108.02 Fermat-like equations for fractional parts

Fermat's Last Theorem [1] states that the equation $x^n + y^n = z^n$ ($n \in \mathbb{Z}, n \geqslant 3$) has no integral (or rational) solutions with $xyz \neq 0$. This was proved by Andrew Wiles in 1994. In this Note we investigate the equation

$$\{x^n\} + \{y^n\} = \{z^n\}, \qquad (n \in \mathbb{Z}, n \geqslant 2), \tag{1}$$

where $\{r\}$ denotes the fractional part of the real number r. We only consider solutions to (1) with $x, y, z \in \mathbb{Q}$ and $x, y, z \notin \mathbb{Z}$. When $n = 2$, it is not difficult to show that (1) has infinitely many solutions. Due to the identity

$$\left(\frac{1}{k+1}\right)^2 + \left(\frac{1}{k}\right)^2 + 1 = \left(\frac{1}{k(k+1)} + 1\right)^2$$

and the fact that if $u, v \in \mathbb{R}$ such that $u - v \in \mathbb{Z}$ then $\{u\} = \{v\}$, we have

$$\left\{\left(\frac{1}{k}\right)^2\right\} + \left\{\left(\frac{1}{k+1}\right)^2\right\} = \left\{\left(\frac{1}{k(k+1)} + 1\right)^2\right\}.$$

Therefore, (1) has infinitely many solutions if $n = 2$.

However, when $n \geqslant 3$, it is not easy to find solutions to (1). In the following, we treat the cases $n = 3$ and $n = 4$. When $n = 3$, we have

Theorem 1: The equation

$$\{x^3\} + \{y^3\} = \{z^3\} \tag{2}$$

has infinitely many solutions.

Proof: First, we have

$$3^3 + 4^3 + 5^3 = 6^3 \text{ and } 3^3 + 4^3 < 5^3.$$

Let $x = \frac{3}{5}(1 + 125k)$, $y = \frac{4}{5}(1 + 125k)$, $z = \frac{6}{5}(1 + 125k)$ with $k \in \mathbb{Z}^+$. Since

$$3^3(1 + 125k)^3 \equiv 3^3 \qquad (\text{mod } 5^3),$$

we have

$$\{x^3\} = \left\{\frac{3^3(1 + 125k)^3}{5^3}\right\} = \left\{\frac{3^3}{5^3}\right\} = \frac{3^3}{5^3}.$$

Similarly,

$$\{y^3\} = \left\{\frac{4^3}{5^3}\right\} = \frac{4^3}{5^3},$$

$$\{z^3\} = \left\{\frac{6^3}{5^3}\right\} = \left\{\frac{3^3 + 4^3 + 5^3}{5^3}\right\} = \left\{\frac{3^3 + 4^3}{5^3}\right\} = \frac{3^3 + 4^3}{5^3}.$$

Therefore,

$$\{x^3\} + \{y^3\} = \{z^3\}.$$

Since k can take infinitely many positive integer values, we have infinitely many solutions to (2).

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When $n = 4$, we have

Theorem 2: The equation
$$\{x^4\} + \{y^4\} = \{z^4\} \tag{3}$$
has infinitely many solutions.

Proof: Let $a = 2682440$, $b = 15365639$, $c = 18796760$, $d = 20615673$. Then
$$a^4 + b^4 + c^4 = d^4 \qquad \text{and} \qquad a^4 + b^4 < c^4.$$
Let $x = \frac{a}{c}(1 + kc^4)$, $y = \frac{b}{c}(1 + kc^4)$, $d = \frac{d}{c}(1 + kc^4)$ with $k \in \mathbb{Z}^+$. Since
$$a^4(1 + kc^4)^4 \equiv a^4 \pmod{c^4},$$
we have
$$\{x^4\} = \left\{\frac{a^4(1 + kc^4)^4}{c^4}\right\} = \left\{\frac{a^4}{c^4}\right\} = \frac{a^4}{c^4}.$$
Similarly,
$$\{y^4\} = \left\{\frac{b^4}{c^4}\right\} = \frac{b^4}{c^4}.$$
$$\{z^4\} = \left\{\frac{d^4}{c^4}\right\} = \left\{\frac{a^4 + b^4 + c^4}{c^4}\right\} = \left\{\frac{a^4 + b^4}{c^4}\right\} = \frac{a^4 + b^4}{c^4}.$$
Therefore,
$$\{x^4\} + \{y^4\} = \{z^4\}.$$
Since k can take infinitely many positive integer values, we have infinitely many solutions to (3).

The proofs of Theorems 1 and 2 depend on the identities
$$3^3 + 4^3 + 5^3 = 6^3 \tag{4}$$
and
$$2682440^4 + 15365639^4 + 18796760^4 = 20615673^4. \tag{5}$$
The first identity is the property of Plato's number 216, see [2]. The second identity was discovered by Noam Elkies [3] in 1988, giving a counter-example to a long-standing conjecture due to Euler on the equation
$$A^4 + B^4 + C^4 = D^4.$$

To end this paper, we do not know if for a positive integer $n > 4$ the equation (1) has solutions or not.

References

1. Fermat's Last Theorem, Wikipedia, https://en.wikipedia.org/wiki/Fermat%27s_Last_Theorem
2. Plato's numbers, Wikipedia, https://en.wikipedia.org/wiki/Plato%27s_number
3. N. D. Elkies, On $A^4 + B^4 + C^4 = D^4$, *Mathematics of Computation*, **51** (1988) pp. 825-835.

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108.03 Remarks on perfect powers

A *perfect power* is a number of the form k^n, where $k \geqslant 1$ and $n \geqslant 2$ are integers; and we say that k^n is a perfect n-th power. Now, consider the first few perfect powers:

$$\begin{array}{cccccccccccccc} 1 & 4 & 8 & 9 & 16 & 25 & 27 & 32 & 36 & 49 & 64 & 81 & 100 & \dots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ 1 & 2^2 & 2^3 & 3^2 & 2^4 = 4^2 & 5^2 & 3^3 & 2^5 & 6^2 & 7^2 & 2^6 = 4^3 = 8^2 & 3^4 = 9^2 & 10^2 & \dots \end{array}$$

Then we observe that seemingly between any two consecutive perfect powers of the same exponent there exists at least one perfect power of lower exponent; also there exist at least two squares between any two successive cubes, and there exist at least two cubes between any two successive quartics. The purpose of this note is to prove such simple observations as the theorem below. Their proofs are completely elementary and straightforward from the following two facts.

1. For any real x and y with $x - y \geqslant k$ for some positive integer k, there exist at least k integers in the interval $[x, y]$.
2. Bernoulli's Inequality: If $x \geqslant 0$ and $r \geqslant 1$, then $(1 + x)^r \geqslant 1 + rx$.

Theorem

Let m and n be positive integers with $m < n$.

(i) There is at least one perfect m-th power between any two perfect n-th powers.

(ii) For $n = 3, 4$, there exist at least two perfect m-th powers between any two perfect n-th powers. But this does not always hold when $n > 4$.

(iii) Given a positive integer k, then there exists an integer $a_0 = a_0(k, m, n)$ such that for any integer $a > a_0$, there exist at least k perfect m-th powers between a^n and $(a + 1)^n$.