7
Theory of complex angular momenta

This theory will allow us to keep track of analyticity and unitarity in the t-channel when analysing s-channel phenomena.

The first step is to write down the amplitude expansion in terms of t-channel partial waves $f_\ell(t)$ rather than $f_\ell(s)$ as we did before to derive the Froissart theorem. We write

$$A(s,t) \;=\; \sum_{n=0}^{\infty}(2n+1)f_n(t)P_n(z), \tag{7.1a}$$

where now

$$z \;\equiv\; z_t = \cos\Theta_t \;=\; \frac{2\,s}{t-4\mu^2} \tag{7.1b}$$

stands for the cosine of the scattering angle in the t-channel process.

The unitarity condition in the t-channel limits the size of each partial amplitude $|f_n(t)| = \mathcal{O}(1)$. In the s-channel, we have obtained the growing amplitude $A(s,t)$ by summing up a large number of terms $\ell \lesssim \ell_0(s)$ in the partial-wave expansion in $f_\ell(s)$. Now we keep t finite and a finite number of partial waves $f_n(t)$ with $n \lesssim n_0(t) = \mathcal{O}(1)$ will contribute.

How will the series (7.1a) behave in the $s \to +\infty$ limit? From the t-channel point of view this region on the Mandelstam plane is absolutely unphysical as it corresponds to large *imaginary* scattering angles (7.1b) $z \gg 1$. We have mentioned before more than once that this unphysical region bears information about high energies in the s-channel. So let us try to imagine what sort of behaviour of the series at large z we could expect.

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The partial wave expansion (7.1a) was written in the physical region of the t-channel. Since partial-wave amplitudes are falling fast at large n, we can split the sum into two pieces,

$$A(s,t) = \sum_{n=0}^{n_0(t)} (2n+1) f_n(t) P_n(z) \; + \; \sum_{n_0(t)}^{\infty} \cdots \, ,$$

and drop the infinite series term. (Formally speaking, this 'tail' will diverge for large z but there is no special reason for it to be large and, more importantly, to change significantly when we will move from positive $t = \mathcal{O}\big(\mu^2\big)$ down to $t < 0$ as to reach the s-channel domain.) Then for $z \to \infty$ we will have a qualitative estimate

$$A(s,t) \; \sim \; z^{n_0(t)} \; \propto \; s^{n_0(t)}. \tag{7.2}$$

Being rather brutal, this estimate nevertheless tells us what we could expect. Namely, that the large-s asymptote of the s-channel amplitude is governed by the *characteristic angular momentum* $n_0(t)$ in the cross-channel. What remains is to learn how to determine this characteristic momentum.

7.1 Sommerfeld–Watson representation

This representation was applied by T. Regge to the problem of analytic continuation of the partial-wave expansion.

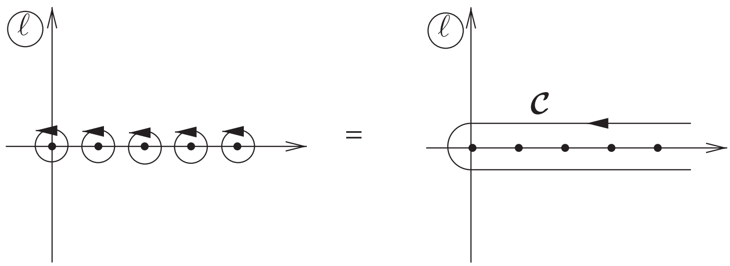
The quest is, how to invent, in a more or less unique way, a function $f_\ell(t)$ that would be *analytic* in ℓ and would coincide with the partial waves in (7.1a) in every integer point,

$$f_\ell(t)|_{\ell=n} \; = \; f_n(t), \quad n = 0, 1, 2, \ldots, \infty.$$

If we succeeded, the problem of analytic continuation of the series (7.1a) to large z would have been relatively easy to solve. Indeed, suppose we knew how to construct such a function. Then I would write a simple formula (the Sommerfeld–Watson integral)

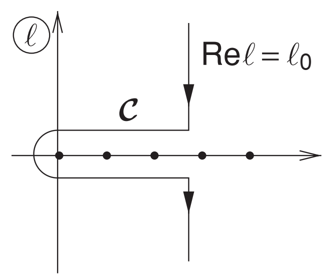
$$A(s,t) \; = \; \frac{1}{2i} \int_{\mathcal{C}} \frac{d\ell}{\sin \pi\ell} f_\ell(t) P_\ell(-z), \tag{7.3}$$

where the contour $\mathcal{C}$ encircles all integer points $n \geq 0$ anti-clockwise:



The function P_ℓ is regular in ℓ. Evaluating the residues of the singular factor $1/\sin\pi\ell$ at $\ell = n$ and bearing in mind that $P_n(-z) = (-1)^n P_n(z)$, we recover the original sum (7.1a).

The idea is as follows. Imagine that we managed to choose f_ℓ 'good enough' so as the contour can be *deformed* as shown here on the right. Then everywhere along $\mathcal{C}$



$$\operatorname{Re}\ell \leq \ell_0 = \text{const.}$$

This gives us an upper bound

$$P_\ell(z) \; \propto \; z^\ell = z^{\operatorname{Re}\ell + i\operatorname{Im}\ell}, \quad A(s,t) \propto |P_\ell(z)| \lesssim \; z^{\ell_0}, \;\; z \to \infty.$$

Such an inequality does not make much sense since it depends on the choice of the integration contour: the boundary gets stronger as we move the contour to the *left*. What prevents us from strengthening the upper bound indefinitely? The function f_ℓ has singularities somewhere in the ℓ plane. Shifting the contour in (7.3) is possible until we hit such a singularity at some point $\ell = \alpha(t)$. Thus it is the position of the *rightmost singularity* of the partial wave $f_\ell(t)$ that will determine the asymptotic behaviour of the amplitude,

$$A(s,t) \; \sim \; z^{\alpha(t)} \; \propto \; s^{\alpha(t)}, \quad s \to \infty. \tag{7.4}$$

Our qualitative expectation has been made precise: we gave definite meaning to the 'characteristic angular momentum' n_0 in (7.2) by having linked it with analytic properties of the partial-wave amplitude considered as a function of a complex variable ℓ.

This is the key idea of the theory of complex angular momenta.

Two things were needed for this programme to succeed namely, that

(1) f_ℓ is analytic in the right half-plane, $\operatorname{Re}\ell > N$; and (7.5a)

(2) f_ℓ falls along each beam $|\ell| \to \infty$ in this half-plane. (7.5b)

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Let us see how fast f_ℓ has to decrease with $|\ell|$. In the physical region $z = \cos\Theta \in [-1, +1]$ where (7.1a) was written,

$$P_\ell(-z) \sim J_0(\ell[\pi - \Theta]) \sim \frac{1}{\sqrt{\ell}} \left[\mathrm{e}^{i\ell(\pi-\Theta)} + \mathrm{e}^{-i\ell(\pi-\Theta)} \right].$$

This means that the ratio

$$\frac{P_\ell(-z)}{\sin \pi\ell} \tag{7.6}$$

in (7.3) falls *exponentially* for all angles but $\Theta = 0$ where

$$\left| \frac{P_\ell(-z)}{\sin \pi\ell} \right| \propto \frac{1}{\sqrt{\ell}}, \qquad |\ell| \to \infty.$$

So it suffices to have

$$|f_\ell| \, < \, \ell^{-3/2}$$

to ensure the convergence of the integral and the possibility of the contour deformation.

It is easy to show that the problem of extrapolating the function from its values in integer points onto the entire complex plane has no more than one solution under a much weaker condition, namely,

$$|f_\ell| \, < \, \mathrm{e}^{|\ell|\pi}, \qquad |\ell| \to \infty. \tag{7.7}$$

This is known as the *Carlson theorem*. We will not prove it. Let us remark, however, that the statement of the theorem is essentially trivial. Suppose we found a solution $f_\ell^{(1)}(t)$. To construct a different one we would have to add a function that vanished in all integers $\ell = n$, that is something of the form

$$f_\ell^{(2)}(t) = f_\ell^{(1)}(t) + \delta f_\ell(t); \qquad \delta f_\ell(t) \, = \, g_\ell(t) \cdot \sin \pi\ell.$$

But the factor $\sin \pi\ell$ grows exponentially along the imaginary axis, $\propto \mathrm{e}^{|\ell|\pi}$, just violating the condition (7.7).

7.2 Non-relativistic theory

One may ask the same question of the behaviour of the amplitude at $\cos\Theta \gg 1$ in the framework of the non-relativistic scattering theory, though it makes not much sense here. Nevertheless, the programme that we have outlined above can be carried out literally and rigorously.

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In the non-relativistic theory we have the Schrödinger equation at our disposal,

$$\left[-\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2 + U(r)\right]\psi(\mathbf{r}) \;=\; E\psi(\mathbf{r}).$$

The radial part, $F_n(r)$, of the wave function $\psi(\mathbf{r}) = F_n(r)Y_{n,m}(\Theta,\phi)$ satisfies the equation

$$\left[-\frac{\hbar^2}{2m}\nabla_r^2 + \frac{n(n+1)}{r^2} + U(r)\right]F_n \;=\; EF_n(r). \tag{7.8}$$

Among the two solutions of (7.8),

$$F_n^{(1)}(r) \propto r^n, \qquad F_n^{(2)}(r) \propto r^{-n-1}, \qquad r \to 0, \tag{7.9}$$

we choose the first one that is regular at $r = 0$. Having fixed the wave function at the origin, at $r \to \infty$ it behaves as

$$F_n(r) \;\propto\; a_n(k)\frac{\mathrm{e}^{-ikr}}{r} + b_n(k)\frac{\mathrm{e}^{ikr}}{r}. \tag{7.10}$$

This corresponds to the S-matrix element $S_n(k) = b_n/a_n$ and to the scattering amplitude

$$f_n(k) \;=\; \frac{1}{2i}\left(\frac{b_n(k)}{a_n(k)} - 1\right). \tag{7.11}$$

Why did we keep the angular momentum n to be an integer? In order to have a non-singular angular dependence of the wave function. However, as long as we are interested in the non-physical region $\cos\Theta > 1$, no-one would forbid us to look upon n in the radial Schrödinger equation (7.8) as an arbitrary continuous parameter.

So we substitute $n \to \ell$ and treat ℓ as a complex number. Then we will solve (7.8), choose $F_\ell(r) \propto r^\ell$ as before and find the functions a and b from the large-r behaviour (7.10). The ratio b_ℓ/a_ℓ does not depend on the normalization of the wave function and describes the 'scattering amplitude' as a function of ℓ.

Have we satisfied the necessary conditions (7.5) for deforming the contour in the Sommerfeld–Watson integral? The condition (7.5a) is fulfilled with $N = \frac{1}{2}$. Indeed, any solution of (7.8) will be an analytic function of ℓ since the parameter ℓ enters the equation *analytically*. Where does the restriction $\mathrm{Re}\,\ell > \frac{1}{2}$ come from in the first place? It emerges from the choice of the solution proper in (7.9): for the prescription to be *unique* we must impose

$$\mathrm{Re}\;\ell \;>\; \mathrm{Re}(-\ell - 1) \qquad \Longrightarrow \qquad \mathrm{Re}\;\ell > \tfrac{1}{2}.$$

The second condition (7.5b) is satisfied as well. It is clear that the scattering amplitude falls fast in the $|\ell| \to \infty$ limit: due to the repulsive centrifugal potential in (7.8), with ℓ increasing, the wave function $F_\ell(r)$ gets more and more suppressed at finite distances $r < r_0$ where the scattering potential $U(r)$ is concentrated.

Thus a_ℓ and b_ℓ are regular analytic functions of ℓ, together with the wave function $F_\ell(r)$, in the right half-plane $\mathrm{Re}\,\ell > \frac{1}{2}$. The amplitude f_ℓ in (7.11) is then regular everywhere but the points $a_\ell(k) = 0$ where it acquires *poles*.

We know already that the poles of the amplitude correspond to *resonances*. The position of such a pole, a complex energy $\overline{E}$, is determined by the equation

$$f_n(\overline{E}) = \infty$$

and depends, obviously, on n as a parameter of the Schrödinger equation, $\overline{E} = E(n)$. The equation

$$f_\ell(\overline{E}) = \infty$$

will have solutions for non-integer values of ℓ as well. In Lecture 3 we were considering the position of the resonance in *energy*, $\overline{E} = E(\ell)$, keeping $\ell = n$ fixed. Now we are studying the same object but from a different angle: we fix E and look at the position of the pole in the ℓ plane, $\bar{\ell} = \ell(E)$.

Having an analytic amplitude which falls properly with ℓ in the right half-plane and has no singularities but poles, we perform the Sommerfeld-Watson trick. Closing the contour around the rightmost pole at $\ell = \alpha(t)$ we will obtain

$$A^{\text{pole}}(s,t) = \frac{-r(t)}{\sin \pi\alpha(t)}(2\alpha(t)+1)P_{\alpha(t)}(-z), \tag{7.12}$$

$$A(s,t) \simeq A^{\text{pole}}(s,t) \propto z^{\alpha(t)}, \qquad z \to \infty. \tag{7.13}$$

as was foreseen in (7.4).

This is a remarkable result.

We have two particles interacting via potential U. In an attractive potential, $U < 0$, there may be bound states or resonance states in various partial waves. Their energies depend on the angular momentum n of the state, $t = t(n)$. By simply *inverting* this dependence, $\ell = \ell(t)$, we get information about the large-z asymptotics: at a given energy t, the behaviour of the amplitude in the $z \to \infty$ limit is determined by the resonance that has *maximal angular momentum* $\ell(t)$ corresponding to this energy.

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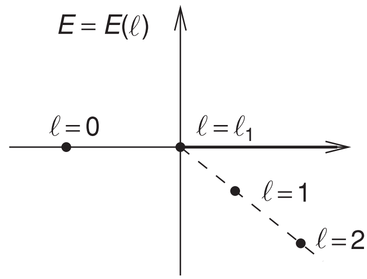

Fig. 7.1 Energy level in a shallow potential well changes continuously with orbital momentum ℓ.

Take not too deep a potential well, such that there exists a level for $\ell = 0$ but not for $\ell = 1$ when the centrifugal repulsion switches on. Imagine that we change ℓ continuously. With ℓ increasing, the energy level will be pushed up until at some $\ell = \ell_1 < 1$ it will cross $E = \overline{E}(\ell_1) = 0$ and move into the continuum, $E > 0$. For $\ell > \ell_1$ there is no *bound state* (a discrete energy level) in our potential any more. However, the level *as a solution of the Schrödinger equation* will not vanish in thin air. What will be its fate then? It can neither belong to the continuous spectrum (by unitarity), nor have a complex energy on the physical sheet (which is forbidden by causality). The only option for the level is to dive onto the *unphysical* sheet and acquire a complex mass there. That is, to become a resonance as displayed in Fig. 7.1.

Redraw the picture now. Let us change the energy and see what will happen to the angular momentum of the level.

At $E = \overline{E}(0)$ we have a pole in the partial wave $\ell = 0$. Increasing the energy, we will find the corresponding value of ℓ. At $E = 0$ we will have $\ell = \ell_1$. If we want to continue keeping ℓ real, we would have to lead E into the complex plane. If instead we continue to keep the *energy* real and increasing, then the pole in the ℓ-plane will move onto the upper half-plane as shown in Fig. 7.2.

It suffices to draw this curve in order to determine the asymptotics of the amplitude at large z. And this was first realized in the framework of a non-relativistic theory. T. Regge (1959) found the way to quantify the value of the characteristic angular momentum n_0 in (7.2),

$$A(s,t) \propto P_{n_0(t)}(z).$$

We may say that in the quantum-mechanical context n_0 measures the strength of the potential. It tells us, what the maximal value of the angular

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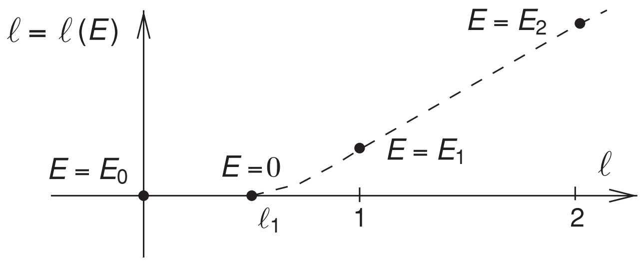

Fig. 7.2 Movement of the pole in the ℓ-plane in non-relativistic theory.

momentum is for which the attraction is still stronger than the centrifugal repulsion and the wave function is still concentrated at small distances so that the partial waves with $n \leq n_0(t)$ are large and contribute significantly to the partial-wave expansion (7.1a).

7.3 Complex ℓ in relativistic theory

It was not at all clear whether this programme could be carried out in a relativistic theory where the potential (if any) depends on particle velocities. Nevertheless, it turned out that the results that we have obtained for potential scattering are *almost* correct in the relativistic framework.

7.3.1 u-channel and a problem with analytic continuation

Why 'almost'? As we have discussed above in Section 7.1, the very supposition that an analytic function f_ℓ existed, immediately allowed me to analytically continue the series (7.1a), originally defined for $z \in [-1, +1]$, to arbitrary $|z| > 1$. We saw that for any complex value of z (but real positive z corresponding to $\Theta = 0$) the ratio (7.6) of $P_\ell(-z)$ and $\sin \pi\ell$ falls *exponentially* along the integration contour. In particular, for $z < 0$ ($\Theta = \pi$) the Legendre function $P_\ell(-z)$ does not increase at all along the imaginary ℓ-axis. The Sommerfeld–Watson integral for A then converges (and so do integrals for its derivatives over z). But this contradicts the fact that $A(z)$ must have singularities at $z < -1$ since we know that our relativistic amplitude has a cut at $s < 0$ corresponding to the u-channel scattering!

Nevertheless, let us try to approach the problem constructively to see where the problem lies and how we might overcome it. So, we start again from two complementary formulae for integer n:

$$A(s,t) \equiv A(z,t) = \sum_n (2n+1) f_n(t) P_n(z), \tag{7.14a}$$

$$f_n(t) = \frac{1}{2}\int_{-1}^{1} dz\, P_n(z) A(z,t), \tag{7.14b}$$

and search for a way of continuing (7.14) to complex angular momenta ℓ.

The task of defining f_ℓ seems easy at the first glance. A straightforward generalization of (7.14b) by a simple substitution $n \to \ell$ will, however, not satisfy us. Indeed, since the z-integral involves all points in the interval $[-1,+1]$, including $z=-1$, it is this end-point corresponding to $\Theta = \pi$ that will make f_ℓ, so defined, behave as $\exp(i\pi\ell)$. But this behaviour violates Carlson's theorem (uniqueness of the analytic continuation) thus forcing us to abandon this bold attempt.

To find a smarter way let us make use of the knowledge of the analytic structure of the amplitude:

$$A(z,t) = \frac{1}{\pi}\int_{z_0}^{\infty} dz' \frac{A_1(z',t)}{z'-z} + \frac{1}{\pi}\int_{z_0}^{\infty} dz_u \frac{A_2(-z_u,t)}{z_u+z}, \tag{7.15}$$

where we represented the contribution of the left cut in terms of a positive integration variable $z_u = -z$. I wrote the dispersion relation without subtractions. They are necessary in principle to have the integrals convergent. 'Subtraction' in the dispersion integral means extracting a polynomial in z of some degree N. However, I am now going to study partial waves with sufficiently large n. If I take $n > N$, then, due to the orthogonality of P_ns, the subtracted polynomial will not affect my partial waves f_n. So we may substitute our 'analytic wisdom' (7.15) into (7.14b) to obtain

$$f_n(t) = \frac{1}{\pi}\int_{z_0}^{\infty} dz\, A_1(z,t) Q_n(z) \; - \; \frac{1}{\pi}\int_{z_0}^{\infty} dz_u\, A_2(-z_u,t) Q_n(-z_u), \tag{7.16}$$

where

$$Q_n(z) \equiv \frac{1}{2}\int_{-1}^{1} dz' \frac{P_n(z')}{z-z'}\,. \tag{7.17}$$

The new expression (7.16) better suits our purpose; it is tempting to try

$$Q_n(z) \Longrightarrow Q_\ell(z)$$

with $Q_\ell(z)$ the second solution of the Legendre equation that is regular at infinity:

$$Q_\ell(z) \propto z^{-\ell-1}, \qquad |z| \to \infty\,.$$

Such a behaviour is perfectly satisfactory for continuing the first term in (7.16). In the second term, however, we get

$$Q_\ell(-z) \propto (-z)^{-\ell-1} = -\mathrm{e}^{-i\pi\ell}|z|^{-\ell-1},$$

and the second part of the partial wave again acquires too fast an exponential increase with $\mathrm{Im}\,\ell$.

So what's the way out? Using the relation

$$Q_n(-z) = (-1)^{n+1} Q_n(z)$$

valid for integer n, we can rewrite (7.16) as

$$f_n(t) = \frac{1}{\pi}\int_{z_0}^{\infty} dz\, A_1(z,t) Q_n(z) + \frac{(-1)^n}{\pi}\int_{z_0}^{\infty} dz\, A_2(-z,t) Q_n(z). \quad (7.18)$$

This way we localize the problem before attempting the analytic continuation to complex n. Actually, we have already derived an analogous formula for s-channel partial-wave amplitudes in Lecture 5 when we discussed the relativistic phenomenon of the appearance of the *backward peak* in the differential angular cross section, cf. (5.18).

7.3.2 Continuing separately even and odd angular momenta

We have to abandon the idea of constructing an analytic continuation of f_n from *all* integer points anyway: as we already know such an attempt is bound to fail because of the existence of the u-channel singularities. We are led to try to continue even and odd angular momenta separately,

$$f_\ell^{(+)}\Big|_{\ell=2n} = f_{2n}, \qquad f_\ell^{(-)}\Big|_{\ell=2n+1} = f_{2n+1}.$$

By so doing we get rid of the oscillating factor in (7.18) and obtain two functions,* both behaving nicely at large $|\ell|$:

$$f_\ell^{(\pm)}(t) = \frac{1}{\pi}\int_{z_0}^{\infty} dz\, Q_\ell(z) A_1(z,t) \pm \frac{1}{\pi}\int_{z_0}^{\infty} dz\, Q_\ell(z) A_2(-z,t). \quad (7.19)$$

Now for the analytic continuation to be unique, a stronger condition than (7.7) must be imposed:

$$\left|f_\ell^{(\pm)}\right| < \exp\!\left(\tfrac{1}{2}|\ell|\pi\right).$$

It is easy to verify that the functions defined by (7.19) do satisfy this condition easily (along the imaginary axis they don't increase at all).

* Equation (7.19) is known as the *Gribov–Froissart projection* (ed.).

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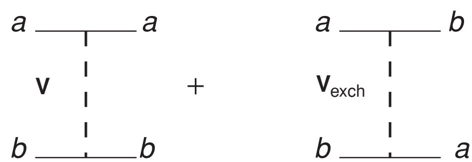
Thus the price we pay for solving the problem of continuation is the introduction of *two* analytic functions in place of one. What is the reason for that?

In a relativistic theory one 'potential' is not enough. There is always another diagram corresponding to what is known in nuclear physics under the name of *exchange potential* $V_{\rm exch}$:



The two graphs differ by the transposition $a \leftrightarrow b$ which, for spinless particles, introduces the factor $(-1)^\ell$. Therefore for *even* and *odd* orbital momenta we have

$$V \Longrightarrow V + V_{\rm exch} \;\; (\ell = 2n) \qquad \text{and} \qquad V \Longrightarrow V - V_{\rm exch} \;\; (\ell = 2n{+}1),$$

correspondingly. Having two different full potentials means that there isn't any analytic relation between partial waves with even and odd angular momenta.

7.3.3 *Sommerfeld–Watson representation for* $f_\ell^{(\pm)}$

Let us split the amplitude into symmetric and anti-symmetric parts with respect to $s \leftrightarrow u$,

$$A(z,t) = A^+(z,t) + A^-(z,t), \quad A^\pm(z,t) \equiv \frac{A(z,t) \pm A(-z,t)}{2}, \tag{7.20}$$

and treat these two amplitudes separately:

$$A^\pm(z,t) \;=\; \sum_{n=\text{even/odd}} (2n+1) f_n^{(\pm)}(t) P_n(z). \tag{7.21}$$

Recall that above we wrote the dispersion relation without subtractions, which was fine for the purpose of analysing partial waves with $n > N$. The integrals (7.19) for $f_\ell^{(\pm)}$ are defined also for sufficiently large angular momenta. This is necessary to ensure convergence: if the amplitude behaves at large z as $|A(z)| = \mathcal{O}\big(z^N\big)$, then

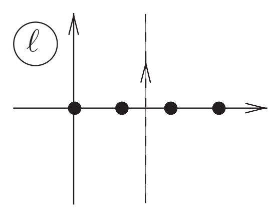
$$f_\ell^{(\pm)} \sim \int^\infty dz\, Q_\ell(z) A(z) \;\sim\; \int^\infty dz \cdot z^{-\ell-1} \cdot z^N < \infty \quad \Longrightarrow \quad \mathrm{Re}\,\ell > N.$$

This means that the series (7.21) for the amplitude $A^\pm$ can be represented by the Sommerfeld–Watson integral only *partially*. Namely, it is the infinite series of partial waves with $n \geq N$ in

$$A = \sum_{n=0}^{N}(2n+1)f_n P_n + \sum_{n=N+1}^{\infty}(2n+1)f_n P_n$$

that can be combined into an analytic function of ℓ, while a few first partial waves remain, generally speaking, arbitrary.

Why did a few partial waves remain unaccounted for? In a non-relativistic language, we may imagine adding to the potential a singular term $\delta_0 V(\mathbf{r}) = \mathrm{const} \cdot \delta(r)$. It contributes to the S-wave scattering only and as a result f_0 would fall out of the family. Introducing $\delta_1 V(\mathbf{r}) \propto \boldsymbol{\nabla}\delta(r)$ we would analogously spoil the P-wave f_1. If we continue these singular series by summing up an infinite number of derivatives we may violate causality (polynomial boundary $|A| \lesssim s^N$). As we shall see soon from the s-channel unitarity condition, the number of 'special' partial waves that remain unaccounted for in fact cannot be larger than two: $\ell = 0$ and $\ell = 1$.



Now that we have removed the oscillating factor $(-1)^\ell$ that used to cause too fast an exponential increase, we can deform the contour $\mathcal{C}$ in the Sommerfeld–Watson representation embracing the points $n \geq N+1$ by straightening it and sending along the imaginary axis at $\mathrm{Re}\,\ell = \ell_0$ so that $N < \ell_0 < N+1$:

$$\begin{aligned} A^{\pm}(s,t) &= \sum_{n=0}^{N}(2n+1)f_n(t)P_n(z)\cdot\tfrac{1}{2}[1 \pm (-1)^n] \\ &\quad + \frac{i}{4}\int_{\ell_0-i\infty}^{\ell_0+i\infty}\frac{d\ell\,(2\ell+1)}{\sin\pi\ell} f_\ell^{(\pm)}(t)[P_\ell(-z) \pm P_\ell(z)]\,. \end{aligned} \tag{7.22}$$

When the contour transformation is done, we are ready to leave the physical region of the t-channel and to study the large-s regime. Taking $z \to \infty$ will affect only the oscillating factor $\exp(i\,\mathrm{Im}\,\ell \cdot \ln z)$ but not the convergence of the integral.

Let us verify that the formula (7.22) is what we have been looking for. To this end, examine the analytic properties of $A^{\pm}$. They obviously must be those that we put in. Namely at positive $s > 4\mu^2$ we should encounter a non-zero *absorptive part* which appears with the opening of the first s-channel threshold:

$$A_1^{\pm} = \frac{A(s+i\epsilon) - A(s-\imath\epsilon)}{2i} = \tfrac{1}{2}[A_1(s) \pm A_2(s)], \tag{7.23a}$$

as it follows directly from the definition of the amplitudes (7.20). Also, if we decrease s, starting from $u > 4\mu^2$ we should see the cross-channel

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absorptive part

$$A_2^\pm \;\equiv\; \frac{A(u+i\epsilon)-A(u-\imath\epsilon)}{2i} = \tfrac{1}{2}[A_2(u)\pm A_1(u)];$$
$$A_2^\pm(z,t) \;=\; \pm A_1^\pm(-z,t). \tag{7.23b}$$

Taking $s>0$ we have $z>1$, and the first Legendre function under the integral in (7.22) becomes complex since $P_\ell(z)$ with non-integer ℓ has a *logarithmic cut* running from -1 to $-\infty$. The phase of the argument in $P_\ell(-z)$ starts to matter. Comparing two ways of defining $(-z)$,

$$\frac{1}{2i}\big[P_\ell(\mathrm{e}^{i\pi}z)-P_\ell(\mathrm{e}^{-i\pi}z)\big] \;=\; \sin\pi\ell\cdot P_\ell(z),$$

and substituting into (7.22) we derive the absorptive part

$$A_1^\pm(s,t) \;=\; \frac{1}{4i}\int_{\mathcal{C}} d\ell(2\ell+1)f_\ell^{(\pm)}(t)P_\ell(z). \tag{7.24}$$

The complexity of $P_\ell(-z)$ has, however, nothing to do with physics. Therefore at $s>0$ the amplitude must stay real until we meet its first physical singularity. Let us see how it happens in our formula.

In (7.24) the poles in the integer points have disappeared inviting us to move the contour back to the right and close it at $+\infty$. If $|z|<1$ we can always do so to obtain $A^\pm_{\rm abs}\equiv 0$ as expected. As for $z>1$, closing the contour will be still possible as long as the integrand falls in the right half-plane.

We have studied the large-ℓ asymptote of f_ℓ in Section 5.3. Applying (5.26) to the t-channel partial waves,

$$f_\ell(t)\propto\exp(-\ell\chi_0), \qquad \cosh\chi_0\equiv z_0 = 1+\frac{2\cdot 4\mu^2}{t-4\mu^2}, \tag{7.25a}$$

and comparing with the asymptote of the Legendre functions (5.25),

$$P_\ell(z)\propto\exp(\ell\Theta), \qquad \cosh\Theta = z, \tag{7.25b}$$

we immediately see that the absorptive part $A_1=0$ as long as $\Theta<\chi_0$, that is up to $s=4\mu^2$ when we hit the s-channel threshold singularity and the partial-wave expansion (7.14a) diverges.

Considering analogously $z<0$, it is easy to obtain the *u-channel* absorptive part independently,

$$A_2^\pm(z,t) = \pm\frac{1}{4i}\int_{\mathcal{C}} d\ell(2\ell+1)f_\ell^{(\pm)}(t)P_\ell(-z) \;=\; \pm A_1^\pm(-z,t), \tag{7.26}$$

in accord with the expectation (7.23b).

Thus we have derived two formulae relating partial waves with a definite *signature* to the absorptive part of the amplitude with a definite $s\leftrightarrow u$

symmetry:

$$A^{\pm}_{\rm abs}(s,t) \;=\; \frac{1}{4i}\int_{\mathcal{C}} d\ell (2\ell+1) f^{(\pm)}_{\ell}(t) P_{\ell}(z); \tag{7.27a}$$

$$f^{(\pm)}_{\ell}(t) \;=\; \frac{2}{\pi}\int_{z_0}^{\infty} dz\, Q_{\ell}(z) A^{\pm}_{\rm abs}(z,t), \tag{7.27b}$$

with z_0 defined in (7.25a). Remember, z in these formulae is a cosine of the *t-channel* scattering angle,

$$z \;\equiv\; z_t(s,t) = 1 + \frac{2\,s}{t-4\mu^2} = \frac{s-u}{t-4\mu^2}. \tag{7.27c}$$

Let us note an attractive feature of (7.27a): the expression for $A_{\rm abs}$ is free from an undetermined sum of few 'non-analytic' terms present in the Sommerfeld–Watson representation for the amplitude A itself. Moreover, $A_{\rm abs}$ is a valuable thing: continuing to $t<0$, we will get hold of the imaginary part of the s-channel amplitude which interests us much.

On its own, the expression (7.27a) for the absorptive part is sort of trivial, something resembling the Mellin transformation. It is complementary to (7.27b) for $f^{(\pm)}_{\ell}$ which expression is slightly less obvious as it exploits analytic properties of the amplitude. Still, if not for the unitarity condition, the translation of $A_{\rm abs}$ into f_{ℓ}, and back again, would have had not much value (although performing Mellin transform may be sometimes useful). The essence of the issue lies in that the singularities in ℓ of $f^{(\pm)}_{\ell}$ are determined by the *physical spectrum* of particles and resonances.

7.4 Analytic properties of partial waves and unitarity

In order to determine the character of the large-s asymptotics of the scattering amplitude, we need to learn what singularities $f^{(\pm)}_{\ell}(t)$ has in the ℓ-plane. Moreover, till now we were sitting at $t>4\mu^2$ while it is the physical region of the s-channel, $t<0$, that really interests us. In non-relativistic quantum mechanics we saw how the unitarity condition has translated the poles of the amplitude on the unphysical sheet (resonances) into singularities in ℓ of the partial wave f_{ℓ} – the Regge trajectories $\ell = \ell(t)$. We are about to try the same path in the relativistic theory.

7.4.1 Redefining partial waves

So we will keep $\mathrm{Re}\,\ell > N$ and discuss the properties of $f_{\ell}(t)$ defined by (7.27b). We have introduced partial waves with complex ℓ at $t>4\mu^2$. There $f^{(\pm)}_{\ell}$ are complex, because so are A_1 and A_2 (see the path 1 in

Fig. 7.3). Below the t-channel threshold the absorptive parts are real and it would have been nice if the partial waves at $t < 4\mu^2$ were real too. However when the sign of $t - 4\mu^2$ in the expression (7.27c) changes, z becomes negative. We have then to watch for $Q_\ell(z) = z^{-\ell-1}\,F(z)$, with F a regular even function of z, which acquires an ℓ-dependent phase. This phase will provide f_ℓ with a 'kinematical' complexity which has nothing to do with analyticity (since $A_{\rm abs}$ is real!). Let us have a look at the vicinity of the threshold, $0 < t - 4\mu^2 \ll \mu^2$. Here we have $z \simeq 2\,s/(t-4\mu^2) \gg 1$, and (7.27b) gives

$$f_\ell^{(\pm)} \;\propto\; (t-4\mu^2)^\ell \cdot \int_{4\mu^2}^{\infty} \frac{ds}{s^{\ell+1}} [\,A_1 \pm A_2\,]\,.$$

For integer n this is nothing but the usual threshold behaviour, $f_n \propto k_c^{2n}$. To get rid of the trivial phase factor it is convenient to redefine partial waves by introducing

$$f_\ell^{(\pm)}(t) \;\equiv\; (t-4\mu^2)^\ell \cdot \phi_\ell^{(\pm)}(t). \tag{7.28}$$

The new partial wave ϕ_ℓ is given by the integral

$$\phi_\ell^{(\pm)}(t) \;=\; \frac{2}{\pi}\int_{4\mu^2}^{\infty} \frac{2\,ds}{(t-4\mu^2)^{\ell+1}} Q_\ell(z) A_{\rm abs}^{\pm},$$

where we choose to integrate over s rather than z as in the original formula (7.19). Moving to $t < 4\mu^2$, we reflect the argument of $Q_\ell(z \to -z)$ and write down a more convenient expression,

$$\phi_\ell^{(\pm)}(t) \;=\; \frac{4}{\pi}\int_{4\mu^2}^{\infty} \frac{ds}{(4\mu^2-t)^{\ell+1}} Q_\ell\left(\frac{2s}{4\mu^2-t}-1\right) A_{\rm abs}^{\pm}(s,t), \tag{7.29}$$

which makes it clear that ϕ_ℓ stays real in the interval $0 < t < 4\mu^2$. If we decrease t further, at $t < 0$ the Legendre function $Q_\ell(z)$ becomes complex ($|z| < 1$) when $4\mu^2 < s < 4\mu^2 - t$ (interval $[a,b]$ on the path #2 in Fig. 7.3). A *physical* singularity will emerge later, when the integration line on the Mandelstam plane crosses the Karplus curve where $\rho_{su} \neq 0$ and the *absorptive part* becomes complex (interval $[c,d]$ on the line #3 in Fig. 7.3).

7.4.2 Two-particle unitarity condition for ϕ_ℓ

Recall how we have used the two-particle unitarity condition to find the discontinuity on the right cut of the partial wave with integer n:

$$\operatorname{Im} f_n(t) \;=\; \tau f_n(t) f_n^*(t), \qquad 4\mu^2 < t < 16\mu^2,$$

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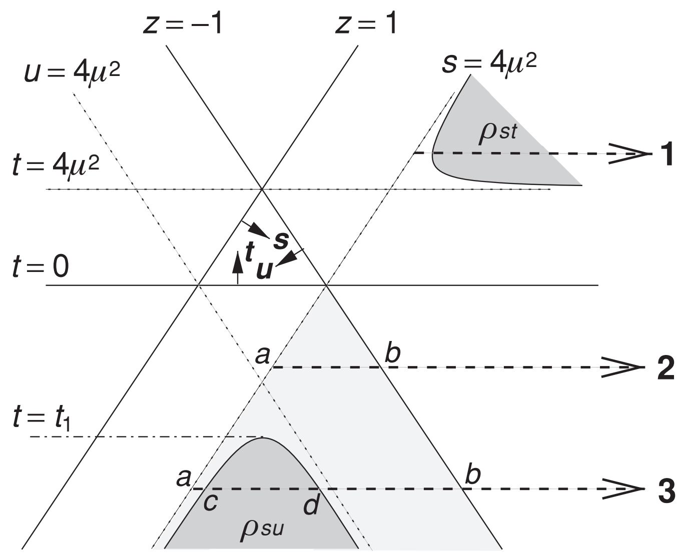

Fig. 7.3 Integration paths in the representation (7.29) for $\phi_\ell(t)$. For $t < 0$, on the interval $[a, b]$ the Legendre function $Q_\ell(z)$ is complex. When $t < t_1$, the third spectral function contributes to complexity of ϕ_ℓ along the $[c, d]$ interval.

or, in terms of ϕ,

$$\begin{aligned} \frac{1}{2i}[\,\phi_n(t+i\epsilon) - \phi_n(t-i\epsilon)\,] \;&=\; C_n \phi_n(t+i\epsilon)\phi_n(t-i\epsilon), \\ C_n \;&\equiv\; \tau \cdot (t-4\mu^2)^n. \end{aligned}$$

Now we can state that the same relation holds for arbitrary complex ℓ:

$$\begin{aligned} \frac{1}{2i}[\,\phi_\ell(t+i\epsilon) - \phi_\ell(t-i\epsilon)\,] \;&=\; C_\ell\, \phi_\ell(t+i\epsilon)\phi_\ell(t-i\epsilon), \\ C_\ell \;&\equiv\; \tau \cdot (t-4\mu^2)^\ell. \end{aligned} \tag{7.30}$$

Why? Thanks to the Carlson theorem. Indeed, as it is easy to see from the properties of ϕ_ℓ, the difference between the l.h.s. and the r.h.s. of (7.30) does not increase at infinity faster than $\exp(\frac{1}{2}\pi|\ell|)$ and equals zero in all even (odd) points. Therefore it is zero on the entire ℓ-plane.

The unitarity condition (7.30) applies to $4\mu^2 < t < 16\mu^2$. Above the four-pion threshold in the unitarity condition there appear *integral terms* the continuation of which to complex angular momenta is not so easy.

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Now that we briefly described the singularities in t for fixed ℓ, it is time to turn to the question which really interests us – that of the structure of singularities in ℓ for fixed t.

Where may they come from? From the divergence of the integral

$$\phi_\ell \sim \int^\infty dz\, Q_\ell(z) A_{\rm abs}(z)$$

at some sufficiently small $\operatorname{Re}\ell$. It is difficult to say anything starting from nothing. Therefore we will begin with the classification.

(1) 'Fixed singularity' whose position $\ell = \ell_0$ does not depend on t.

(2) 'Moving singularity', $\ell = \ell_0(t)$.

For the time being we will discuss only the rightmost singularity in the ℓ-plane, the one with the maximal $\operatorname{Re}\ell_0$.

7.4.3 Fixed singularities in the ℓ plane

The first statement we can make about the fixed singularity (should it happen to be the rightmost one) is that it may occur only on the *real axis*. Indeed, whatever the nature of the singular point, its contribution to the asymptote of $A_{\rm abs}(z)$ is proportional to z^{ℓ_0}. If $\operatorname{Im}\ell_0 \neq 0$, this factor *oscillates* fast at large z. But this would contradict the positivity of the cross section, $\sigma_{\rm tot} \propto A_{\rm abs}$ (in the next lecture we will verify that the t-*derivative* of $A_{\rm abs}(s,t)$ must also be positive in the interval $0 < t < 4\mu^2$).

Now, since $\ell_0 \neq \ell_0(t)$, we can choose $4\mu^2 < t < 16\mu^2$ to see what the unitarity condition would tell us. The r.h.s. of (7.30), for *real* ℓ_0, reduces to $|\phi_\ell|^2$ and we have

$$\operatorname{Im}\phi_\ell = C_\ell \cdot |\phi_\ell|^2 \quad \Longrightarrow \quad |\phi_\ell| < C_\ell^{-1}. \tag{7.31}$$

Therefore the singularity may be only a *weak* one, namely such that in the singular point the partial-wave amplitude stays finite like, for example, $\phi_\ell \sim \sqrt{\ell - \ell_0}$. But this is exactly the case when, as we have discussed in the previous lecture, the cross section *must fall* at large s.

Let us check that, indeed, $|\phi_{\ell_0}| < \infty$ implies a falling cross section. Dropping irrelevant factors we write

$$A_{\rm abs}(s) \sim \int_{\mathcal{C}} d\ell\, \phi_\ell \cdot s^\ell$$

and shift the contour to the left in search for singularity. If ϕ_ℓ had a pole we would have taken the residue and obtained a power asymptote

$$A_{\rm abs}^{\rm pole} = {\rm const} \cdot s^{\ell_0}.$$

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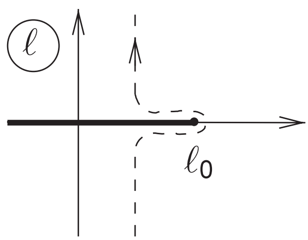
The pole is, however, forbidden by the t-channel unitarity restriction (7.31). (By the way, a particular case of this veto is the familiar classical diffraction picture, $A_{\rm abs}(s,t) = sF(t)$, which corresponds to the fixed pole singularity at $\ell_0 = 1$.) Therefore ϕ_ℓ may only have a *branch cut* starting at ℓ_0 and running to the left. Integrating the discontinuity $\Delta\phi_\ell$ along the cut we get



$$A_{\rm abs} \simeq s^{\ell_0} \cdot \int_0^\infty dx\, \Delta\phi(x)\, e^{-x \ln s},$$

where we have introduced $x = \ell_0 - \ell > 0$ as an integration variable. When s is large, the integral converges at $\langle x \rangle \sim 1/\ln s \ll 1$ so that only the very tip of the cut matters. Parametrizing the discontinuity of the partial wave as $\Delta\phi(x) \propto x^\gamma$ and evaluating the integral we obtain

$$A_{\rm abs} \propto \frac{s^{\ell_0}}{(\ln s)^{\gamma+1}}. \tag{7.32}$$

The *finiteness* condition (7.31) tells us that $\gamma \geq 0$. As a result

$$\sigma \propto s^{-1} A_{\rm abs} \sim \frac{s^{\ell_0 - 1}}{(\ln s)^{\gamma+1}} < \frac{1}{\ln s}, \tag{7.33}$$

where we have used the maximal power value $\ell_0 = 1$ allowed by the Froissart theorem.

Fixed singularities in NQM. Do fixed singularities exist in quantum mechanics? Yes, and they are related to the '*falling on the centre*' phenomenon, with the behaviour of the potential at small distances. As we have already discussed, in non-relativistic quantum mechanics a singularity appears in ℓ when the choice between the two solutions of the Schrödinger equation at the origin, $r = 0$, becomes ambiguous. When discussing NQM scattering, we have tacitly implied that the interaction potential was less singular than the centrifugal barrier, $r^2 \cdot V(r) \to 0$ at $r \to 0$, in which case the singularity was at $\ell_0 = -\frac{1}{2}$ (when $r^{-\ell} \sim r^{\ell+1}$). In the opposite case $V(r)$ itself will govern the $r \to 0$ asymptotic behaviour of the wave function $\psi_\ell(r)$, and a fixed singularity may emerge at any ℓ_0.

It is important to stress that such fixed singularities correspond to definite physics, namely a super-singular behaviour of the interaction at small distances. It seems they are very unlikely to have any relation with the approximate constancy of the total cross section.

Strictly speaking, the question of a possible rôle of fixed singularities remained unsolved.

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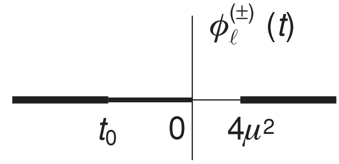
7.4.4 Moving singularities

More difficult to analyse are moving singularities. At the same time they are much more interesting. Given the function $\ell = \ell_0(t)$, we might invert it and consider $t = t_0(\ell)$ as a singularity on the t-plane about which we have already learned a thing or two!

Above we have described *all* singularities of the partial wave amplitude on the t-plane: there is nothing but the right cut, $t > 4\mu^2$ and the left cut, $t < 0$. So where are then these new ℓ-dependent singularities?



Recall that our analysis of $\phi_\ell^{(\pm)}(t)$ was carried out for $\mathrm{Re}\,\ell > N$. New singularities show up at smaller ℓ. How can this happen? It is clear that a singularity cannot just 'pop up' suddenly with ℓ decreasing. Our amplitude has cuts on the complex t-plane, and the possibility arises for the singularity to move from beneath a cut and appear on the physical sheet at some $\ell < N$. This means that such a moving singularity is always present but 'hidden' on the *unphysical sheets* of the amplitude at large $\mathrm{Re}\,\ell$. Therefore, in order to learn which singularities the partial wave with $\mathrm{Re}\,\ell < N$ may have on the t-plane, it suffices to find out (as we did before when we studied resonances in Lecture 3) what the singularities are on the unphysical sheets at $\mathrm{Re}\,\ell > N$.

Left cut. The first important statement: No moving singularities emerge from the sheets linked to the *left cut*.

We take $t < 0$, write

$$\phi_\ell^{(\pm)}(t + i\epsilon) = \phi_\ell^{(\pm)}(t - i\epsilon) + \Delta\phi_\ell^{(\pm)}(t)$$

and move the argument $t + i\epsilon$ down under the cut to explore the corresponding sheet. The amplitude $\phi_\ell^{(\pm)}(t - i\epsilon)$ on the r.h.s. of the equation stays on the physical sheet. Since it is regular there, the partial wave on the l.h.s. diving under the cut will exhibit singularities of the discontinuity over the left cut, $\Delta\phi_\ell^{(\pm)}(t)$. The latter, however, cannot have any (moving) singularities. This is a consequence of a simple fact that, as we have repeatedly stressed before, it is given by integrals over *finite* intervals. Indeed, one contribution to $\Delta_t\phi$ comes from the complexity of Q_ℓ at $t < 0$: $\Delta_t Q_\ell(-z) = -\frac{\pi}{2}P_\ell(-z)$ (for $-1 < z < 1$); the other one appears at $t < t_1$ due to the discontinuity of $A^\pm_{\mathrm{abs}}$: $\mathrm{Im}_t\,\mathrm{Im}_s\,A^\pm \equiv \rho^\pm_{su}$. From (7.29) we obtain

$$\begin{aligned}\Delta\phi_\ell^{(\pm)}(t) = &-2\int_{s_a}^{s_b} \frac{ds}{(4\mu^2 - t)^{\ell+1}} P_\ell\left(\frac{2s}{4\mu^2 - t} - 1\right) A^\pm_{\mathrm{abs}}(s, t + i\epsilon) \\ &+ \frac{4}{\pi}\int_{s_c}^{s_d} \frac{ds}{(4\mu^2 - t)^{\ell+1}} Q_\ell\left(\frac{2s}{4\mu^2 - t + i\epsilon} - 1\right) \rho^\pm_{su}(s, t).\end{aligned} \tag{7.34}$$

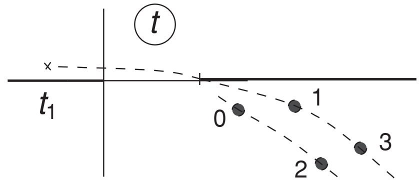


Fig. 7.4 Resonances on Regge trajectories $\ell^{\pm}(t)$ and the movement of a Regge pole onto the physical sheet.

The integrals run over finite regions; they converge and cannot produce singularities. As for an explicit ℓ-dependence of the integrands, P_ℓ is regular on the entire ℓ plane; Q_ℓ is 'almost regular': strictly speaking, it has *poles* in negative integers, $\ell = -1, -2, \ldots$, but these do not concern us here as they are not related to moving singularities.

Right cut. Exactly as it was the case of integer angular momenta that we have explored in Lecture 3, on the unphysical sheet linked to the two-particle cut there may be only poles:

$$\phi_\ell^{(\pm)}(+) = \frac{\phi_\ell^{(\pm)}(-)}{1 - 2iC_\ell\, \phi_\ell^{(\pm)}(-)}, \quad \phi_{\ell_0}^{(\pm)}(t) = \frac{1}{2iC_{\ell_0}(t)} \implies \ell_0 = \ell^{\pm}(t).$$

A remarkable thing! We knew that resonances with different spins n live on the unphysical sheet. Now not only have we got the statement about the large-s behaviour but also about the resonances themselves. In Lecture 3 we had independent equations for resonance masses,

$$\phi_n^{(\pm)}(t) = [\, 2iC_n(t)\,]^{-1} \quad \implies \quad m_n^2 = t(n).$$

Now we see that all these resonances are analytically linked to each other as shown in Fig. 7.4. This discovery laid the basis for the classification of all hadrons according to 'Regge trajectories' they belong to. Real and imaginary parts of the position of the pole on the t-plane give the squared mass and the width of the resonance.

Moreover, two (generally speaking different) analytic curves $\ell^{\pm}(t)$ that combine together resonances with even spins and those with odd spins, are those very same curves that determine the asymptotic behaviour of the symmetric and anti-symmetric parts of the scattering amplitude, correspondingly.

What sort of information may this give us in practice? By studying t-channel particle scattering at relatively small energies t, experimenters find resonances, measure their masses, decay widths and determine their spins. Imagine that we put 'many points' on a Regge trajectory and in so doing approximately found $\mathrm{Re}\,\ell(t)$. This is a mere classification at this point. Now, let us extrapolate the curve to $t = 0$, and below. This will

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tell us the *characteristic angular momentum*, $\ell^{\pm}(t_1)$, corresponding to a given value of the momentum transfer $t_1 < 0$ and immediately give the energy behaviour of the scattering amplitude,

$$A^{\pm}(s, t_1 \leq 0) \ \propto \ s^{\mathrm{Re}\,\ell^{\pm}(t_1)},$$

in a completely different – crossing – channel! Understanding this cross-channel relation constitutes the main achievement of the theory of complex angular momenta.