for } c \in \Lambda\}, \quad \overline{\alpha} \mapsto (\bar{e})\overline{\alpha}.$$

Proof. (1) Let $U := uAu$ and $\mathbf{P}_1(Ae)$ be the full subcategory of A-Mod consisting of all modules X such that there is an exact sequence $P_1 \to P_0 \to X \to 0$ of A-modules with $P_0, P_1 \in \mathrm{Add}(Ae)$, where $\mathrm{Add}(Ae)$ is the full subcategory of A-Mod consisting of direct summands of direct sums of copies of Ae. We identify the functor $\mathrm{Hom}_A(Au, -) : A\text{-Mod} \to U\text{-Mod}$ with the functor $u\cdot : A\text{-Mod} \to U\text{-Mod}$, given by the left multiplication of u. Let $\mu : Au \otimes_U u(-) \to \mathrm{Id}$ be the counit of the adjunction of the adjoint pair $(Au \otimes_U -, u\cdot)$. Then, for an A-module X, the map μ_X is an isomorphism if and only if $X \in \mathbf{P}_1(Au)$. Applying $Ae \otimes_\Lambda -$ to a projective presentation of ${}_\Lambda eA$, we obtain an exact sequence $P_1 \to P_0 \to Ae \otimes_\Lambda eA \to 0$ of A-modules with $P_1, P_0 \in \mathrm{Add}(Ae)$. This shows $Ae \otimes_\Lambda eA \in \mathbf{P}_1(Ae)$. Due to $\mathrm{add}({}_AAu) = \mathrm{add}({}_AAe)$, we have $Ae \otimes_\Lambda eA \in \mathbf{P}_1(Au)$, and therefore $\mu_{Ae\otimes_\Lambda eA} : Au \otimes_U u(Ae \otimes_\Lambda eA) \to Ae \otimes_\Lambda eA$ is an isomorphism of A-A-modules. Since the multiplication map $\rho : Ae \otimes_\Lambda eA \to A, ae \otimes eb \mapsto aeb$ for $a, b \in A$, satisfies $e\,\mathrm{Ker}(\rho) = 0 = e\,\mathrm{Coker}(\rho)$, it follows from $\mathrm{add}({}_AAu) = \mathrm{add}({}_AAe)$ that $u\,\mathrm{Ker}(\rho) = 0 = u\,\mathrm{Coker}(\rho)$. Then $u\rho : u(Ae \otimes_\Lambda eA) \to uA$ is an isomorphism of U-A-bimodules, and $u\rho u : u(Ae \otimes_\Lambda eA)u \to uAu$ is an

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isomorphism of U-U-bimodules. Consequently, there is an isomorphism of A-A-bimodules

$$\mathrm{Id}_{Au} \otimes_U u\rho : \ Au \otimes_U u(Ae \otimes_\Lambda eA) \xrightarrow{\simeq} Au \otimes_U uA.$$

Thus, $\psi := (\mathrm{Id}_{Au} \otimes_U u\rho)^{-1}\mu_{Ae\otimes_\Lambda eA} : Au \otimes_U uA \to Ae \otimes_\Lambda eA$ is an isomorphism of A-A-bimodules. In fact, if $x_i \in uAe$ and $y_i \in eAu$ with $1 \leq i \leq n$ such that $\sum_{i=1}^n x_i y_i = u$, then $(a(u \otimes u)b)\psi = a(\sum_{i=1}^n x_i \otimes y_i)b$. This induces an isomorphism of A-A-bimodules:

$$\mathrm{Id}_A \oplus \psi : R(A,u,u) = A \oplus Au \otimes_U uA \longrightarrow R = A \oplus Ae \otimes_\Lambda eA,$$

$$(a, x \otimes y) \mapsto (a, (x \otimes y)\psi) \quad \text{for } a \in A, \ x \in Au, \ y \in uA.$$

It can be verified that this is an isomorphism of algebras

(2) Suppose that ${}_AAe$ is a generator. Then $\mathrm{add}({}_AAe) = \mathrm{add}({}_AA)$. Let $B := R(A,1,1)$. By (1), $R \simeq B$ as algebras. Now, identifying $A \otimes_A A$ with A, we get $B = A \oplus A$ with multiplication given by

$$(a_1, a_2)(b_1, b_2) := (a_1b_1, a_1b_2 + a_2b_1 + a_2b_2) \quad \text{for } a_1, a_2, b_1, b_2 \in A.$$

Clearly, $(1, 0)$ is the identity of B and $(1, -1)$ is a central idempotent element of B. Thus, the map $B \to A \times A$, $(a_1, a_2) \mapsto (a_1, a_1 + a_2)$, is an algebra isomorphism. Thus, $R \simeq B \simeq A \times A$ as algebras.

(3) The algebra Γ can be regarded as an A-A-bimodule via α, and any A-A-bimodule can be considered as a module over the enveloping algebra $A^e := A \otimes_k A^{\mathrm{op}}$. Define $F = Ae \otimes_\Lambda - \otimes_\Lambda eA : \Lambda^e\text{-Mod} \to A^e\text{-Mod}$ and $G = e(-)e : A^e\text{-Mod} \to \Lambda^e\text{-Mod}$. Then there are isomorphisms of k-modules

$$\begin{aligned}\mathrm{Hom}_{A^e}(Ae \otimes_\Lambda eA, \Gamma) &\simeq \mathrm{Hom}_{A^e}(F(\Lambda), \Gamma) \simeq \mathrm{Hom}_{\Lambda^e}(\Lambda, G(\Gamma)) = \mathrm{Hom}_{\Lambda^e}(\Lambda, (e)\alpha\,\Gamma(e)\alpha) \\ &= \mathrm{Hom}_{\Lambda^e}(\Lambda, f\Gamma f) = \{y \in f\Gamma f \mid (c)\alpha\, y = y(c)\alpha \text{ for any } c \in \Lambda\} =: \Gamma'.\end{aligned}$$

Let $\overline{\alpha} \in \mathrm{Hom}_{\alpha\text{-Alg}}(R, \Gamma)$ and $x := (\overline{e})\overline{\alpha} \in \Gamma$. Since the restriction of $\overline{\alpha}$ to A equals α, the restriction of $\overline{\alpha}$ to $Ae \otimes_\Lambda eA$ is an homomorphism of A-A-bimodules. By $\overline{e}^2 = \overline{e}$, we have $x^2 = x$ and $(ae \otimes eb)\overline{\alpha} = (a)\alpha\, x\,(b)\alpha$ for any $a, b \in A$. This means that $x \in \Gamma'$ and $\overline{\alpha}$ is determined by α and x. Thus, the map in (3) is well defined and injective.

Conversely, let $y \in \Gamma'$ and let $h : Ae \otimes_\Lambda eA \to \Gamma$ be the homomorphism of A-A-bimodules sending $ae \otimes eb$ to $(a)\alpha\, y(b)\alpha$. Define $\overline{h} := (\alpha, h) : R \to \Gamma$. Then $\overline{h}$ is an algebra homomorphism if and only if $((ae \otimes eb) * (a'e \otimes eb'))h = (ae \otimes eb)h(a'e \otimes eb')h$ for any $a, a', b, b' \in A$ if and only if $y(ba')\alpha\, y = (eba')\alpha\, y$ for any $b, a' \in A$. Now suppose $y^2 = y$. Since α is an algebra homomorphism and $fy = y = yf$, we see that $(eba')\alpha\, y = (eba'e)\alpha\, y = (eba'e)\alpha\, y^2 = y(eba'e)\alpha y = y(ba')\alpha\, y$. Thus, $\overline{h}$ is an algebra homomorphism with $y = (\overline{e})h = (\overline{e})\overline{h}$. This shows that the map in (3) is surjective. Hence, (3) holds. $\square$

Proposition 3.7. *Let A be an indecomposable algebra. Then:*

(1) *R is a decomposable algebra if and only if ${}_AAe$ is a generator. In this case, $R \simeq A \times A$ as algebras.*
(2) *If $\mathrm{add}(Ae) \cap \mathrm{add}(A(1-e)) = 0$ and $(1-e)A(1-e)$ is an indecomposable algebra, then S is an indecomposable algebra.*

Proof. (1) If ${}_AAe$ is a generator, then $R \simeq R(A,1,1) \simeq A \times A$ as algebras by Lemma 3.6(2), and therefore R is decomposable. Conversely, assume that R is a decomposable algebra. Then there is an element $z = z^2 \in Z(R)$ of R such that $z \neq 0, 1$. Since $\pi_1 : R \to A$ is a surjective homomorphism of algebras, it restricts to an algebra homomorphism $Z(R) \to Z(A)$. This implies $(z)\pi_1 \in Z(A)$. Since A is indecomposable, we have $(z)\pi_1 = 0$ or 1. If $(z)\pi_1 = 0$,

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then $z \in \mathrm{Ker}(\pi_1) = I$. If $(z)\pi_1 = 1$, then $1 - z \in I$. Similarly, by the surjective homomorphism π_2, we know $z \in J$ or $1 - z \in J$. Assume $z \in I$. If $z \in J$, then $z = z^2 \in IJ = 0$ by Lemma 3.4(1). This is a contradiction. Thus, $1 - z \in J$ and $1 = z + (1 - z) \in I + J = ReR$ by Lemma 3.4(1). This shows $ReR = R$. It then follows from π_1 that $AeA = A$. Hence, ${}_AAe$ is a generator. For the case $1 - z \in I$, we can show similarly that ${}_AAe$ is a generator.

(2) Let $J_1 := S \cap J$. In the proof of (1), we replace π_1 and π_2 with $\pi_1' : S \to (1-e)A(1-e)$ and $\pi_2' : S \to A$ (see Lemma 3.4(4)), respectively, and show similarly that if $(1-e)A(1-e)$ is indecomposable and S is decomposable, then $S = I + J_1$. In this case, the equality $A = AeA$ still holds because π_2' is surjective with $\mathrm{Ker}(\pi_2') = J_1$ and $(\overline{e})\pi_2' = e$. Consequently, ${}_AAe$ is a generator, and therefore the assumption $\mathrm{add}(Ae) \cap \mathrm{add}(A(1-e)) = 0$ forces $e = 1$. Thus, $S = I \simeq A$ as algebras. This contradicts A being indecomposable. □

3.2 Examples of mirror-reflective algebras: quivers with relations

In this subsection we describe explicitly the mirror-reflective algebras for algebras presented by quivers with relations. This explains the terminology 'mirror-reflective algebras' (see Example 3.10 below).

Throughout this subsection we assume that k is a field.

Let $Q := (Q_0, Q_1)$ be a quiver with the vertex set Q_0 and arrow set Q_1. For an arrow $\alpha : i \to j$, we denote by $s(\alpha)$ and $t(\alpha)$ the starting vertex i and the terminal vertex j, respectively. Composition of an arrow $\alpha : i \to j$ with an arrow $\beta : j \to m$ is written as $\alpha\beta$. A *path* of length $n \geq 0$ in Q is a sequence $p := \alpha_1 \cdots \alpha_n$ of n arrows α_i in Q_1 such that $t(\alpha_i) = s(\alpha_{i+1})$ for $1 \leq i < n \in \mathbb{N}$. Set $s(p) = s(\alpha_1)$ and $t(p) = t(\alpha_n)$. In the case where $n = 0$, we understand the trivial path as an vertex $i \in Q_0$, denoted by e_i, and set $s(e_i) = i = t(e_i)$. We write $\mathscr{P}(Q)$ for the set of all paths of finite length in Q. For a field k, we write kQ for the path algebra of Q over k. Clearly, it has $\mathscr{P}(Q)$ as a k-basis.

A *relation* σ on Q over k is a k-linear combination of paths p_i of length at least 2. We may assume that all paths in a relation have the same starting vertex and terminal vertex, and define $s(\sigma) = s(p_i)$ and $t(\sigma) = t(p_i)$. If $\rho = \{\sigma_i\}_{i \in T}$ is a set of relations on Q over k with T an index set, the pair (Q, ρ) is called a *quiver with relations* over k. In this case, we have a k-algebra $k(Q, \rho) := kQ/\langle\rho\rangle$, the quotient algebra of the path algebra kQ modulo the ideal $\langle\rho\rangle$ generated by the relations $\sigma_i, i \in T$.

LEMMA 3.8. *Let B be a k-algebra, $\{f_i \mid i \in Q_0\}$ a set of orthogonal idempotent elements in B with $1_B = \sum_{i \in Q_0} f_i$, and $\{f_\alpha \mid \alpha \in Q_1\}$ a set of elements in B. If $f_{s(\alpha)} f_\alpha = f_\alpha = f_\alpha f_{t(\alpha)}$ for $\alpha \in Q_1$, then there is a unique algebra homomorphism $f : kQ \to B$ which sends $e_i \mapsto f_i$ and $\alpha \mapsto f_\alpha$ for $i \in Q_0$ and $\alpha \in Q_1$.*

Let $Q' := (Q_0', Q_1')$ be a full subquiver of Q, that is, $Q_0' \subseteq Q_0$ and $Q_1' = \{\alpha \in Q_1 \mid s(\alpha), t(\alpha) \in Q_0'\}$. Define

$$A := k(Q, \rho), \quad V_0 := Q_0 \setminus Q_0' \quad \text{and} \quad e := \sum_{i \in V_0} e_i \in A.$$

We shall explicitly describe the quiver and relations for the mirror-reflective algebra $R(A, e)$.

Let $\overline{Q}$ be a copy of the quiver Q, say $\overline{Q}_0 = \{\bar{i} \mid i \in Q_0\}$ and $\overline{Q}_1 = \{\bar{\alpha} \mid \alpha \in Q_1\}$, with $s(\bar{\alpha}) = \bar{i}$ and $t(\bar{\alpha}) = \bar{j}$ if $s(\alpha) = i$ and $t(\alpha) = j$. Consider Q' as a full subquiver of $\overline{Q}$ by identifying $\bar{i}$ with i for $i \in Q_0'$, and $\bar{\alpha}$ with α for $\alpha \in Q_1'$. So $Q_0 \cap \overline{Q}_0 = Q_0'$ and $Q_1 \cap \overline{Q}_1 = Q_1'$. Let $\Delta := (\Delta_0, \Delta_1)$ be the pullback of the quivers Q and $\overline{Q}$ over Q', that is,

$$\Delta_0 := Q_0 \dot{\cup} (\overline{Q_0} \setminus Q_0') \quad \text{and} \quad \Delta_1 := Q_1 \dot{\cup} (\overline{Q_1} \setminus Q_1').$$

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We define a map $(-)^+ : \{e_i \mid i \in Q_0\} \cup Q_1 \to k\Delta$ by

$$e_i^+ := \begin{cases} e_i, & i \in Q'_0, \\ e_i + e_{\overline{i}}, & i \in V_0, \end{cases} \qquad \alpha^+ := \begin{cases} \alpha, & \alpha \in Q'_1, \\ \alpha + \overline{\alpha}, & \alpha \in Q_1 \setminus Q'_1. \end{cases}$$

Since $e^+_{s(\alpha)}\alpha^+ = \alpha^+ = \alpha^+ e^+_{t(\alpha)}$ for any $\alpha \in Q_1$, it follows from Lemma 3.8 that $(-)^+$ can be extended to an algebra homomorphism

$$(-)^+ : \ kQ \longrightarrow k\Delta, \quad p \mapsto p^+ := \alpha_1^+ \cdots \alpha_n^+ \quad \text{for } p = \alpha_1 \cdots \alpha_n \in \mathscr{P}(Q).$$

Given a relation $\sigma := \sum_{i=1}^n a_i p_i$ on Q with $a_i \in k$, $p_i \in \mathscr{P}(Q)$ for $1 \leq i \leq n \in \mathbb{N}$, and $s(\sigma), t(\sigma) \in Q'_0$, we define

$$\sigma_+ := \sum_{1 \leq j \leq n,\, p_j \in \mathscr{P}(Q')} a_j p_j + \sum_{1 \leq i \leq n,\, p_i \notin \mathscr{P}(Q')} a_i(p_i + \overline{p_i}) = \sigma + \sum_{1 \leq i \leq n,\, p_i \notin \mathscr{P}(Q')} a_i \overline{p_i}.$$

Now let $\psi := \psi_1 \cup \psi_2 \cup \psi_3 \cup \psi_4$ with

$$\psi_1 := \{\overline{a}pb, ap\overline{b} \mid a, b \in Q_1, s(a), t(b) \in V_0, p \in \mathscr{P}(Q'), apb \in \mathscr{P}(Q)\},$$
$$\psi_2 := \{\sigma \in \rho \mid s(\sigma) \in V_0 \text{ or } t(\sigma) \in V_0\},$$
$$\psi_3 := \{\overline{\sigma} \mid \sigma \in \psi_2\},$$
$$\psi_4 := \{\sigma_+ \mid \sigma \in \rho, s(\sigma), t(\sigma) \in Q'_0\}.$$

Then ψ is a set of relations on Δ over k, and we consider the k-algebra $k(\Delta, \psi)$.

Proposition 3.9.

(1) *The homomorphism $(-)^+ : kQ \to k\Delta$ of algebras is injective and induces an injective homomorphism $\mu : A \to k(\Delta, \psi)$ of algebras.*
(2) *There exists an isomorphism $\theta : R(A, e) \xrightarrow{\simeq} k(\Delta, \psi)$ of algebras such that $(e_i \otimes e_i)\theta = e_{\overline{i}}$ for $i \in V_0$, and the restriction of θ to A coincides with μ in (1).*

Proof. (1) For a subset $\mathcal{U} \subseteq k\Delta$, let $\langle \mathcal{U} \rangle$ be the ideal of $k\Delta$ generated by $\mathcal{U}$. Set $E := \{e_{\overline{i}} \mid i \in V_0\}$ and denote by $\delta : k\Delta \to k\Delta/\langle E \rangle$ the canonical surjection. Then $k\Delta/\langle E \rangle \xrightarrow{\sim} kQ$ as algebras and there are homomorphisms of algebras

$$kQ \xrightarrow{(-)^+} k\Delta \xrightarrow{\delta} k\Delta/\langle E \rangle \xrightarrow{\sim} kQ$$

such that their composition is the identity map of kQ. This shows that $(-)^+$ is injective. We define

$$\rho^+ := \{\sigma^+ \mid \sigma \in \rho\} \quad \text{and} \quad \psi' := \rho^+ \cup \Bigg(\bigcup_{i,j \in V_0} (e_i k\Delta e_{\overline{j}} \cup e_{\overline{j}} k\Delta e_i) \Bigg).$$

We shall show $\langle \psi' \rangle = \langle \psi \rangle$ in $k\Delta$.

In fact, let $\varphi = \bigcup_{i,j \in V_0} (e_i k\Delta e_{\overline{j}} \cup e_{\overline{j}} k\Delta e_i) \subseteq \psi'$. Clearly, $\langle \varphi \rangle = \langle \psi_1 \rangle$. Now consider the image of a path under $(-)^+$.

(i) For $p \in \mathscr{P}(Q)$ of length at least 1, we have the following statements.
  (1) If $p \in \mathscr{P}(Q')$, then $p^+ = p$.
  (2) If $p \notin \mathscr{P}(Q')$, then $p^+ = p + \overline{p} + p'$ with p' in the k-space $k\varphi$ generated by elements of φ.
(ii) For $\sigma \in \rho$, we write $\sigma = \sum_{i=1}^s a_i p_i + \sum_{j=s+1}^n a_j p_j$ with p_i a path in kQ for $1 \leq i \leq n$ such that $p_i \in \mathscr{P}(Q')$ for $1 \leq i \leq s$ and $p_j \notin \mathscr{P}(Q')$ for $s+1 \leq j \leq n$. It follows from (i)

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that

$$\sigma^+ = \sum_{i=1}^{s} a_i p_i^+ + \sum_{j=s+1}^{n} a_j p_j^+ = \sum_{i=1}^{s} a_i p_i + \sum_{j=s+1}^{n} a_j (p_j + \overline{p_j} + p'_j)$$
$$= \sigma + \sum_{j=s+1}^{n} a_j \overline{p_j} + \sum_{j=s+1}^{n} a_j p'_j. \qquad (*)$$

If $\sigma \in \psi_2$, then $s = 0$ and $\sigma^+ = \sigma + \overline{\sigma} + \sum_{j=1}^n a_j p'_j$ with $\overline{\sigma} \in \psi_3$, and therefore $\sigma^+ \in \langle\psi\rangle$. If $\sigma \notin \psi_2$, that is, $s(\sigma), t(\sigma) \in Q'_0$, then $\sigma_+ \in \psi_4$ and $\sigma^+ = \sigma_+ + \sum_{j=s+1}^n a_j p'_j \in \langle\psi\rangle$. Thus, $\langle\psi'\rangle \subseteq \langle\psi\rangle$ in $k\Delta$.

Conversely, pick up $\tau \in \psi$, we show $\tau \in \langle\psi'\rangle$. If $\tau = \sigma_+ \in \psi_4$, then $\tau = \sigma^+ - \sum_{j=s+1}^n a_j p'_j \in \langle\psi'\rangle$. If $\tau = \sigma \in \psi_2$ and $s(\sigma) \in V_0$, then $e_{s(\sigma)}\overline{\sigma} = 0$ and therefore $\sigma = e_{s(\sigma)}\sigma = e_{s(\sigma)}\sigma^+ - e_{s(\sigma)} \sum_{j=1}^n a_j p'_j \in \langle\psi'\rangle$. If $\tau = \sigma \in \psi_2$ and $t(\sigma) \in V_0$, then $\overline{\sigma} e_{t(\sigma)} = 0$ and $\sigma = \sigma e_{t(\sigma)} = \sigma^+ e_{t(\sigma)} - \sum_{j=1}^n a_j p'_j e_{t(\sigma)} \in \langle\psi'\rangle$. If $\tau = \overline{\sigma} \in \psi_3$ with $\sigma \in \psi_2$, then $\overline{\sigma} = \sigma^+ - \sigma - \sum_{j=1}^n a_j p'_j$. By what we have just proved, $\sigma \in \langle\psi'\rangle$, and therefore $\overline{\sigma} \in \langle\psi'\rangle$. Thus, $\langle\psi\rangle \subseteq \langle\psi'\rangle$, and therefore $\langle\psi'\rangle = \langle\psi\rangle$ and $k(\Delta, \psi') = k(\Delta, \psi)$.

Since $\varphi \subseteq \langle E\rangle$, it is clear that $\langle\psi'\rangle \subseteq \langle\rho^+ \cup E\rangle$. By the third equality in $(*)$ and the fact that $\sum_{j=s+1}^n a_j \overline{p_j}$ and $\sum_{j=s+1}^n a_j p'_j$ belong to $\langle E\rangle$, we obtain $\langle\rho^+ \cup E\rangle = \langle\rho \cup E\rangle$ in $k\Delta$. Thus, $k\Delta/\langle\rho^+ \cup E\rangle = k\Delta/\langle\rho \cup E\rangle \simeq kQ/\langle\rho\rangle = A$ as algebras. Moreover, since $\langle\rho^+\rangle \subseteq \langle\psi'\rangle \subseteq \langle\rho^+ \cup E\rangle \subseteq k\Delta$, the homomorphisms $(-)^+$ and δ induce algebra homomorphisms $\mu : A \to k\Delta/\langle\psi'\rangle$ and $\overline{\delta} : k\Delta/\langle\psi'\rangle \to k\Delta/\langle\rho^+ \cup E\rangle$, respectively. Now we identify $k\Delta/\langle\rho^+ \cup E\rangle$ with A. Then $\mu\overline{\delta} = \mathrm{Id}_A$ and μ is injective.

(2) We first construct a map θ by applying Lemma 3.6(3). For simplicity, let

$$R := R(A, e), \quad S := k(\Delta, \psi), \quad x := \sum_{i \in V_0} e_{\overline{i}} \in S.$$

Then $x^2 = x$. By (1), $(e)\mu = e^+ = \sum_{j \in V_0}(e_j + e_{\overline{j}})$. Since $e^+ e_{\overline{i}} = e_{\overline{i}} = e_{\overline{i}} e^+$, we have $e^+ x = x = x e^+$ and $x \in e^+ S e^+$. Recall that $e_j S e_{\overline{i}} = e_{\overline{i}} S e_j = 0$ for $i, j \in V_0$, due to the relation set ψ_1. Thus, for $s \in S$, we have

$$e^+ s e^+ x = e^+ s x = \sum_{j \in V_0} \sum_{i \in V_0} (e_j + e_{\overline{j}}) s e_{\overline{i}} = \Big(\sum_{j \in V_0} e_{\overline{j}}\Big) s \Big(\sum_{i \in V_0} e_{\overline{i}}\Big),$$
$$x e^+ s e^+ = x s e^+ = \sum_{i \in V_0} \sum_{j \in V_0} e_{\overline{i}} s (e_j + e_{\overline{j}}) = \Big(\sum_{i \in V_0} e_{\overline{i}}\Big) s \Big(\sum_{j \in V_0} e_{\overline{j}}\Big).$$

This shows $e^+ s e^+ x = x e^+ s e^+$. Since $\Lambda = eAe$ and $(\Lambda)\mu \subseteq e^+ S e^+$, we have $(c)\mu x = x e^+ (c)\mu$ for any $c \in \Lambda$. By Lemma 3.6(3), there is a unique algebra homomorphism $\theta : R \to S$ such that the restriction of θ to A equals μ and $(\overline{e})\theta = x$. Let $\overline{e_i} := e_i \otimes e_i \in R$. Then $\overline{e_i} = e_i \overline{e} e_i$ and $(\overline{e_i})\theta = e_i^+ x e_i^+ = e_i^+ (\sum_{i \in V_0} e_{\overline{i}}) e_i^+ = e_{\overline{i}}$.

Next, we prove that θ is surjective. It suffices to show that $\Delta_1 \subseteq \mathrm{Im}(\theta)$ and $e_t \in \mathrm{Im}(\theta)$ for $t \in \Delta_0$.

In fact, if $t \in Q'_0$, then $(e_t)\theta = (e_t)\mu = e_t$; if $t \in V_0$, then $(\overline{e_t})\theta = e_{\overline{t}}$ and $(e_t - \overline{e_t})\theta = e_t + e_{\overline{t}} - e_{\overline{t}} = e_t$. This implies that e_t belongs to $\mathrm{Im}(\theta)$ for any $t \in \Delta_0$. Now let $\alpha : u \to v$ be an arrow in Q_1. If $u, v \in Q'_0$, then $(\alpha)\theta = \alpha$. If $u \in V_0$ or $v \in V_0$, then $(\alpha)\theta = (\alpha)\mu = \alpha + \overline{\alpha}$. In the case where $u \in V_0$, we get

$$(\overline{e_u}\alpha)\theta = (\overline{e_u})\theta(\alpha)\theta = e_{\overline{u}}(\alpha)\mu = e_{\overline{u}}(\alpha + \overline{\alpha}) = \overline{\alpha} \quad \text{and} \quad (\alpha - \overline{e_u}\alpha)\theta = \alpha.$$

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In the case where $v \in V_0$, we have $(\alpha\overline{e_v})\theta = \overline{\alpha}$ and $(\alpha - \alpha\overline{e_v})\theta = \alpha$. Thus, $Q_1 \subseteq \mathrm{Im}(\theta)$ and $\overline{Q_1} \setminus Q_1' \subseteq \mathrm{Im}(\theta)$.

Finally, we construct an algebra homomorphism $\pi : S \to R$ such that $\theta\pi = \mathrm{Id}_R$, the identity map of R. This means that θ is injective. Hence, it is bijective.

We define a map $\{e_t \mid t \in \Delta_0\} \cup \Delta_1 \to R$ by $e_i \mapsto e_i - \overline{e_i}$, $e_{\bar{i}} \mapsto \overline{e_i}$ for $i \in V_0$; $e_j \mapsto e_j$ for $j \in Q_0'$; and for $\alpha \in Q_1$,

$$i \xrightarrow{\alpha} j \mapsto \begin{cases} \alpha & i, j \in Q_0', \\ \alpha - \alpha\overline{e_j}, & i \in Q_0, j \in V_0, \\ \alpha - \overline{e_i}\alpha, & i \in V_0, j \in Q_0; \end{cases} \qquad \bar{i} \xrightarrow{\overline{\alpha}} \bar{j} \mapsto \begin{cases} \alpha\overline{e_j}, & i \in Q_0,\ j \in V_0, \\ \overline{e_i}\alpha, & i \in V_0,\ j \in Q_0. \end{cases}$$

Note that $\overline{e_i}\alpha = e_i \otimes \alpha = \alpha \otimes e_j = \alpha\overline{e_j}$ in R for $i, j \in V_0$. By Lemma 3.8, the map can be extended to a unique homomorphism $\gamma : k\Delta \to R$ of algebras. Clearly, γ preserves the idempotent elements corresponding to the vertices in Q_0' and also the arrows in Q_1'. Further, if $i \in V_0$, then $(e_i^+)\gamma = (e_i + e_{\bar{i}})\gamma = e_i$; if $\alpha \in Q_1 \setminus Q_1'$, then $(\alpha^+)\gamma = (\alpha + \overline{\alpha})\gamma = \alpha$. This implies $(\sigma^+)\gamma = \sigma$ for any $\sigma \in \rho$. Moreover, by Lemma 3.4(1),

$$(e_i k\Delta e_{\bar{j}})\gamma \subseteq (e_i - \overline{e_i})R\overline{e_j} \subseteq (e - \overline{e})R\overline{e} = 0 \quad \text{and} \quad (e_{\bar{j}} k\Delta e_i)\varphi \subseteq \overline{e_j}R(e_i - \overline{e_i}) \subseteq \overline{e}R(e - \overline{e}) = 0$$

for any $i, j \in V_0$. Consequently, we have $\langle \psi' \rangle \subseteq \mathrm{Ker}(\gamma)$, and therefore γ induces an algebra homomorphism $\pi : S \to R$. Now let $g := \theta\pi : R \to R$ and $h := (-)^+\,\gamma : kQ \to R$. Since the restriction of θ to A equals μ, the restriction $g|_A : A \to R$ of g to A is induced from h. As γ preserves the idempotent elements corresponding to the vertices in Q_0 and also the arrows in Q_1, we see that $g|_A$ has its image in A and factorizes through Id_A. Since $(\overline{e_i})g = (e_{\bar{i}})\pi = \overline{e_i}$ for $i \in V_0$ and $\overline{e} = \sum_{i \in V_0} \overline{e_i}$, we have $(\overline{e})g = \overline{e}$. Thus, $g = \mathrm{Id}_R$ by Lemma 3.6(3). $\square$

Let us now illustrate the construction of $R(A, e)$ by an example.

Example 3.10. Suppose that A is an algebra over a field k presented by the quiver with relations:

$$\begin{tikzcd}
1 \arrow[d, bend right, "\alpha"'] \arrow[d, bend left, "\beta"] & 4 \arrow[l, "\delta"'] \arrow[d, "\sigma"] \\
2 \arrow[r, "\tau"] \arrow[d, "\gamma"'] & 5 \arrow[loop right, "\eta"] \arrow[dl, "\theta"] \\
3 &
\end{tikzcd}, \qquad \eta^2 = \sigma\eta = \tau\eta = \alpha\gamma = \delta\beta\tau = 0, \quad \beta\gamma = \beta\tau\theta.$$

Let Q' be the full subquiver of Q consisting of the vertex set $\{1, 2, 3\}$ and let $e = e_4 + e_5$. By Proposition 3.9(2), the algebra $R(A, e)$ is isomorphic to the algebra presented by the following quiver with relations:

$$\begin{tikzcd}
\overline{4} \arrow[r, "\overline{\delta}"] \arrow[d, "\overline{\sigma}"'] & 1 \arrow[d, bend right, "\alpha"'] \arrow[d, bend left, "\beta"] & 4 \arrow[l, "\delta"'] \arrow[d, "\sigma"] \\
\overline{5} \arrow[loop left, "\overline{\eta}"] \arrow[dr, "\overline{\theta}"'] & 2 \arrow[l, "\overline{\tau}"'] \arrow[r, "\tau"] \arrow[d, "\gamma"] & 5 \arrow[loop right, "\eta"] \arrow[dl, "\theta"] \\
& 3 &
\end{tikzcd}, \qquad \begin{aligned} &\delta\beta\overline{\tau} = \overline{\delta}\beta\tau = \delta\alpha\overline{\tau} = \overline{\delta}\alpha\tau = 0, \\ &\eta^2 = \sigma\eta = \tau\eta = \delta\beta\tau = 0, \\ &\overline{\eta}^2 = \overline{\sigma}\,\overline{\eta} = \overline{\tau}\,\overline{\eta} = \overline{\delta}\beta\,\overline{\tau} = 0, \\ &\alpha\gamma = 0, \quad \beta\gamma = \beta\tau\theta + \beta\overline{\tau}\,\overline{\theta}. \end{aligned}$$

This quiver is the mirror reflection of that of A along the full subquiver Q' of Q.

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4. Derived recollements

In this section we start by recalling recollements of triangulated categories, introduced by Beilinson, Bernstein and Deligne in [BBD82], and introduce the notion of stratified dimensions of algebras. We also construct recollements of mirror-reflective algebras.

4.1 Stratifying ideals and recollements

Definition 4.1. Let $\mathcal{T}$, $\mathcal{T}'$ and $\mathcal{T}''$ be triangulated categories. $\mathcal{T}$ admits a *recollement* of $\mathcal{T}'$ and $\mathcal{T}''$ (or there is a recollement among $\mathcal{T}''$, $\mathcal{T}$ and $\mathcal{T}'$) if there are six triangle functors

$$\begin{tikzcd} \mathcal{T}'' \arrow[r, "i_*=i_!"] & \mathcal{T} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^!=j^*"] & \mathcal{T}' \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

among the three categories such that the following four conditions are satisfied.

(1) $(i^*, i_*), (i_!, i^!), (j_!, j^!)$ and (j^*, j_*) are adjoint pairs.
(2) i_*, j_* and $j_!$ are fully faithful functors.
(3) $j^! i_! = 0$ (and thus also $i^! j_* = 0$ and $i^* j_! = 0$).
(4) For an object $X \in \mathcal{T}$, there are triangles $i_! i^!(X) \to X \to j_* j^*(X) \to i_! i^!(X)[1]$ and $j_! j^!(X) \to X \to i_* i^*(X) \to j_! j^!(X)[1]$ induced by the counits and units of the adjunctions, where $[1]$ is the shift functor of $\mathcal{T}$.

Recollements of derived module categories of rings are called *derived recollements*. Quasi-hereditary algebras, introduced by Cline, Parshall and Scott (see [CPS88, CPS96]), provide such a special class of derived recollements. For a heredity ideal I of an algebra A over a commutative ring, we have $\mathrm{Ext}^i_{A/I}(X, Y) \simeq \mathrm{Ext}^i_A(X, Y)$ for (A/I)-modules X, Y and $i \geq 0$. A slight generalization of heredity ideals is the n-idempotent ideals defined in [APT92].

Definition 4.2 [APT92]. Let A be an algebra, I an ideal of A, and n a positive integer. The ideal I of A is said to be *n-idempotent* if, for $X, Y \in (A/I)$-Mod, the canonical homomorphism $\mathrm{Ext}^i_{A/I}(X, Y) \to \mathrm{Ext}^i_A(X, Y)$ of k-modules is an isomorphism for all $1 \leq i \leq n$.

The ideal I is said to be a *strong idempotent ideal* if I is n-idempotent for all $n \geq 1$. In this case, if $I = AeA$ for an idempotent element $e \in A$, then e is called a *strong idempotent element* of A.

A strong idempotent ideal generated by an idempotent element is precisely a *stratifying ideal* as introduced in [CPS96, Definition 2.1.1]. We use the term 'stratifying ideals' throughout the paper. To emphasize the idempotent elements considered, we also retain the terminology of strong idempotent elements of algebras.

By a *trivial strong idempotent* element of A we mean the idempotent element 0 or an idempotent element e with $AeA = A$. Clearly, an ideal I is 1-idempotent if and only if I is idempotent. Moreover, stratifying ideals are closely related to homological ring epimorphisms. A ring homomorphism $\lambda : A \to B$ is called a *homological ring epimorphism* if the multiplication map $B \otimes_A B \to B$ is an isomorphism and $\mathrm{Tor}^A_i(B, B) = 0$ for all $i \geq 1$. This is equivalent to saying that the derived restriction functor $D(\lambda_*) : \mathscr{D}(B) \to \mathscr{D}(A)$, induced by the restriction functor $\lambda_* : B\text{-Mod} \to A\text{-Mod}$, is fully faithful. Note that an ideal I of A is a stratifying ideal if and only if the canonical surjection $A \to A/I$ is a homological ring epimorphism.

Lemma 4.3 [APT92]. *Let $I = AeA$ for an idempotent element e in A.*

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(1) *Let n be a positive integer. Then I is $(n+1)$-idempotent if and only if the multiplication map*

$$Ae \otimes_{eAe} eA \longrightarrow I, \quad ae \otimes eb \mapsto aeb, \quad a, b \in A$$

is an isomorphism of A-A-bimodules and $\mathrm{Tor}_i^{eAe}(Ae, eA) = 0$ for all $1 \leq i \leq n-1$.

(2) *If I is 2-idempotent, then*

$$\begin{aligned} &\sup\{n \in \mathbb{N} \mid \mathrm{Ext}_A^i(A/I, A/I) = 0, 1 \leq i \leq n\} \\ &\geq \sup\{n \in \mathbb{N} \mid \mathrm{Tor}_i^{eAe}(Ae, eA) = 0, 1 \leq i \leq n\} + 2. \end{aligned}$$

Proof. (1) Although all the results in [APT92] are stated for finitely generated modules over Artin algebras, many of them, such as Theorem 2.1, Lemma 3.1 and Propositions 2.4 and 3.7(b), hold for arbitrary modules over rings if we modify $\mathbf{P}_n$ in [APT92, Definition 2.3] as follows.

Let $\mathbf{P}_n(Ae)$ be the full subcategory of A-Mod consisting of all modules X such that there is an exact sequence $P_n \to \cdots \to P_1 \to P_0 \to X \to 0$ of A-modules with $P_i \in \mathrm{Add}(Ae)$ for $0 \leq i \leq n$, where $\mathrm{Add}(Ae)$ is the full subcategory of A-Mod consisting of direct summands of direct sums of copies of Ae.

By [APT92, Theorem 2.1], $I := AeA$ is $(n+1)$-idempotent if and only if $I \in \mathbf{P}_n(Ae)$. In particular, I is 2-idempotent if and only if $I \in \mathbf{P}_1(Ae)$. By [APT92, Lemma 3.1], the adjoint pair $(Ae \otimes_{eAe} -, \mathrm{Hom}_A(Ae, -))$ between (eAe)-Mod and A-Mod induces additive equivalences between (eAe)-Mod and $\mathbf{P}_1(Ae)$. Note that $\mathrm{Hom}_A(Ae, I) \simeq eI = eA$. Thus, $I \in \mathbf{P}_1(Ae)$ if and only if the multiplication map $Ae \otimes_{eAe} eA \to AeA$ is an isomorphism of A-A-bimodules. Assume now that I is 2-idempotent. By [APT92, Proposition 3.7(b)], $I \in \mathbf{P}_n(Ae)$ if and only if $\mathrm{Tor}_i^{eAe}(Ae, eA) = 0$ for all $1 \leq i < n$. This shows (1).

(2) If I is $(n+1)$-idempotent, then $\mathrm{Ext}_A^i(A/I, A/I) \simeq \mathrm{Ext}_{A/I}^i(A/I, A/I) = 0$ for all $1 \leq i \leq n+1$. Now (2) follows from (1). □

Corollary 4.4.

(1) *Let e and f be idempotent elements of A such that $ef = e = fe$. If AeA is an $(n+1)$-idempotent ideal of A for a positive integer n, then $fAeAf$ is an $(n+1)$-idempotent ideal of fAf. In particular, if e is a strong idempotent element of A, then it is also a strong idempotent element of fAf.*

(2) *Let $\{e, e_1, e_2\}$ be a set of pairwise orthogonal idempotent elements of A such that e is a strong idempotent element of A. Define $f := e + e_1$, $g := e + e_1 + e_2$ and $\overline{A} := A/AeA$. Let $\overline{f} := f + AeA$ denote the image of f in $\overline{A}$. If $\overline{f}$ is a strong idempotent element of $\overline{g}\overline{A}\overline{g}$, then f is a strong idempotent element of gAg.*

Proof. (1) Transparently, $e \in fAf$, $efAfe = eAe$, $fAeAfe = fAe$ and $efAeAf = eAf$. If $Ae \otimes_{eAe} eA \simeq AeA$, then $fAe \otimes_{eAe} eAf \simeq fAeAf$. Since $Ae = fAe \oplus (1-f)Ae$ and $eA = eAf \oplus eA(1-f)$, we see that the abelian group $\mathrm{Tor}_i^{eAe}(fAe, eAf)$ is a direct summand of $\mathrm{Tor}_i^{eAe}(Ae, eA)$ for $i \in \mathbb{N}$. Then (1) follows from Lemma 4.3(1).

(2) Clearly, $AeA \subseteq AfA \subseteq AgA$, and $\overline{g}\overline{A}\overline{g} \simeq gAg/gAeAg$ and $\overline{g}\overline{A}\overline{g}/\overline{g}\overline{A}\overline{f}\overline{A}\overline{g} \simeq gAg/gAfAg$ as algebras. Suppose that $\overline{f}$ is a strong idempotent element of $\overline{g}\overline{A}\overline{g}$. Then the canonical surjection $\pi_2 : gAg/gAeAg \to gAg/gAfAg$ is homological. Since e is a strong idempotent element of A and $ge = e = eg$, the canonical surjection $\pi_1 : gAg \to gAg/gAeAg$ is also homological by (1). Observe that compositions of homological ring epimorphisms are again homological ring epimorphisms. Thus, $\pi_1\pi_2 : gAg \to gAg/gAfAg$ is homological. This implies that f is a strong idempotent element in gAg. □

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Let $e = e^2 \in A$. If AeA is a stratifying ideal in A, then the recollement of derived module categories of algebras,

$$\begin{tikzcd} \mathscr{D}(A/AeA) \arrow[r] & \mathscr{D}(A) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}(eAe) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd} ,$$

is called a *standard recollement* induced by AeA. If ${}_AAeA$ or AeA_A is projective (for example, AeA is a heredity ideal in A), then AeA is a stratifying ideal in A. In the case where ${}_AAeA$ is projective, the recollement restricts to a recollement $(\mathscr{D}^-(A/AeA), \mathscr{D}^-(A), \mathscr{D}^-(eAe))$ of bounded-above derived categories.

For constructing finitely generated (one-sided) projective idempotent ideals of the endomorphism algebras of objects in additive categories (see [CX16b, Lemmas 3.2 and 3.4]), we have the following lemma.

Lemma 4.5. *Suppose that R is an algebra and I is an ideal of R.*

(1) *Let $A := \mathrm{End}_R(R \oplus\ R/I)$ and $e^2 = e \in A$ correspond to the direct summand R/I of the R-module $R \oplus\ R/I$. Then AeA_A is finitely generated and projective, and there is a recollement $(\mathscr{D}(R/\mathrm{Ann}_{R^{\mathrm{op}}}(I)), \mathscr{D}(A), \mathscr{D}(R/I))$, with $\mathrm{Ann}_{R^{\mathrm{op}}}(I) := \{r \in R \mid Ir = 0\}$.*
(2) *Let $B := \mathrm{End}_R(R \oplus I)$ and $f = f^2 \in B$ correspond to the direct summand I of the R-module $R \oplus I$. If I is idempotent, then ${}_BBfB$ is finitely generated and projective, and there is a recollement $(\mathscr{D}(R/I), \mathscr{D}(B), \mathscr{D}(\mathrm{End}_R(I)))$.*

Another way to produce finitely generated projective ideals comes from Morita context algebras, as explained below.

Let R be an algebra and let I and J be ideals of R with $IJ = 0$. Define

$$M_l(R, I, J) := \begin{pmatrix} R & I \\ R/J & R/J \end{pmatrix} \quad \left(\text{respectively, } M_r(R, I, J) := \begin{pmatrix} R & R/I \\ J & R/I \end{pmatrix}\right),$$

which is the *Morita context algebra* with the bimodule homomorphisms given by the canonical ones:

$$I \otimes_{R/J} (R/J) \simeq I \hookrightarrow R, \quad (R/J) \otimes_R I \simeq I/JI \twoheadrightarrow (I + J)/J \hookrightarrow R/J$$

(respectively, $(R/I) \otimes_{R/I} J \simeq J \hookrightarrow R,\ J \otimes_R (R/I) \simeq J/JI \twoheadrightarrow (I + J)/I \hookrightarrow R/I$). Note that $M_r(R, I, \mathrm{Ann}_{R^{\mathrm{op}}}(I)) \simeq \mathrm{End}_R(R \oplus R/I)$ as algebras. Moreover, if ${}_RR$ is injective and $I^2 = I$, then $M_l(R, I, \mathrm{Ann}_{R^{\mathrm{op}}}(I)) \simeq \mathrm{End}_R(R \oplus I)$ as algebras. This is due to $\mathrm{Hom}_R(I, R/I) = 0$.

Let

$$e := \begin{pmatrix} 0 & 0 \\ 0 & 1 + J \end{pmatrix} \in M_l(R, I, J), \quad f := \begin{pmatrix} 0 & 0 \\ 0 & 1 + I \end{pmatrix} \in M_r(R, I, J).$$

Then the next lemma is easy to verify.

Lemma 4.6. *Let $A := M_l(R, I, J)$ and $B := M_r(R, I, J)$. Then ${}_AAeA$ and BfB_B are finitely generated and projective. Moreover, there are recollements $(\mathscr{D}(R/I), \mathscr{D}(A), \mathscr{D}(R/J))$ and $(\mathscr{D}(R/J), \mathscr{D}(B), \mathscr{D}(R/I))$.*

4.2 Stratified dimensions of algebras

Now, we introduce stratified dimensions of algebras over a commutative ring, which measure how many steps the given algebras can be stratified by their nontrivial strong idempotent elements.

Definition 4.7. By an *idempotent stratification* of length n of an algebra A, we mean a set $\{e_i \mid 0 \leq i \leq n\}$ of $n+1$ nonzero (not necessarily primitive) pairwise orthogonal idempotent elements of A satisfying the following conditions:

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(a) $1 = \sum_{j=0}^{n} e_j$ and $e_{i+1} \notin Ae_{\leq i}A$ (or equivalently, $Ae_{\leq i}A \subsetneq Ae_{\leq(i+1)}A$) for all $0 \leq i \leq n-1$, where $e_{\leq m} := \sum_{j=0}^{m} e_j$ for $0 \leq m \leq n$; and
(b) $e_{\leq i}$ is a strong idempotent element of the algebra $e_{\leq(i+1)}Ae_{\leq(i+1)}$ for $0 \leq i \leq n-1$.

The *stratified dimension* of A, denoted by $\mathrm{stdim}(A)$, is defined to be the supremum of the lengths of all idempotent stratifications of A.

Clearly, $\mathrm{stdim}(A) = 0$ if and only if A has no stratifying ideal apart from itself and 0. If $\mathrm{stdim}(A) = n > 0$, then there are nontrivial standard recollements $(\mathscr{D}(A_i/I_i), \mathscr{D}(A_i), \mathscr{D}(A_{i-1}))$, $1 \leq i < n+1$, where $A_0 := e_0Ae_0$, $A_i := e_{\leq i}Ae_{\leq i}$ and $I_i := e_{\leq i}Ae_{\leq(i-1)}Ae_{\leq i}$ are as defined in Definition 4.7. Moreover, for any two algebras Γ_1 and Γ_2, $\mathrm{stdim}(\Gamma_1 \times \Gamma_2) = \mathrm{stdim}(\Gamma_1) + \mathrm{stdim}(\Gamma_2) + 1$. This implies that the stratified dimension of the direct product of countably many copies of a field k is infinite.

Stratifications of algebras in the sense of Cline, Parshall and Scott are idempotent stratifications. But the converse is not true. Following [CPS96, Chapter 2], a *stratification* of length $n+1$ of an algebra A is a chain of ideals, $0 = U_{-1} \subsetneq U_0 \subsetneq U_1 \subsetneq \cdots \subsetneq U_{n-1} \subsetneq U_n = A$, generated by idempotent elements such that U_i/U_{i-1} is a stratifying ideal in A/U_{i-1} for $0 \leq i \leq n$. In this case, A is said to be *CPS stratified*. If $\{e_i \mid 0 \leq i \leq n\}$ is a complete set of nonzero primitive pairwise orthogonal idempotent elements of A and $U_i = Ae_{\leq i}A$ for $0 \leq i \leq n$, then A is called a *fully CPS-stratified* algebra. Standardly stratified algebras with respect to an order of simple modules are fully CPS stratified.

Lemma 4.8. *Let $\{e_i \mid 0 \leq i \leq n\}$ be a set of nonzero pairwise orthogonal idempotent elements of A satisfying condition (a) in Definition 4.7. Define $U_i := Ae_{\leq i}A$ for $0 \leq i \leq n$ and $U_{-1} := 0$. If U_i/U_{i-1} is a stratifying ideal in A/U_{i-1} for $0 \leq i \leq n$, then condition (b) in Definition 4.7 holds.*

Proof. Since U_i/U_{i-1} is a stratifying ideal in A/U_{i-1} by assumption, the canonical surjection $A/U_{i-1} \to A/U_i$ is homological. As the composition of homological ring epimorphisms is still a homological ring epimorphism, the canonical surjection $A \to A/U_i$ is homological. This implies that $e_{\leq i}$ is a strong idempotent element of A. By Corollary 4.4, $e_{\leq i}$ is a strong idempotent element of $e_{\leq(i+1)}Ae_{\leq(i+1)}$. Thus, Definition 4.7(b) holds. □

Proposition 4.9. *Let A be an Artin algebra over a commutative Artin ring k. Then:*

(1) $\mathrm{stdim}(A) \leq \#(A) - 1$.
(2) *If A has a stratification of length $n+1$ with $n \in \mathbb{N}$, then* $\mathrm{stdim}(A) \geq n$. *In particular, if A is a fully CPS-stratified algebra, then* $\mathrm{stdim}(A) = \#(A) - 1$.
(3) *If* $\mathrm{stdim}(A) \geq 1$, *then* $\mathrm{stdim}(A) = \sup_{e \in A}\{\mathrm{stdim}(eAe) + \mathrm{stdim}(A/AeA) + 1\}$, *where e runs over all nonzero strong idempotent elements of A with $AeA \neq A$.*
(4) *If k is a field and B is a finite-dimensional k-algebra, then*

$$\mathrm{stdim}(A \otimes_k B) \geq (\mathrm{stdim}(A) + 1)(\mathrm{stdim}(B) + 1) - 1.$$

Proof. (1) This is clear by Definition 4.7(a).

(2) The first part of (2) follows from Lemma 4.8. If A is a fully CPS-stratified algebra, then it has a stratification of length $\#(A) - 1$. By (1), we obtain $\mathrm{stdim}(A) = \#(A) - 1$.

(3) An Artin algebra has only finitely many nonisomorphic, indecomposable, projective modules. This implies:

($*$) If f is an idempotent element of A and I is an idempotent ideal of A such that $AfA \subseteq I$, then there is an idempotent element f' of A which is orthogonal to f such that $I = A(f + f')A$.

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Now let $n := \operatorname{stdim}(A) \geq 1$. On the one hand, since $e_{\leq n-1}$ in Definition 4.7(b) is a strong idempotent element of A, we have $\operatorname{stdim}(A) = \operatorname{stdim}(e_{\leq n-1}Ae_{\leq n-1}) + 1$ and $\operatorname{stdim}(A/Ae_{\leq n-1}A) = 0$ by ($*$) and Corollary 4.4(2). On the other hand, for each nontrivial strong idempotent element e of A, it follows again from ($*$) and Corollary 4.4(2) that $\operatorname{stdim}(eAe) + \operatorname{stdim}(A/AeA) + 1 \leq n$. Thus, (3) holds.

(4) Let $m := \operatorname{stdim}(B)$ and $\ell := n + m$. If $\ell = 0$ (that is, $n = 0 = m$), then the inequality obviously holds. Let $\ell \geq 1$. Without loss of generality, suppose $n \geq 1$. By the proof of (3), there is a nonzero strong idempotent element e of A with $AeA \neq A$ such that $\operatorname{stdim}(eAe) = n - 1$ and $\operatorname{stdim}(A/AeA) = 0$. Then the canonical surjection $\pi : A \to A/AeA$ is homological. For two homological ring epimorphisms $\lambda_i : R_i \to S_i$ of algebras over the field k with $i = 1, 2$, the tensor product $\lambda_1 \otimes_k \lambda_2 : R_1 \otimes_k R_2 \to S_1 \otimes_k S_2$ is again a homological ring epimorphism. This is due to the isomorphism

$$\operatorname{Tor}_j^{R_1 \otimes_k R_2}(S_1 \otimes_k S_2, S_1 \otimes_k S_2) \simeq \bigoplus_{p+q=j} \operatorname{Tor}_p^{R_1}(S_1, S_1) \otimes_k \operatorname{Tor}_q^{R_2}(S_2, S_2) \quad \text{for all } j \in \mathbb{N}.$$

Now let $C := A \otimes_k B$ and $e' := e \otimes 1 \in C$. Then the surjection $\pi \otimes 1 : C \to (A/AeA) \otimes_k B$ is homological. Clearly, there are algebra isomorphisms $(A/AeA) \otimes_k B \simeq C/(AeA \otimes_k B) \simeq C/Ce'C$. It follows that the canonical surjection $C \to C/Ce'C$ is homological, and therefore e' is a nontrivial strong idempotent element of C. By (3), $\operatorname{stdim}(C) \geq \operatorname{stdim}(eAe \otimes_k B) + \operatorname{stdim}((A/AeA) \otimes_k B) + 1$. Moreover, by induction, $\operatorname{stdim}(eAe \otimes_k B) \geq (\operatorname{stdim}(eAe) + 1)(\operatorname{stdim}(B) + 1) - 1$ and $\operatorname{stdim}((A/AeA) \otimes_k B) \geq \operatorname{stdim}(B)$. Thus, $\operatorname{stdim}(C) \geq (n+1)(m+1) - 1$. $\square$

Definition 4.10. Let A be an Artin algebra over a commutative Artin ring k. The rational number $\operatorname{stdim}(A)/\#(A)$ is called the *stratified ratio* of A and denoted by $\operatorname{sr}(A)$.

By Proposition 4.9(1), $\operatorname{sr}(A) \in \mathbb{Q} \cap [0, 1)$. Let A^n denote the product of n copies of A. Then

$$\lim_{n\to\infty} \operatorname{sr}(A^n) = \lim_{n\to\infty} \frac{n(\operatorname{stdim}(A)) + n - 1}{n\ \#(A)} = \frac{\operatorname{stdim}(A) + 1}{\#(A)} \leq 1.$$

In particular, if $\operatorname{stdim}(A) = \#(A) - 1$ (for example, A is quasi-hereditary or local), then $\lim_{n\to\infty} \operatorname{sr}(A^n) = 1$. In § 5, for a gendo-symmetric algebra with infinite dominant dimension, we construct a series of indecomposable symmetric algebras S_n such that $\lim_{n\to\infty} \operatorname{sr}(S_n) = 1$ (see Corollary 5.12 for details).

4.3 Construction of recollements from mirror-reflective algebras

In this subsection we construct explicitly derived recollements from mirror-reflective algebras.

Throughout this subsection we retain all notation in Section 3.1. Recall that $R := R(A, e)$, $S := S(A, e)$ and $\bar{e} := e \otimes e \in R$.

Proposition 4.11. *Let $A_2 := \operatorname{End}_R(R \oplus R/I)$ and $B_2 := \operatorname{End}_S(S \oplus S/I)$. Suppose that the right A-module eA_A is faithful. Then the following statements hold true.*

(1) *There are standard recollements of derived module categories*

$$\begin{tikzcd} \mathscr{D}(A) \arrow[r] & \mathscr{D}(A_2) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}(A), \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd} \qquad \begin{tikzcd} \mathscr{D}(A) \arrow[r] & \mathscr{D}(B_2) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}((1-e)A(1-e)) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

induced by idempotent ideals that are finitely generated and projective as right modules over A_2 and B_2, respectively.

(2) $\operatorname{stdim}(A_2) \geq 2\operatorname{stdim}(A) + 1$ *and* $\operatorname{stdim}(B_2) \geq \operatorname{stdim}(A) + \operatorname{stdim}((1-e)A(1-e)) + 1$.

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(3) $\mathrm{gldim}(A^{\mathrm{op}}) \leq \mathrm{gldim}(A_2^{\mathrm{op}}) \leq 2\mathrm{gldim}(A^{\mathrm{op}}) + 2$,
$\mathrm{findim}(A^{\mathrm{op}}) \leq \mathrm{findim}(A_2^{\mathrm{op}}) \leq 2\mathrm{findim}(A^{\mathrm{op}}) + 2$.

Proof. (1) Let e_2 be the idempotent of A_2 corresponding to the direct summand R/I of the R-module $R \oplus R/I$. Then $e_2A_2e_2 \simeq R/I$ and $A_2/A_2e_2A_2 \simeq R/\mathrm{Ann}_{R^{\mathrm{op}}}(I)$ as algebras. By Lemma 4.5(1), the A_2^{op}-module $A_2e_2A_2$ is finitely generated and projective. This implies that e_2 is a strong idempotent of A_2.

Let f_2 be the idempotent of B_2 corresponding to the direct summand S/I of the S-module $S \oplus S/I$. Similarly, by Lemma 4.5(1), $f_2B_2f_2 \simeq S/I$ and $B_2/B_2f_2B_2 \simeq S/\mathrm{Ann}_{S^{\mathrm{op}}}(I)$ as algebras, the B_2^{op}-module $B_2f_2B_2$ is finitely generated and projective, and thus f_2 is a strong idempotent of B_2.

Since eA_A is faithful, $J = \mathrm{Ann}_{R^{\mathrm{op}}}(I)$ by Lemma 3.5(1). Note that I is an ideal of S and $\mathrm{Ann}_{S^{\mathrm{op}}}(I) = S \cap \mathrm{Ann}_{R^{\mathrm{op}}}(I) = S \cap J$. By Lemma 3.4(4), there are algebra isomorphisms $A \simeq R/I \simeq R/J \simeq S/(S \cap J)$ and $S/I \simeq (1-e)A(1-e)$, and therefore

$$e_2A_2e_2 \simeq A \simeq A_2/A_2e_2A_2, \quad f_2B_2f_2 \simeq (1-e)A(1-e) \quad \text{and} \quad B_2/B_2f_2B_2 \simeq A.$$

Since e_2 is a strong idempotent of A_2 and f_2 is a strong idempotent of B_2, (1) holds.

(2) By the proof of (1), e_2 and f_2 are strong idempotents in A_2 and B_2, respectively. Thus, $\mathrm{stdim}(A_2) \geq 1$ and $\mathrm{stdim}(B_2) \geq 1$. Then (2) follows from Proposition 4.9(3).

(3) This will be shown by some general formulas on the global and finitistic dimensions of rings.

Let Γ be a ring and f a strong idempotent element of Γ. By Definition 4.2, we have

$$\text{(a)} \quad \mathrm{gldim}(\Gamma/\Gamma f\Gamma) \leq \mathrm{gldim}(\Gamma).$$

Applying [CX17, Theorem 3.17(2)] to the standard recollement

$$\begin{tikzcd} \mathscr{D}(\Gamma/\Gamma f\Gamma) \arrow[r, "i_*"] & \mathscr{D}(\Gamma) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}(f\Gamma f), \arrow[l, bend right, "j!"'] \arrow[l, bend left] \end{tikzcd}$$

where i_* is the derived restriction functor induced from the canonical surjection $\Gamma \to \Gamma/\Gamma f\Gamma$ and $j_!$ is the left-derived functor $\Gamma f \otimes^{\mathbb{L}}_{f\Gamma f} -$, we obtain

$$\text{(b)} \quad \mathrm{gldim}(\Gamma) \leq \mathrm{gldim}(f\Gamma f) + \mathrm{gldim}(\Gamma/\Gamma f\Gamma) + \mathrm{pdim}({}_\Gamma\Gamma/\Gamma f\Gamma) + 1.$$

Moreover, by [CX17, Corollary 3.12], if ${}_\Gamma\Gamma/\Gamma f\Gamma$ has a finite projective resolution by finitely generated projective Γ-modules, then

$$\text{(c)} \quad \mathrm{findim}(\Gamma/\Gamma f\Gamma) \leq \mathrm{findim}(\Gamma) \leq \mathrm{findim}(f\Gamma f) + \mathrm{findim}(\Gamma/\Gamma f\Gamma) + \mathrm{pdim}({}_\Gamma\Gamma/\Gamma f\Gamma) + 1.$$

Let $\Gamma := A_2^{\mathrm{op}}$ and $f := e_2^{\mathrm{op}}$. Then $f\Gamma f \simeq A^{\mathrm{op}} \simeq \Gamma/\Gamma f\Gamma$ as rings. By the proof of (1) (see the first paragraph), the Γ-module $\Gamma f\Gamma$ is finitely generated and projective, and the element f is a strong idempotent of Γ. Thus, (a) and (b) imply (3) on global dimensions, while (c) gives (3) on finitistic dimensions. □

We now consider n-idempotent and stratifying ideals of mirror-reflective algebras.

Proposition 4.12.

(1) *The ideals I and J of R are 2-idempotent.*
(2) *Let $n \geq 1$ be an integer. Then I is $(n+2)$-idempotent if and only if so is J if and only if $\mathrm{Tor}_i^{eAe}(Ae, eA) = 0$ for all $1 \leq i \leq n$.*

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(3) *If* $\mathrm{Tor}_i^{eAe}(Ae, eA) = 0$ *for all* $i \geq 1$, *then there are standard recollements of derived module categories induced by* $I := R\bar{e}R$:

$$\begin{tikzcd} \mathscr{D}(A) \arrow[r] & \mathscr{D}(R) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}(eAe) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathscr{D}((1-e)A(1-e)) \arrow[r] & \mathscr{D}(S) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}(eAe). \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

Proof. (1) There is a commutative diagram

$$\begin{tikzcd} R\bar{e} \otimes_{\bar{e}R\bar{e}} \bar{e}R \arrow[r, "\mu"] \arrow[d, "\pi_2\otimes\pi_2"'] & R\bar{e}R = Ae \otimes_{eAe} eA \arrow[d, "\pi_2"] \\ Ae \otimes_{eAe} eA \arrow[r, "\mu'"] & AeA \end{tikzcd}$$

where μ and μ' are the multiplication maps. By Lemma 3.5(2), $\pi_2 \otimes \pi_2$ is an isomorphism. Note that the composition of the inverse of $\pi_2 \otimes \pi_2$ with μ is the identity of $Ae \otimes_{eAe} eA$. Thus, μ is an isomorphism. This shows that I is 2-idempotent by Lemma 4.3(1). Similarly, we can show that J is 2-idempotent by using the idempotent element $e - \bar{e}$ and the algebra homomorphism π_1.

(2) By Lemma 3.4(3), I is $(n+2)$-idempotent if and only if so is J. Since I is 2-idempotent by (1), it follows from Lemma 4.3(1) that I is $(n+2)$-idempotent if and only if $\mathrm{Tor}_i^{\bar{e}R\bar{e}}(R\bar{e}, \bar{e}R) = 0$ for $1 \leq i \leq n$. By Lemma 3.5(2), π_2 induces isomorphisms of abelian groups $\mathrm{Tor}_i^{\bar{e}R\bar{e}}(R\bar{e}, \bar{e}R) \simeq \mathrm{Tor}_i^{eAe}(Ae, eA)$ for all $i \in \mathbb{N}$. Thus, I is $(n+2)$-idempotent if and only if $\mathrm{Tor}_i^{eAe}(Ae, eA) = 0$ for $1 \leq i \leq n$.

(3) By (2), I is a stratifying ideal in R if and only if $\mathrm{Tor}_i^{eAe}(Ae, eA) = 0$ for all $i \geq 1$. According to Corollary 4.4(1), if I is a stratifying ideal in R, then $e_0 I e_0$ is a stratifying ideal in S. By Lemmas 3.4 and 3.5(2), $e_0 I e_0 = I$, $S/I \simeq (1-e)A(1-e)$, $R/I \simeq A$ and $\bar{e}R\bar{e} \simeq eAe \simeq \bar{e}S\bar{e}$. Thus, the recollements in (3) exist. $\square$

5. Iterated mirror-reflective algebras and Tachikawa's second conjecture

This section is devoted to proofs of all results stated in the introduction. We first show that mirror-reflective algebras of gendo-symmetric algebras at any levels are symmetric (see Proposition 5.2). Based on this result, we construct not only gendo-symmetric algebras of strictly increasing dominant dimensions and higher minimal Auslander–Gorenstein algebras (see Theorem 1.4), but also recollements of derived module categories of these algebras (see Theorem 1.2). The recollements constructed are then applied to give a new formulation of Tachikawa's second conjecture for symmetric algebras in terms of stratified dimensions and ratios (see Theorem 5.13). Consequently, a sufficient condition is given for the conjecture to hold for symmetric algebras (see Theorem 1.1(II)).

Throughout this section all algebras considered are finite-dimensional algebras over a field k.

5.1 Relations among mirror-reflective, symmetric and gendo-symmetric algebras

Let A be an algebra, $e^2 = e \in A$ and $\Lambda := eAe$. Suppose that there is an isomorphism $\iota : eA \to D(Ae)$ of Λ-A-bimodules. Let $\iota_e := (e)\iota \in D(Ae) = \mathrm{Hom}_k(Ae, k)$. Then $\iota_e = e\iota_e = \iota_e e$. Moreover, ι is nothing other than the left multiplication map by ι_e. Define $\zeta : Ae \otimes_\Lambda eA \to k$ to be the composition of the maps

$$Ae \otimes_\Lambda eA \xrightarrow{\mathrm{id}\otimes\iota} Ae \otimes_\Lambda D(Ae) \xrightarrow{\mathrm{ev}} k$$

where ev stands for the evaluation map: $ae \otimes f \mapsto (ae)f$ for $a \in A$ and $f \in D(Ae)$. Then ζ is given by $(ae \otimes eb)\zeta = (bae)\iota_e = (ebae)\iota_e$ for $a, b \in A$. To any element $\lambda \in Z(\Lambda)$, two maps of

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k-spaces are associated:

$$\chi: \quad R(A,e,\lambda) = A \oplus Ae \otimes_\Lambda eA \longrightarrow k,$$
$$a + \sum_{i=1}^{n} a_i e \otimes e b_i \mapsto \sum_{i=1}^{n} (a_i e \otimes e b_i)\zeta = \sum_{i=1}^{n} (e b_i a_i e)\iota_e \quad \text{for } a_i, b_i \in A,$$
$$\gamma: \quad Ae \otimes_\Lambda eA \longrightarrow D(A),$$
$$ae \otimes eb \mapsto [a' \mapsto (eba'ae)\iota_e \text{ for } a, a', b \in A].$$

Lemma 5.1.

(1) *For any $r_1, r_2 \in R(A,e,\lambda)$, $(r_1 * r_2)\chi = (r_2 * r_1)\chi$, where $*$ denotes the multiplication of $R(A,e,\lambda)$.*
(2) *The map γ is a homomorphism of A-A-bimodules. It is an isomorphism if and only if the map $(\cdot e): \mathrm{End}_{A^{\mathrm{op}}}(A) \to \mathrm{End}_{\Lambda^{\mathrm{op}}}(Ae)$ induced from right multiplication by e is an isomorphism of algebras.*
(3) *If $\epsilon : D(A) \to k$ denotes the map sending $f \in D(A)$ to $(1)f$, then $\zeta = \gamma\,\epsilon$.*

Proof. (1) It suffices to show $((a_1 + ae \otimes eb) * (a_2 + a'e \otimes eb'))\chi = ((a_2 + a'e \otimes eb') * (a_1 + ae \otimes eb))\chi$ for any $a, a', b, b', a_1, a_2 \in A$. Indeed, this follows from $(a'(ae \otimes eb))\zeta = ((ae \otimes eb)a')\zeta$ and $((ae \otimes eb) \otimes (a'e \otimes eb'))\omega_\lambda\zeta = ((a'e \otimes eb') \otimes (ae \otimes eb))\omega_\lambda\zeta$, by the definitions of ζ and ω_λ in Section 3.1.

(2) There is a canonical isomorphism $\varphi: \; Ae \otimes_\Lambda D(Ae) \to D(\mathrm{End}_{\Lambda^{\mathrm{op}}}(Ae))$, $ae \otimes f \mapsto [g \mapsto (ae)gf]$ for $a \in A$, $f \in D(Ae)$ and $g \in \mathrm{End}_{\Lambda^{\mathrm{op}}}(Ae)$. Let $\vartheta : A \to \mathrm{End}_{A^{\mathrm{op}}}(A)$ be the isomorphism which sends a to $(a\cdot)$. Then the composition of the maps

$$Ae \otimes_\Lambda eA \xrightarrow{Ae \otimes \iota} Ae \otimes_\Lambda D(Ae) \xrightarrow{\varphi} D(\mathrm{End}_{\Lambda^{\mathrm{op}}}(Ae)) \xrightarrow{D(\cdot e)} D(\mathrm{End}_{A^{\mathrm{op}}}(A)) \xrightarrow{D(\vartheta)} D(A)$$

coincides with γ. Clearly, all the maps above are homomorphisms of A-A-bimodules. Thus, γ is a homomorphism of A-A-bimodules. Since $D : k\text{-mod} \to k\text{-mod}$ is a duality, γ is an isomorphism if and only if the map $(\cdot e)$ in (2) is an isomorphism of algebras.

(3) This follows from $(ae \otimes eb)\zeta = (ebae)\iota_e$ for $a, b \in A$. $\square$

From now on, let (A, e) be a gendo-symmetric algebra. Then $\mathrm{add}(Ae)$ coincides with the full subcategory of A-mod consisting of projective-injective A-modules. If e' is another idempotent element of A such that $\mathrm{add}(Ae) = \mathrm{add}(Ae')$, then the mirror-reflective algebras $R(A, e)$ and $R(A, e')$ are isomorphic as algebras by Lemma 3.6(1). So, for simplicity, we write $R(A)$ for $R(A, e)$.

In the following, we describe $R(A)$ as a deformation of the trivial extension of A. Let $\Lambda := eAe$ and $\iota : eA \to D(Ae)$ be an isomorphism of Λ-A-bimodules (see Lemma 2.2(2)). Then Λ is symmetric and eA is a generator over Λ. Moreover, there are algebra isomorphisms $A \simeq \mathrm{End}_\Lambda(eA)$ and $A^{\mathrm{op}} \simeq \mathrm{End}_{\Lambda^{\mathrm{op}}}(Ae)$. By Lemma 5.1(2), there is an isomorphism of A-A-bimodules: $\gamma: \;\; Ae \otimes_\Lambda eA \xrightarrow{\simeq} D(A)$. Since $A \simeq \mathrm{End}_\Lambda(eA)$ and eA is a generator over Λ, the functor $e(-)e : A^e\text{-Mod} \to \Lambda^e\text{-Mod}$ between the categories of bimodules induces an algebra isomorphism $Z(A) \to Z(\Lambda)$. So, for $\lambda \in Z(\Lambda)$, there exists a unique element $\lambda' \in Z(A)$ such that $e\lambda' e = \lambda$. Define $\overline{\omega_e} := (\gamma \otimes \gamma)^{-1}\omega_e\gamma : \;\; D(A) \otimes_A D(A) \xrightarrow{\simeq} D(A)$ and $F = Ae \otimes_\Lambda - \otimes_\Lambda eA : \Lambda^e\text{-Mod} \to A^e\text{-Mod}$.

We obtain the following commutative diagram.

$$\begin{tikzcd}
(Ae\otimes_\Lambda eA)\otimes_A(Ae\otimes_\Lambda eA) \arrow[r, "\omega_e", "\simeq"'] \arrow[d, "\gamma\otimes\gamma"'] & Ae\otimes_\Lambda eA \arrow[r, "F(\cdot\lambda)"] \arrow[d, "\gamma"] & Ae\otimes_\Lambda eA \arrow[d, "\gamma"] \\
D(A)\otimes_A D(A) \arrow[r, "\overline{\omega_e}"] & D(A) \arrow[r, "(\cdot\lambda')"] & D(A)
\end{tikzcd}$$

Define $\overline{\omega_\lambda}:=\overline{\omega_e}(\cdot\lambda'):\ D(A)\otimes_A D(A)\longrightarrow D(A)$. We now extend $\overline{\omega_\lambda}$ to a multiplication on the direct sum $A\oplus D(A)$ by setting

$$(A\oplus D(A))\times(A\oplus D(A))\longrightarrow A\oplus D(A),\quad ((a,f),(b,g))\mapsto(ab,ag+fb+(f\otimes g)\overline{\omega_\lambda})$$

for $a,b\in A$ and $f,g\in D(A)$. Denote by $A\ltimes_\lambda D(A)$ the abelian group $A\oplus D(A)$ with the above-defined multiplication. By Lemma 3.3(1), $A\ltimes_\lambda D(A)$ is an algebra with an algebra isomorphism

$$\overline{\gamma}:=\begin{pmatrix}\mathrm{Id}_A & 0\\ 0 & \gamma\end{pmatrix}:\ R(A,e,\lambda)\xrightarrow{\simeq}A\ltimes_\lambda D(A).$$

Compared with the trivial extension $A\ltimes D(A)$, the following result, suggested by Kunio Yamagata, shows that $A\ltimes_\lambda D(A)$ is also a symmetric algebra for any λ.

Proposition 5.2. *If (A,e) is a gendo-symmetric algebra, then $R(A,e,\lambda)$ is symmetric for $\lambda\in Z(\Lambda)$.*

Proof. Let $R:=R(A,e,\lambda)$. Applying $\chi:R\to k$, we define a bilinear form $\widetilde{\chi}:R\times R\to k$, $(r_1,r_2)\mapsto(r_1*r_2)\chi$ for $r_1,r_2\in R$. By Lemma 5.1(1), $\widetilde{\chi}$ is symmetric. To show that R is a symmetric algebra, it suffices to show that $\widetilde{\chi}$ is nondegenerate.

Let $T:=A\ltimes_\lambda D(A)$ and $\psi:=\overline{\gamma}^{-1}\chi:T\to k$. Since $\overline{\gamma}:R\to T$ is an algebra isomorphism, ψ induces a symmetric bilinear form $\widetilde{\psi}:T\times T\to k$, $(t_1,t_1)\in T\times T\mapsto(t_1t_2)\psi$. Clearly, $\widetilde{\chi}$ is nondegenerate if and only if so is $\widetilde{\psi}$. Further, by Lemma 5.1(3), ψ is given by $(a,f)\mapsto(1)f$ for $a\in A$ and $f\in D(A)$. This implies that $((a,f),(b,g))\widetilde{\psi}=(a)g+(b)f+(1)(f\otimes g)\overline{\omega_\lambda}$ for $b\in A$ and $g\in D(A)$. We now show that $\widetilde{\psi}$ is nondegenerate.

Let $(a,f)\neq 0$. Then $a\neq 0$ or $f\neq 0$. If $f\neq 0$, then there is an element $b\in A$ such that $(b)f\neq 0$, and therefore $((a,f),(b,0))\widetilde{\psi}=(b)f\neq 0$. If $f=0$ and $a\neq 0$, then the canonical isomorphism $A\simeq DD(A)$ implies that there is an element $g\in D(A)$ such that $(a)g\neq 0$. In this case, $((a,0),(0,g))\widetilde{\psi}=(a)g\neq 0$. Thus, $\widetilde{\psi}$ is nondegenerate. □

Compared with $R(A)$, the algebra $S(A,e)$ depends on the choice of e, that is, if $f=f^2\in A$ such that (A,f) is gendo-symmetric, then $S(A,e)$ and $S(A,f)$ do not have to be isomorphic in general. The following result collects basic homological properties of $S(A,e)$.

Proposition 5.3. *Let $S:=S(A,e)$ and $B_0:=(1-e)A(1-e)$. Then:*

(1) *S is a symmetric algebra.*
(2) *B_0 can be regarded as an S-module via the surjective homomorphism $S\to S/S\overline{e}S\simeq B_0$ and contains no nonzero projective S-modules as direct summands.*
(3) *If $\mathrm{add}({}_AAe)\cap\mathrm{add}({}_AA(1-e))=0$, then $\#(S)=\#(A)$. Moreover, if B_0 is indecomposable as an algebra, then so is S.*

Proof. (1) Let $R:=R(A)$, $\bar{e}:=e\otimes e\in R$ and $e_0:=(1-e)+\overline{e}\in R$. Since R is symmetric by Proposition 5.2(1) and $S=e_0Re_0$ by Lemma 3.4(1), S is symmetric.

(2) Since π_1 induces a surjective algebra homomorphism $\pi_1':S\to B_0$ such that $S/S\overline{e}S\simeq B_0$ (see Lemma 3.4 for notation), B_0 can be regarded as an S-module. Assume that the S-module B_0

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contains an indecomposable projective direct summand X. Then there is a primitive idempotent element $f \in A$ such that $1-e = f + f'$ with f and f' orthogonal idempotent elements in A, and $X \simeq Sf$ as S-modules. Clearly, $S\bar{e}Sf = 0$, $(f)\pi_2' = f$, $(1-e)\pi_2' = 1-e$ and $(S\bar{e}Sf)\pi_2' = AeAf$. Consequently, $\mathrm{Hom}_A(Ae, Af) \simeq eAf = 0$, and therefore $\mathrm{Hom}_A(Af, Ae) \simeq D\,\mathrm{Hom}_A(Ae, Af) = 0$. By Lemma 2.2(2), Af can be embedded into $(Ae)^n$ for some $n \geq 1$. This implies $Af = 0$, a contradiction.

(3) Since $\bar{e}S\bar{e} \simeq eAe$ by Lemma 3.5(2), it follows from (2) that $\#S(A) = \#(eAe) + \#(B_0)$. Due to $\mathrm{add}(Ae) \cap \mathrm{add}(A(1-e)) = 0$, we have $\#(A) = \#(eAe) + \#(B_0)$ and $\#S(A) = \#(A)$. The second assertion in (3) follows from Proposition 3.7(2). □

5.2 Mirror-reflective algebras and Auslander–Gorenstein algebras

In this subsection we construct new gendo-symmetric algebras from minimal Auslander–Gorenstein algebras. Consequently, there is a series of algebras such that in each step the dominant dimensions increase at least by 2. This is based on study of mirror-reflective algebras. Finally, we will give a proof of Theorem 1.4, which will be partially used to prove Theorem 1.2(2).

By Lemma 3.4, we have an algebra automorphism $\phi : R(A) \to R(A)$ and two surjective algebra homomorphisms $\pi_1, \pi_2 : R(A) \to A$ such that $\pi_2 = \phi\pi_1$. Through π_1 we regard A-modules as $R(A)$-modules in the following discussion. Thus, A-mod is a Serre subcategory of $R(A)$-mod, that is, it is closed under direct summands, submodules, quotients and extensions in $R(A)$-mod. Let

$$\phi_* : R(A)\text{-mod} \longrightarrow R(A)\text{-mod} \quad \text{and} \quad (\pi_2)_* : A\text{-mod} \longrightarrow R(A)\text{-mod}$$

be the restriction functors induced by ϕ and π_2, respectively. Then ϕ_* is an auto-equivalence and $\phi_*(X) = (\pi_2)_*(X)$ for $X \in A$-mod.

Lemma 5.4. *Suppose that Λ is a symmetric algebra and N is a basic Λ-module without nonzero projective direct summands. Let $A := \mathrm{End}_\Lambda(\Lambda \oplus N)$, e be an idempotent element of A corresponding to the direct summand Λ of $\Lambda \oplus N$, and $R := R(A, e)$. If ${}_\Lambda N$ is m-rigid for a natural number m, then the following statements hold.*

(1) *The R-module $A(1-e)$ is $(m+2)$-rigid and there are isomorphisms of R-modules:*

$$\Omega_R^{m+3}(A(1-e)) \simeq \Omega_R^{m+2}(\phi_*(Ae \otimes_\Lambda N)) \simeq \phi_*(\mathrm{Hom}_\Lambda(eA, \Omega_\Lambda^{m+2}(N))).$$

(2) *If $\Omega_\Lambda^{m+2}(N) \simeq N$, then $\Omega_R^{m+3}(A(1-e)) \simeq \phi_*(A(1-e))$ and the R-module $A(1-e)$ is $(2m+4)$-rigid. In this case, $\Omega_R^{2m+6}(A(1-e)) \simeq A(1-e)$.*

Proof. (1) By the proof of Proposition 4.12(2), π_2 induces an isomorphism $\mathrm{Tor}_i^{\bar{e}R\bar{e}}(R\bar{e}, \bar{e}R) \simeq \mathrm{Tor}_i^\Lambda(Ae, eA)$ for all $i \geq 1$. Since Λ is symmetric and $D(Ae) \simeq eA$ by Lemma 2.2(2), we have

$$D\mathrm{Tor}_i^\Lambda(Ae, eA) \simeq \mathrm{Ext}_\Lambda^i(eA, D(Ae)) \simeq \mathrm{Ext}_\Lambda^i(eA, eA) = \mathrm{Ext}_\Lambda^i(\Lambda \oplus N, \Lambda \oplus N) \simeq \mathrm{Ext}_\Lambda^i(N, N).$$

As ${}_\Lambda N$ is m-rigid, we have $\mathrm{Tor}_i^{\bar{e}R\bar{e}}(R\bar{e}, \bar{e}R) = 0$ for $1 \leq i \leq m$. By Proposition 4.12(1), $I := R\bar{e}R$ is 2-idempotent. Therefore, I is $(m+2)$-idempotent by Lemma 4.3(1). Further, it follows from Lemma 4.3(2) that ${}_R R/I$ is $(m+2)$-rigid. Since $R/I \simeq A$ as R-modules, ${}_R A$ is $(m+2)$-rigid. Note that ${}_R A \simeq R(e - \bar{e}) \oplus A(1-e)$ by Lemma 3.5(2). As R is symmetric by Proposition 5.2, we see that $R(e - \bar{e})$ is projective-injective. Consequently, ${}_R A(1-e)$ is $(m+2)$-rigid.

The proof of Proposition 4.12(1) implies $I \simeq R\bar{e} \otimes_{\bar{e}R\bar{e}} \bar{e}R$ as R-R-bimodules. By Lemma 3.5(2), π_2 restricts to an algebra isomorphism $\bar{e}R\bar{e} \to \Lambda$ and also an isomorphism $R\bar{e} \to Ae$ of abelian groups. Via the algebra isomorphism, we can regard $R\bar{e}$ as an R-Λ-bimodule. Then $R\bar{e} \simeq (\pi_2)_*(Ae) = \phi_*(Ae)$ as R-Λ-bimodules. This gives a natural isomorphism

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$R\overline{e}\otimes_\Lambda - \xrightarrow{\simeq} \phi_*(Ae)\otimes_\Lambda -$ of functors from Λ-proj to R-proj. Since N has no nonzero projective direct summands, $\mathrm{add}({}_AAe)\cap\mathrm{add}({}_AA(1-e))=0$. From $A\otimes_R R\overline{e}\simeq Ae\simeq R\overline{e}$ and $A\otimes_R R(1-e)\simeq A(1-e)$, we obtain $\mathrm{add}(R\overline{e})\cap\mathrm{add}(R(1-e))=0$. Since $I(1-e)$ is isomorphic to $R\overline{e}\otimes_{\overline{e}R\overline{e}}\overline{e}R(1-e)$, which is a quotient module of $(R\overline{e})^n$ for some n, we deduce that $I(1-e)$ does not contain nonzero direct summands in $\mathrm{add}(R(1-e))$. Thus, the surjection ${}_RR(1-e)\to A(1-e)$ induced by π_1 is a projective cover of the R-module $A(1-e)$, and therefore $\Omega_R(A(1-e))=I(1-e)$. Since π_2 induces an isomorphism $\overline{e}R\to eA$ and sends $1-e$ to $1-e$ by Lemma 3.5(2), we have $\overline{e}R(1-e)\simeq eA(1-e)$ and

$$\Omega_R(A(1-e))\simeq R\overline{e}\otimes_{\overline{e}R\overline{e}} eA(1-e)\simeq R\overline{e}\otimes_\Lambda eA(1-e)\simeq \phi_*(Ae)\otimes_\Lambda N=\phi_*(Ae\otimes_\Lambda N).$$

Let $\cdots\to Q_{m+1}\xrightarrow{\partial} Q_m\to\cdots\to Q_1\to Q_0\to N\to 0$ be a minimal projective resolution of ${}_\Lambda N$. Then it follows from $eA(1-e)=N$ and $\mathrm{Tor}_i^\Lambda(Ae,N)\simeq D\,\mathrm{Ext}^i_\Lambda(N,N)=0$ for $1\le i\le m$ that the sequence

$$Ae\otimes_\Lambda Q_{m+1}\xrightarrow{Ae\otimes\partial} Ae\otimes_\Lambda Q_m\longrightarrow\cdots\longrightarrow Ae\otimes_\Lambda Q_1\longrightarrow Ae\otimes_\Lambda Q_0\longrightarrow Ae\otimes_\Lambda N\longrightarrow 0$$

is exact. As the composition of ${}_AAe\otimes_\Lambda -$ with $(e\cdot)$ is isomorphic to the identity functor of Λ-mod, we have $\Omega_A^{m+2}(Ae\otimes_\Lambda N)\simeq\mathrm{Ker}(Ae\otimes\partial)$. Note that $Ae\otimes_\Lambda -\simeq \mathrm{Hom}_\Lambda(eA,-):\Lambda\text{-proj}\xrightarrow{\simeq}\mathrm{add}({}_AAe)$ since $Ae=\mathrm{Hom}_\Lambda(\Lambda\oplus N,\Lambda)$. This shows $\mathrm{Ker}(Ae\otimes\partial)\simeq \mathrm{Hom}_\Lambda(eA,\mathrm{Ker}(\partial))=\mathrm{Hom}_\Lambda(eA,\Omega_\Lambda^{m+2}(N))$, and therefore

$$\Omega_R^{m+3}(A(1-e))\simeq\Omega_R^{m+2}(\phi_*(Ae\otimes_\Lambda N))\simeq\phi_*(\Omega_R^{m+2}(Ae\otimes_\Lambda N))\simeq\phi_*(\mathrm{Hom}_\Lambda(eA,\Omega_\Lambda^{m+2}(N))).$$

(2) Let $X:=A(1-e)$. Suppose $\Omega_\Lambda^{m+2}(N)\simeq N$. Then $\Omega_R^{m+3}(X)\simeq\phi_*(\mathrm{Hom}_\Lambda(eA,eX))$. Since the functor $(e\cdot):A\text{-mod}\to\Lambda\text{-mod}$ induces an algebra isomorphism $\mathrm{End}_A(A)\simeq\mathrm{End}_\Lambda(eA)$, we have $X\simeq\mathrm{Hom}_A(A,X)\simeq\mathrm{Hom}_\Lambda(eA,eX)$. It follows that $\Omega_R^{m+3}(X)\simeq\phi_*(X)$. Since ϕ is an algebra isomorphism with $\phi^2=\mathrm{Id}_R$ by Lemma 3.4(3) and since Ω_R commutes with ϕ_*, we obtain $\Omega_R^{2m+6}(X)\simeq X$. It remains to show that ${}_RX$ is $(2m+4)$-rigid.

Since R is symmetric, the stable module category $R\text{-}\underline{\mathrm{mod}}$ of R is a triangulated category with the shift functor $[1]=\Omega_R^-:R\text{-}\underline{\mathrm{mod}}\to R\text{-}\underline{\mathrm{mod}}$, where Ω_R^- is the cosyzygy functor on $R\text{-}\underline{\mathrm{mod}}$. Clearly, $\mathrm{Ext}^n_R(X_1,X_2)\simeq\underline{\mathrm{Hom}}_R(X_1,X_2[n])$ for all $n\ge1$ and $X_1,X_2\in R$-mod, where $\underline{\mathrm{Hom}}_R(X,Y)$ denotes the morphism set from X to Y in $R\text{-}\underline{\mathrm{mod}}$. Since the Auslander–Reiten translation on R-mod coincides with Ω_R^2, it follows from the Auslander–Reiten formula that there is a natural isomorphism $D\,\underline{\mathrm{Hom}}_R(X_1,X_2)\simeq\underline{\mathrm{Hom}}_R(X_2,X_1[-1])$. Consequently, for $i\in\mathbb{N}$, there are isomorphisms

$$\begin{aligned}\mathrm{Ext}_R^{m+3+i}(X,X)&\simeq\underline{\mathrm{Hom}}_R(\Omega_R^{m+3}(X),X[i])\simeq\underline{\mathrm{Hom}}_R(\phi_*(X),X[i])\\&\simeq D\,\underline{\mathrm{Hom}}_R(X[i],\phi_*(X)[-1]).\end{aligned}$$

By Lemma 3.4(3), ϕ is an algebra isomorphism with $\phi^2=\mathrm{Id}_R$. Thus,

$$\begin{aligned}\underline{\mathrm{Hom}}_R(X[i],\phi_*(X)[-1])&\simeq\underline{\mathrm{Hom}}_R(\phi_*(X)[i],X[-1])\simeq\underline{\mathrm{Hom}}_R(\Omega_R^{m+3}(X),X[-1-i])\\&\simeq\mathrm{Ext}_R^{m+2-i}(X,X)\end{aligned}$$

for $0\le i\le m+1$. This implies $\mathrm{Ext}_R^{m+3+i}(X,X)\simeq D\,\mathrm{Ext}_R^{m+2-i}(X,X)$ for $0\le i\le m+1$. Since X is $(m+2)$-rigid by (1), it is actually $(2m+4)$-rigid. □

PROPOSITION 5.5. *Suppose that Λ is a symmetric algebra and N is a basic Λ-module without nonzero projective direct summands. Let $A:=\mathrm{End}_\Lambda(\Lambda\oplus N)$, e be an idempotent element of A corresponding to the direct summand Λ of $\Lambda\oplus N$, and $R:=R(A,e)$.*

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(1) *If ${}_\Lambda\Lambda \oplus N$ is m-rigid, then ${}_RR \oplus A(1-e)$ is $(m+2)$-rigid.*
(2) *If ${}_\Lambda\Lambda \oplus N$ is m-ortho-symmetric, then ${}_RR \oplus A(1-e)$ is $(2m+4)$-ortho-symmetric.*
(3) *If ${}_\Lambda\Lambda \oplus N$ is maximal m-orthogonal, then ${}_RR \oplus A(1-e)$ is maximal $(2m+4)$-orthogonal.*

Proof. (1) Since R is a symmetric algebra by Proposition 5.2, (1) follows from Lemma 5.4(1).

(2) By assumption, ${}_\Lambda N$ is basic and contains no nonzero projective direct summands. This implies that ${}_AA(1-e)$ is basic and contains no nonzero projective-injective direct summands. We claim that ${}_RA(1-e)$ contains no nonzero projective direct summands. In fact, by the proof of Lemma 5.4(1), ${}_RR(1-e)$ is a projective cover of ${}_RA(1-e)$. If ${}_RA(1-e)$ contains an indecomposable projective direct summand Y, then Y is a direct summand of $R(1-e)$. Since R is symmetric, ${}_RY$ must be projective-injective. However, since A-mod $\subseteq R$-mod is a Serre subcategory, ${}_AY$ is also a nonzero projective-injective direct summand of ${}_AA(1-e)$. This is a contradiction and shows that the above claim holds. Then (2) follows from Lemmas 5.4 and 2.3.

(3) Maximal orthogonal modules over an algebra B are exactly ortho-symmetric B-modules such that their endomorphism algebras have finite global dimension. Let $A_1 := \mathrm{End}_R(R \oplus A(1-e))$. By (2), to show (3), it suffices to show that $\mathrm{gldim}(A_1) < \infty$ if $\mathrm{gldim}(A) < \infty$

Let $B_1 := \mathrm{End}_R(R \oplus A)$. Since ${}_RA \simeq R(e-\bar{e}) \oplus A(1-e)$ by the proof of Lemma 5.4(1), we know that A_1 and B_1 are Morita equivalent, and therefore $\mathrm{gldim}(A_1) = \mathrm{gldim}(B_1)$. Since the right A-module eA_A is faithful, it follows from Proposition 4.11(3) that if $\mathrm{gldim}(A) < \infty$ then $\mathrm{gldim}(B_1) = \mathrm{gldim}(B_1^{\mathrm{op}}) < \infty$. Hence, $\mathrm{gldim}(A_1) < \infty$. □

Proof of Theorem 1.4. The statement (1) follows from Proposition 5.2. Let $R := R(A)$ and $S := S(A,e)$. Then R and S are symmetric by (1) and Proposition 5.3(1). Let $A_2 := \mathcal{A}(A,e)$ and $B_2 := \mathcal{B}(A,e)$. Then A_2 and B_2 are gendo-symmetric.

Next, we show that (2) and (3) hold for A_2. In fact, since A is gendo-symmetric, we can identify A with $\mathrm{End}_\Lambda(\Lambda \oplus X)$, where $\Lambda := eAe$ is symmetric and $X = eA(1-e)$. As global, dominant and injective dimensions are invariant under Morita equivalences, the classes of minimal Auslander–Gorenstein algebras and of higher Auslander algebras are closed under Morita equivalences. Moreover, for a self-injective algebra Γ and $M \in \Gamma$-mod, it follows from [Mul68, Lemma 3] that $\mathrm{domdim}(\mathrm{End}_\Gamma(\Gamma \oplus M))$ equals the maximal natural number $n \geq 2$ or ∞ such that M is $(n-2)$-rigid. So, for a basic module X that has no nonzero projective direct summands, the inequality $\mathrm{domdim}(A_2) \geq \mathrm{domdim}(A) + 2$ and the statement (3) follow immediately from Proposition 5.5. Further, for an arbitrary module X, the consideration can be reduced by a series of Morita equivalences, as shown below.

We take a direct summand N of X such that N is basic, has no nonzero projective direct summands and satisfies $\mathrm{add}(\Lambda \oplus N) = \mathrm{add}(\Lambda \oplus X)$. Let $B := \mathrm{End}_\Lambda(\Lambda \oplus N)$ and $f^2 = f \in A$ correspond to the direct summand $\Lambda \oplus N$ of $\Lambda \oplus X$. Then ${}_AAf$ is a progenerator (that is, a projective generator), and therefore $B = fAf$ is Morita equivalent to A. Since $ef = e = fe$, we have $R(B) = fAf \oplus fAe \otimes_\Lambda eAf = fRf$. Due to $R \otimes_A Af \simeq Rf$, the module ${}_RRf$ is a progenerator. Thus, R and $R(B)$ are Morita equivalent. Now let $H := \mathrm{End}_{R(B)}(R(B) \oplus B(f-e))$. If A is n-minimal Auslander–Gorenstein (respectively, n-Auslander), then so is B, and therefore, so is H by the case proved above. Next, we shall show that A_2 and H are Morita equivalent. Actually, the restriction of π_1 to A is the identity map of A. This implies $A \otimes_R Rf = Af$ as R-modules, and therefore $\mathrm{add}({}_RA) = \mathrm{add}({}_RAf)$. Let $A_2' := \mathrm{End}_R(Rf \oplus A(1-e)f) = \mathrm{End}_R(Rf \oplus A(f-e))$. Then A_2 and A_2' are Morita equivalent. Since the functor $(f\cdot) : R\text{-mod} \to R(B)\text{-mod}$ is an equivalence and $f(Rf \oplus A(f-e)) = R(B) \oplus B(f-e)$, there is an algebra isomorphism $A_2' \simeq H$. Hence, A_2 and H are Morita equivalent. Thus, (2) and (3) hold true for A_2.

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It remains to show $\mathrm{domdim}(B_2) \geq \mathrm{domdim}(A)+2$. Up to Morita equivalence, we assume $A = \mathrm{End}_\Lambda(\Lambda \oplus N)$. If ${}_\Lambda\Lambda \oplus N$ is m-rigid for some $m \in \mathbb{N}$, then it follows from the first part of the proof of Lemma 5.4(1) that I is an $(m+2)$-idempotent ideal of R. Let $e_0 := (1-e)+\overline{e} \in R$. By Lemma 3.4, we have $\overline{e}e_0 = \overline{e} = e_0\overline{e}$, $I := R\overline{e}R = S\overline{e}S$ and $S/I \simeq (1-e)A(1-e)$ as algebras. Thanks to Corollary 4.4(1), I is an $(m+2)$-idempotent ideal of S. Further, by Lemma 4.3(2), ${}_SS/I$ is $(m+2)$-rigid, and therefore ${}_SS \oplus S/I$ is $(m+2)$-rigid since S is symmetric by Proposition 5.3(1). Thus, $\mathrm{domdim}(B_2) \geq \mathrm{domdim}(A)+2$, due to [Mul68, Lemma 3]. □

5.3 Recollements of mirror-reflective algebras and Tachikawa's second conjecture

In this subsection we study the iterated process of constructing (reduced) mirror-reflective algebras from gendo-symmetric algebras and prove Theorems 1.1 and 1.2.

Throughout this subsection, let (A,e) be a gendo-symmetric algebra over a field. For $n \geq 1$, we inductively define

$$A_1 = B_1 := A, \quad R_1 := R(A_1, e_1), \quad S_1 := S(A_1, f_1),$$
$$A_{n+1} := \mathrm{End}_{R_n}(R_n \oplus A_n(1_{A_n} - e_n)), \quad R_{n+1} := R(A_{n+1}, e_{n+1}),$$
$$B_{n+1} := \mathrm{End}_{S_n}(S_n \oplus (1_{B_n} - f_n)B_n(1_{B_n} - f_n)), \quad S_{n+1} := S(B_{n+1}, f_{n+1}),$$

where $e_1 = f_1 := e$, and for $n \geq 1$, $e_{n+1} \in A_{n+1}$ is the idempotent element corresponding to the direct summand R_n of the R_n-module $R_n \oplus A_n(1_{A_n} - e_n)$, and $f_{n+1} \in B_{n+1}$ is the idempotent element corresponding to the direct summand S_n of the S_n-module $S_n \oplus (1_{B_n} - f_n)B_n(1_{B_n} - f_n)$. In other words,

$$A_{n+1} = \mathcal{A}(A_n, e_n), \quad B_{n+1} = \mathcal{B}(B_n, f_n) \quad \text{for } n \geq 1$$

(see Section 1.3 for notation). For convenience, we set

$$R_0 = S_0 := eAe \quad \text{and} \quad B_0 := (1-e)A(1-e).$$

Definition 5.6. For $n \geq 1$, the algebras R_n, S_n, A_n and B_n are called the nth mirror-reflective, reduced mirror-reflective, gendo-symmetric and reduced gendo-symmetric algebras of (A,e), respectively.

By Propositions 5.2 and 5.3(1), the algebras R_n and S_n are symmetric. Thus, A_n and B_n are gendo-symmetric. They are characterized in terms of Morita context algebras in § 4, just before Lemma 4.6. Moreover, it follows from Theorem 1.4(2) that $\mathrm{domdim}(A_{n+1}) \geq \mathrm{domdim}(A_n)+2$ and $\mathrm{domdim}(B_{n+1}) \geq \mathrm{domdim}(B_n)+2$. Thus, $\min\{\mathrm{domdim}(A_n), \mathrm{domdim}(B_n)\} \geq \mathrm{domdim}(A) + 2(n-1) \geq 2n$.

In the next result we describe the relation between the families A_n and B_n on the one hand and the families R_n and S_n on the other hand by derived and stable equivalences of Morita type. For the definitions and constructions of derived and stable equivalences of Morita type, we refer to the survey article [Xi18].

Lemma 5.7.

(1) *Let $I_n := R_n\overline{e}_nR_n$ and $J_n := R_n(e_n - \overline{e}_n)R_n$ with $\overline{e}_n = e_n \otimes e_n \in R_n$ for $n \geq 1$. Then A_{n+1} is derived equivalent and stably equivalent of Morita type to the Morita context algebra $M_l(R_n, I_n, J_n)$.*

(2) *Let $K_n := S_n\overline{f}_nS_n$ and $L_n := S_n \cap (R(B_n)(f_n - \overline{f}_n)R(B_n))$ for $n \geq 1$. Then B_{n+1} is derived equivalent and stably equivalent of Morita type to the Morita context algebra $M_l(S_n, K_n, L_n)$.*

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Proof. (1) There is a surjective algebra homomorphism $\pi_{1,n}: R_n \to A_n$ with $\mathrm{Ker}(\pi_{1,n}) = I_n$ which induces an isomorphism $R_n(e_n - \overline{e}_n) \simeq A_n e_n$ of R_n-modules. Thus, $I_n \simeq \Omega_{R_n}(A_n) \oplus Q_n$ with Q_n a projective R_n-module, and $A_n e_n$ is a projective R_n-module. Hence, A_{n+1} is Morita equivalent to $A'_{n+1} := \mathrm{End}_{R_n}(R_n \oplus A_n)$. Let $C_{n+1} := \mathrm{End}_{R_n}(R_n \oplus I_n)$. By [HX13, Corollary 1.2], for any self-injective algebra Λ and $M \in \Lambda$-mod, the algebras $\mathrm{End}_\Lambda(\Lambda \oplus M)$ and $\mathrm{End}_\Lambda(\Lambda \oplus \Omega_\Lambda(M))$ are almost ν-stable derived equivalent. Since R_n is symmetric, it follows that A'_{n+1} and C_{n+1} are almost ν-stable derived equivalent. By [HX10, Theorem 1.1], each almost ν-stable derived equivalence between finite-dimensional algebras over a field gives rise to a stable equivalence of Morita type. Consequently, A_{n+1} and C_{n+1} are both derived equivalent and stably equivalent of Morita type. It remains to show $C_{n+1} \simeq M_l(R_n, I_n, J_n)$ as algebras.

In fact, since $I_n^2 = I_n$, the inclusion $\lambda_n : I_n \hookrightarrow R_n$ induces $\mathrm{End}_{R_n}(I_n) \simeq \mathrm{Hom}_{R_n}(I_n, R_n)$. As R_n is symmetric and $J_n = \mathrm{Ann}_{R_n^{\mathrm{op}}}(I_n)$ by Lemma 3.5(1), we get $R_n/J_n \simeq \mathrm{End}_{R_n}(I_n)$ as algebras via the restriction of λ_n. This yields a series of isomorphisms

$$\begin{aligned} C_{n+1} &\simeq \begin{pmatrix} R_n & I_n \\ \mathrm{Hom}_{R_n}(I_n, R_n) & \mathrm{End}_{R_n}(I_n) \end{pmatrix} \simeq \begin{pmatrix} R_n & I_n \\ \mathrm{End}_{R_n}(I_n) & \mathrm{End}_{R_n}(I_n) \end{pmatrix} \\ &\simeq \begin{pmatrix} R_n & I_n \\ R_n/J_n & R_n/J_n \end{pmatrix}, \end{aligned}$$

of which the composition is an isomorphism from C_{n+1} to $M_l(R_n, I_n, J_n)$ of algebras. This shows (1).

(2) By Lemma 3.4(4), $K_n = R(B_n)\overline{f_n}R(B_n)$ and $S_n/K_n \simeq (1_{B_n} - f_n)B_n(1_{B_n} - f_n)$. By the proof of Proposition 4.11(1), $\mathrm{Ann}_{S_n^{\mathrm{op}}}(K_n) = L_n$. Similarly, since S_n is symmetric, we can show that B_{n+1} and $\mathrm{End}_{S_n}(S_n \oplus K_n)$ are both derived equivalent and stably equivalent of Morita type, and that $\mathrm{End}_{S_n}(S_n \oplus K_n)$ is isomorphic to $M_l(S_n, K_n, L_n)$ as algebras. □

Remark 5.8. By the proof of Lemma 5.7, B_{n+1} and $\mathrm{End}_{S_n}(S_n \oplus S_n/K_n)$ are isomorphic, while A_{n+1} and $\mathrm{End}_{R_n}(R_n \oplus A_n)$ are Morita equivalent. It follows from Proposition 4.11(1) that there are recollements of derived module categories $(\mathscr{D}(A_n), \mathscr{D}(A_{n+1}), \mathscr{D}(A_n))$ and $(\mathscr{D}(B_n), \mathscr{D}(B_{n+1}), \mathscr{D}(B_0))$, which are induced by finitely generated and right-projective idempotent ideals of A_{n+1} and B_{n+1}, respectively.

Proof of Theorem 1.2. We retain all the notation introduced in Lemma 5.7 and its proof.

(1) By Lemma 4.6, there is a recollement $(\mathscr{D}(R_n/I_n), \mathscr{D}(M_l(R_n, I_n, J_n)), \mathscr{D}(R_n/J_n))$ induced by a finitely generated, left-projective idempotent ideal of $M_l(R_n, I_n, J_n)$. Thus, the recollement restricts to a recollement of bounded-above derived categories. Since $R_n/I_n \simeq A_n \simeq R_n/J_n$ as algebras and since A_{n+1} and $M_l(R_n, I_n, J_n)$ are derived equivalent by Lemma 5.7(1), there is a recollement $(\mathscr{D}^-(A_n), \mathscr{D}^-(A_{n+1}), \mathscr{D}^-(A_n))$.

Similarly, we can apply Lemmas 5.7(2) and 4.6 to show the existence of the recollement $(\mathscr{D}^-(S_n/K_n), \mathscr{D}^-(B_{n+1}), \mathscr{D}^-(S_n/L_n))$. Note that there are isomorphisms of algebras $S_n/L_n \simeq B_n$ and

$$\begin{aligned} S_n/K_n &\simeq (1_{B_n} - f_n)B_n(1_{B_n} - f_n) \simeq (1_{B_{n-1}} - f_{n-1})B_{n-1}(1_{B_{n-1}} - f_{n-1}) \simeq \cdots \\ &\simeq (1 - f_1)B_1(1 - f_1) = B_0. \end{aligned}$$

This implies the existence of the second recollement in (1).

(2) Note that R_0 is symmetric, $A \simeq \mathrm{End}_{R_0}(eA)$ and $D(eA) \simeq Ae$. Suppose $\mathrm{domdim}(A) = \infty$. By [Mul68, Lemma 3], $\mathrm{Ext}^i_{R_0}(eA, eA) = 0$ for all $i \geq 1$. It follows from $\mathrm{Ext}^i_{R_0}(eA, eA) \simeq \mathrm{Ext}^i_{R_0}(eA, D(Ae)) \simeq D\,\mathrm{Tor}_i^{R_0}(Ae, eA)$ that $\mathrm{Tor}_i^{R_0}(Ae, eA) = 0$ for all $i \geq 1$. By Proposition 4.12(3), the recollements in (2) exist for $n = 1$. If $n \geq 1$, then R_n and S_n are

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symmetric algebras, while A_n and B_n are gendo-symmetric algebras. Moreover, $\mathrm{domdim}(A_n) = \infty = \mathrm{domdim}(B_n)$ by Theorem 1.4(2) and $(1_{B_n} - f_n)B_n(1_{B_n} - f_n) \simeq B_0$ as algebras. Thus, by induction we can show the existence of recollements for $n \geq 1$. $\square$

Theorem 1.2 can be applied to investigate homological dimensions and higher algebraic K-groups. As usual, for a ring R and $m \in \mathbb{N}$, we denote by $K_m(R)$ the mth algebraic K-group of R in the sense of Quillen, and by $nK_m(R)$ the direct sum of n copies of $K_m(R)$ for $n \geq 0$. If R is an Artin algebra, then $K_0(R)$ is a finitely generated free abelian group of rank $\#(R)$.

LEMMA 5.9. *Let R be a ring with $f^2 = f \in R$ such that $I := RfR$ is a stratifying ideal in R. Suppose that one of the following conditions holds.*

(a) *Either ${}_RI$ or I_R is finitely generated and projective.*
(b) *There is a ring homomorphism $\lambda : R/I \to R$ such that the composition of λ with the canonical surjection $R \to R/I$ is an isomorphism. Then $K_n(R) \simeq K_n(fRf) \oplus K_n(R/I)$ for $n \in \mathbb{N}$.*

Proof. When (a) holds, the isomorphisms of algebraic K-groups in Lemma 5.9 follow from [CX16b, Corollary 1.3] or [CX12, Corollary 1.2].

Let $\pi : R \to R/I$ be the canonical surjection. Clearly, π is the universal localization of R at the map $0 \to Rf$. Since I is a stratifying ideal in R, π is a homological ring epimorphism (also called *stably flat* in [NR04]). By [NR04, Theorem 0.5] and [CX16b, Lemma 2.6], the tensor functors $Rf \otimes_{fRf} - : (fRf)\text{-proj} \to R\text{-proj}$ and $(R/I) \otimes_R - : R\text{-proj} \to (R/I)\text{-proj}$ induce a long exact sequence of algebraic K-groups of rings

$$\cdots \to K_{n+1}(R/I) \to K_n(fRf) \to K_n(R) \to K_n(R/I) \to \cdots \to K_0(fRf) \to K_0(R) \to K_0(R/I).$$

Suppose (b) holds. Then the composition of the functors $R \otimes_{R/I} - : (R/I)\text{-proj} \to R\text{-proj}$ with $(R/I) \otimes_R - : R\text{-proj} \to (R/I)\text{-proj}$ is an equivalence. This implies that the composition of the maps $K_n(R \otimes_{R/I} -) : K_n(R/I) \to K_n(R)$ with $K_n((R/I) \otimes_R -) : K_n(R) \to K_n(R/I)$ induced from tensor functors is an isomorphism. Consequently, $0 \to K_n(fRf) \to K_n(R) \to K_n(R/I) \to 0$ is split-exact. Thus, $K_n(R) \simeq K_n(fRf) \oplus K_n(R/I)$. $\square$

COROLLARY 5.10. *Let n be a positive integer. Then:*

(1) $\mathrm{findim}(A_n) \leq \mathrm{findim}(A_{n+1}) \leq 2\mathrm{findim}(A_n) + 2$ *and*

$$\mathrm{findim}(B_0) \leq \mathrm{findim}(B_{n+1}) \leq \mathrm{findim}(B_0) + \mathrm{findim}(B_n) + 2.$$

Thus,

$$\mathrm{findim}(A_{n+1}) \leq 2^n \,\mathrm{findim}(A) + 2^{n+1} - 2 \text{ and}$$
$$\mathrm{findim}(B_{n+1}) \leq \mathrm{findim}(A) + n(\mathrm{findim}(B_0) + 2).$$

Analogous inequalities hold true when finitistic dimension is replaced by global dimension.

(2) $K_*(A_{n+1}) \simeq 2^n\, K_*(A)$ *and* $K_*(B_{n+1}) \simeq nK_*(B_0) \oplus K_*(A)$ *for* $* \in \mathbb{N}$.
(3) *If* $\mathrm{domdim}(A) = \infty$, *then* $K_*(R_n) \simeq K_*(\Lambda) \oplus (2^n - 1)K_*(A)$ *and* $K_*(S_n) \simeq K_*(\Lambda) \oplus nK_*(B_0)$ *for any* $* \in \mathbb{N}$.

Proof. (1) By Lemma 5.7(1), A_{n+1} and $M_l(R_n, I_n, J_n)$ are stably equivalent of Morita type. Since global and finitistic dimensions are invariant under stably equivalences of Morita type, A_{n+1} and $M_l(R_n, I_n, J_n)$ have the same global and finitistic dimensions. Then the statements on A_{n+1} in (1) hold by (c) in the proof of Proposition 4.11(3) (or by applying [CX17, Corollary 3.12 and Theorem 3.17] to the recollement $(\mathscr{D}(R_n/I_n), \mathscr{D}(M_l(R_n, I_n, J_n)), \mathscr{D}(R_n/J_n))$ in Theorem 1.2(1)).

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In a similar way, we show the statements on B_n by the recollement $(\mathscr{D}(B_0), \mathscr{D}(B_{n+1}), \mathscr{D}(B_n))$ in Theorem 1.2(1).

(2) Derived equivalent algebras have isomorphic algebraic K-groups (see [DS04]). By Lemma 5.9(a) and the proof of Theorem 1.2(1), we have

$$K_*(A_{n+1}) \simeq K_*(M_l(R_n, I_n, J_n)) \simeq K_*(R_n/I_n) \oplus K_*(R_n/J_n) \simeq 2K_*(A_n)$$

and

$$K_*(B_{n+1}) \simeq K_*(M_l(S_n, K_n, L_n)) \simeq K_*(S_n/K_n) \oplus K_*(S_n/L_n) \simeq K_*(B_0) \oplus K_*(B_n).$$

Starting with $A_1 = A = B_1$, we can show the isomorphisms in (2) by induction.

(3) By Lemma 5.9(b) and Theorem 1.2(2), $K_*(R_n) \simeq K_*(R_{n-1}) \oplus K_*(A_n)$ and $K_*(S_n) \simeq K_*(S_{n-1}) \oplus K_*(B_0)$ for $n \geq 1$. Together with (2), we can show the isomorphisms in (3) by induction. □

Remark 5.11. Without assuming $\mathrm{domdim}(A) = \infty$, the isomorphisms in Corollary 5.10(3) still hold for $* = 0$. This follows from Corollary 5.10(2) and the fact that if R is a finite-dimensional algebra over a field and $f^2 = f \in R$, then $K_0(R) \simeq K_0(fRf) \oplus K_0(R/RfR)$. Thus, $\#(R_n) = \#(\Lambda) + (2^n - 1)\,\#(A)$ and $\#(S_n) = \#(\Lambda) + n\,\#(B_0)$.

As a consequence of Theorem 1.2, we obtain bounds for the stratified dimensions and ratios of iterated mirror-reflective algebras of gendo-symmetric algebras which are not symmetric. This provides a new approach to attacking Tachikawa's second conjecture.

Corollary 5.12. *Let n be a positive integer, and let (A, e) be a gendo-symmetric algebra with $\mathrm{domdim}(A) = \infty$. If A is not symmetric, then the following statements hold.*

(1) $2^n - 1 \leq \mathrm{stdim}(eAe) + (2^n - 1)(\mathrm{stdim}(A) + 1) \leq \mathrm{stdim}(R_n) \leq \#(eAe) + (2^n - 1)\,\#(A) - 1$ *and*

$$n \leq \mathrm{stdim}(eAe) + n(\mathrm{stdim}(B_0) + 1) \leq \mathrm{stdim}(S_n) \leq \#(eAe) + n\,\#(B_0) - 1.$$

(2) $(\mathrm{stdim}(A) + 1)/\#(A) \leq \underline{\lim}_{n\to\infty} \mathrm{sr}(R_n) \leq 1$ *and*

$$(\mathrm{stdim}(B_0) + 1)/\#(B_0) \leq \underline{\lim}_{n\to\infty} \mathrm{sr}(S_n) \leq 1.$$

In particular, if B_0 is local, then $\lim_{n\to\infty} \mathrm{sr}(S_n) = 1$, where $\underline{\lim}$ means the limit inferior.

Proof. (1) By Theorem 1.2(2) and Proposition 4.9(3), $\mathrm{stdim}(R_n) \geq \mathrm{stdim}(R_{n-1}) + \mathrm{stdim}(A_n) + 1$ and $\mathrm{stdim}(S_n) \geq \mathrm{stdim}(S_{n-1}) + \mathrm{stdim}(B_0) + 1$. Similarly, by Remark 5.8 and Proposition 4.9(3), we have $\mathrm{stdim}(A_{n+1}) \geq 2\,\mathrm{stdim}(A_n) + 1$, that is, $\mathrm{stdim}(A_{n+1}) + 1 \geq 2(\mathrm{stdim}(A_n) + 1)$. Moreover, by Proposition 4.9(1), $\mathrm{stdim}(R_n) \leq \#(R_n) - 1$ and $\mathrm{stdim}(S_n) \leq \#(S_n) - 1$. Combining these inequalities with Remark 5.11, we get (1) by induction.

(2) This follows from (1) and Remark 5.11. □

Finally, we state the promised connections between (TC2) and stratified dimensions of algebras in the following theorem, which is the combination of Theorems 1.1(1) and 1.3.

Theorem 5.13. *Let k be a field. The following statements are equivalent.*

(1) (TC2) *holds for all symmetric k-algebras.*
(2) *No indecomposable symmetric k-algebra has a stratifying ideal apart from itself and* 0.
(3) *The supremum of stratified ratios of all indecomposable symmetric k-algebras is less than* 1.

Proof. (1) ⇒ (2) This is shown in Introduction.

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(2) $\Rightarrow$ (3) An algebra S has no stratifying ideal apart from itself and 0 if and only if $\mathrm{stdim}(S) = 0$ if and only if $\mathrm{sr}(S) = 0$. Thus, (3) follows.

(3) $\Rightarrow$ (1) Suppose that (TC2) does not hold for an indecomposable symmetric algebra S over k. Then there exists an indecomposable, nonprojective self-orthogonal S-module M. Then $A := \mathrm{End}_S(S \oplus M)$ is a gendo-symmetric, but not a symmetric algebra. Let S_n be the nth reduced mirror symmetric algebra of A for $n \geq 1$. Then S_n is symmetric by Proposition 5.3(1). As M is indecomposable, $\mathrm{End}_S(M)$ is local. Since M contains no nonzero projective direct summands, S_1 is indecomposable by Proposition 5.3(3). Further, by the proof of Theorem 1.2(1), $\mathrm{End}_S(M) \simeq (1_{B_n} - f_n)B_n(1_{B_n} - f_n)$ as algebras for any $n \geq 1$. Combining this fact with Proposition 5.3(2), we show that S_n is indecomposable by induction. Since M is self-orthogonal, we see $\mathrm{domdim}(A) = \infty$ by [Mul68, Lemma 3]. It follows from Corollary 5.12(2) that $\lim_{n\to\infty} \mathrm{sr}(S_n) = 1$. Thus, the supremum in (3) must be 1, a contradiction to the assumption (3). This shows that (3) implies (1). $\square$

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Conflicts of interest

None.

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References

Aus71 M. Auslander, *Representation dimension of Artin algebras*, Queen Mary College Mathematics Notes (Queen Mary College, 1971).

APT92 M. Auslander, I. M. Platzeck and G. Todorov, *Homological theory of idempotent ideals*, Trans. Amer. Math. Soc. **332** (1992), 667–692.

BBD82 A. A. Beilinson, J. Bernstein and P. Deligne, *Faisceaux pervers*, Asterisque **100** (1982), 5–171.

CK16 H. X. Chen and S. Koenig, *Ortho-symmetric modules, Gorenstein algebras, and derived equivalences*, Int. Math. Res. Not. IMRN **2016** (2016), 6979–7037.

CX12 H. X. Chen and C. C. Xi, *Recollements of derived categories, II: Additive formulas of algebraic K-groups*, Preprint (2012), arXiv:1212.1879v2.

CX16a H. X. Chen and C. C. Xi, *Dominant dimensions, derived equivalences and tilting modules*, Israel J. Math. **215** (2016), 349–395.

CX16b H. X. Chen and C. C. Xi, *Higher algebraic K-theory of ring epimorphisms*, Algebr. Represent. Theory **19** (2016), 1347–1367.

https://doi.org/10.1112/S0010437X24007395 Published online by Cambridge University Press

CX17 H. X. Chen and C. C. Xi, *Recollements of derived categories, III: Finitistic dimensions*, J. Lond. Math. Soc. **95** (2017), 633–658.

CX22 H. X. Chen and C. C. Xi, *Homological theory of orthogonal modules*, Preprint (2022), 1–40, arXiv:2208.14712.

CPS88 E. Cline, B. Parshall and L. Scott, *Finite dimensional algebras and highest weight categories*, J. Reine Angew. Math. **391** (1988), 277–291.

CPS96 E. Cline, B. Parshall and L. Scott, *Stratifying endomorphism algebras*, Memoirs of the American Mathematical Society, vol. 124, no. 591 (American Mathematical Society, 1996).

dlPX06 J. A. de la Peña and C. C. Xi, *Hochschild cohomology of algebras with homological ideals*, Tsukuba J. Math. **30** (2006), 61–80.

DS04 D. Dugger and B. Shipley, *K-theory and derived equivalences*, Duke Math. J. **124** (2004), 587–617.

FK11 M. Fang and S. Koenig, *Endomorphism algebras of generators over symmetric algebras*, J. Algebra **332** (2011), 428–433.

FK16 M. Fang and S. Koenig, *Gendo-symmetric algebras, canonical comultiplication, bar cocomplexes and dominant dimension*, Trans. Amer. Math. Soc. **368** (2016), 5037–5055.

HX10 W. Hu and C. C. Xi, *Derived equivalences and stable equivalences of Morita type, I*, Nagoya Math. J. **200** (2010), 107–152.

HX13 W. Hu and C. C. Xi, *Derived equivalences for Φ-Auslander-Yoneda algebras*, Trans. Amer. Math. Soc. **365** (2013), 5681–5711.

Iya07 O. Iyama, *Auslander correspondence*, Adv. Math. **210** (2007), 51–82.

IS18 O. Iyama and Ø. Solberg, *Auslander-Gorenstein algebras and precluster tilting*, Adv. Math. **326** (2018), 200–240.

KY13 O. Kerner and K. Yamagata, *Morita algebras*, J. Algebra **382** (2013), 185–202.

LY12 Q. H. Liu and D. Yang, *Blocks of group algebras are derived simple*, Math. Z. **272** (2012), 913–920.

Mul68 B. Müller, *The classification of algebras by dominant dimension*, Canad. J. Math. **20** (1968), 398–409.

Nak58 T. Nakayama, *On algebras with complete homology*, Abh. Math. Semin. Univ. Hambg **22** (1958), 300–307.

NR04 A. Neeman and A. Ranicki, *Noncommutative localization in algebraic K-theory I*, Geom. Topol. **8** (2004), 1385–1425.

Tac73 H. Tachikawa, *Quasi-Frobenius rings and generalizations*, Lecture Notes in Mathematics, vol. 351 (Springer-Verlag, Berlin, 1973).

Xi18 C. C. Xi, *Derived equivalences of algebras*, Bull. Lond. Math. Soc. **50** (2018), 945–985.

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