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Existence and non-existence of solutions to the coboundary equation for measure-preserving systems

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Abstract. A fundamental question in the field of cohomology of dynamical systems is to determine when there are solutions to the coboundary equation:

$$f = g - g \circ T.$$

In many cases, T is given to be an ergodic invertible measure-preserving transformation on a standard probability space $(X, \mathcal{B}, \mu)$ and $f : X \to \mathbb{R}$ is contained in L^p for $p \geq 0$. We extend previous results by showing for any measurable f that is non-zero on a set of positive measure, the class of measure-preserving T with a measurable solution g is meager (including the case where $\int_X f\, d\mu = 0$). From this fact, a natural question arises: given f, does there always exist a solution pair T and g? In regards to this question, our main results are as follows. Given measurable f, there exist an ergodic invertible measure-preserving transformation T and measurable function g such that $f(x) = g(x) - g(Tx)$ for almost every (a.e.) $x \in X$, if and only if $\int_{f>0} f\, d\mu = -\int_{f<0} f\, d\mu$ (whether finite or ∞). Given mean-zero $f \in L^p(\mu)$ for $p \geq 1$, there exist an ergodic invertible measure-preserving T and $g \in L^{p-1}(\mu)$ such that $f(x) = g(x) - g(Tx)$ for a.e. $x \in X$. In some sense, the previous existence result is the best possible. For $p \geq 1$, there exists a dense G_δ set of mean-zero $f \in L^p(\mu)$ such that for any ergodic invertible measure-preserving T and any measurable g such that $f(x) = g(x) - g(Tx)$ almost everywhere, then $g \notin L^q(\mu)$ for $q > p - 1$. Finally, it is shown that we cannot expect finite moments for solutions g, when $f \in L^1(\mu)$. In particular, given any $\phi : \mathbb{R} \to \mathbb{R}$ such that $\lim_{x\to\infty} \phi(x) = \infty$, there exist mean-zero $f \in L^1(\mu)$ such that for any solutions T and g, the transfer function g satisfies:

$$\int_X \phi(|g(x)|)\, d\mu = \infty.$$

Key words: coboundaries, measure preserving systems, integrability conditions, solution existence
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1. *Introduction*

We give new fundamental results concerning solutions to the coboundary equation:

$$f = g - g \circ T. \tag{1.1}$$

There has been substantial progress in many cases such as homogeneous spaces, smooth actions, Lie groups, as well as many other important families of dynamical systems. Most previous research focuses on the case where a measurable transformation, or topological dynamical system is specified, and a solution g is sought for individual f or families of functions f (e.g., Hölder f). In this paper, we study the situation from the general perspective of solutions T and g where f may be any real-valued measurable function, or function $f \in L^p$ for $p \geq 0$.

In this paper, we assume all measurable dynamical systems are defined on a Lebesgue space $(X, \mathcal{B}, \mu)$. Thus, by the Rokhlin isomorphism theorem [38, 39], we take $X = [0, 1)$, μ is Lebesgue measure, and $\mathcal{B}$ is the collection of Lebesgue measurable sets. For $p > 0$, define the standard L^p space, $L^p = \{f : X \to \mathbb{R} | \ f \text{ is measurable and } \int_X |f|^p \, d\mu < \infty\}$. For $p \geq 1$, define $L_0^p = \{f \in L^p : \int_X f \, d\mu = 0\}$. Also, L^∞ is the set of essentially bounded measurable functions on $(X, \mathcal{B}, \mu)$ and similarly, L_0^∞ are functions in L^∞ with zero integral. The space L^0 is the set of measurable functions on $(X, \mathcal{B}, \mu)$. Let $\mathcal{M}$ be the family of invertible measure-preserving transformations defined on $(X, \mathcal{B}, \mu)$ and $\mathcal{E}$ is the family of ergodic invertible measure-preserving transformations on $(X, \mathcal{B}, \mu)$. We obtain the following main positive result.

THEOREM 1.1. (Existence of solutions) *Let $1 \leq p \leq \infty$ and suppose $f \in L_0^p$. There exist $T \in \mathcal{E}$ and $g \in L^{p-1}$ such that $f(x) = g(x) - g(Tx)$ for almost every (a.e.) $x \in X$.*

In some sense, Theorem 1.1 gives the best possible positive result. The following theorem demonstrates a major limitation for solutions to the coboundary equation. In particular, typically, there is no solution g in the same integrability class as f, even when allowing T to range over all of $\mathcal{E}$.

THEOREM 1.2. (L^q non-existence) *Given $1 \leq p < \infty$, there exist $f \in L_0^p$ such that for any solution $T \in \mathcal{E}$ and measurable g to the coboundary equation $f = g - g \circ T$, then $g \notin L^q$ for $q > p - 1$. More generally, there exists a dense G_δ set $\mathcal{G}_p \subset L_0^p$ such that for any $f \in \mathcal{G}_p$, and any solution pair T, g with $T \in \mathcal{E}$, then $g \notin L^q$ for $q > p - 1$.*

The solution g is referred to as the transfer function for coboundary f. Theorem 1.2 implies that for generic mean-zero $f \in L^p$ for $p < 2$, any transfer function is not integrable, regardless of $T \in \mathcal{E}$. However, for $f \in L_0^1$, we can always find a solution with measurable $g \in L^0$.

For the case where f is only assumed to be measurable, we give a straightforward equivalent condition for the existence of a measurable transfer function. Also, Theorem 1.3 highlights the need to control T, or the inter-dependence of T and f, if one hopes to find a measurable transfer function.

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THEOREM 1.3. (Measurable transfer functions) *Suppose* $(X, \mathcal{B}, \mu)$ *is a standard probability space and* $f \in L^0$ *is non-zero on a set of positive measure.*

- *The coboundary equation* $f = g - g \circ T$ *has a solution pair,* $T \in \mathcal{E}$, $g \in L^0$, *if and only if* $\int_{f>0} f \, d\mu = -\int_{f<0} f \, d\mu$, *whether both integrals are* ∞ *or finite.*
- *The class of ergodic invertible measure-preserving transformations* T *such that* $f = g - g \circ T$ *has a measurable solution* g *is first category (that is, meager).*

2. *Connections to previous research*

There has been substantial interest in the study of the cohomology of dynamical systems. Much of the recent focus is on smooth dynamics including hyperbolic actions or actions of Lie groups. Powerful rigidity or local rigidity results have been obtained involving cocycles. Some of the earliest results include [27, 28]. Cocycle rigidity depends closely on solving the coboundary equation, since the difference between cohomologous cocycles is a coboundary. Livšic [34] provided one of the earliest regularity results in this setting by demonstrating Hölder cocycle rigidity for families of U-systems, topological Markov chains, and Smale systems. More recently, this Hölder regularity has been extended to non-uniformly expanding Markov maps [22], and to Weyl chamber flows or twisted Weyl chamber flows [42]. In [41], Veech proves that the coboundary equation $f = g - g \circ T$ admits a C^∞ solution g for $C^\infty f$ when T is an ergodic toral endomorphism and f sums to zero over every periodic orbit. Also, a connection is made to the generalized Riemann hypothesis.

We will consider the coboundary equation in a general context. In the setting of topological dynamics, the following early result was observed in Gottschalk and Hedlund [21]: a bounded continuous function f is a coboundary for a minimal homeomorphism on a compact space if and only if the following is uniformly bounded for positive n,

$$\left| \sum_{i=0}^{n-1} f(T^i x) \right|.$$

More recently, Quas [37] proves for a μ-invariant minimal homeomorphism on a compact probability space, if a continuous f is a coboundary with an $L^\infty(\mu)$ transfer function, then f is a coboundary with a continuous transfer function. Also, we would like to mention a result of Baggett, Medina, and Merrill which is in the same spirit of Theorem 1.3. They prove in [5] that if $f \in L_0^1(S^1)$ is not a trigonometric polynomial, then the set of irrational rotations of the unit circle S^1, for which f is a coboundary with an integrable transfer function, is of first category.

2.1. *The Halász–Schmidt condition.*

The following associated condition for measurable dynamics can be found in [25, 40]. A measurable function f is a coboundary for $T \in \mathcal{E}$ if and only if for each $\delta > 0$, there exists $M_\delta \in \mathbb{N}$ such that for $n \in \mathbb{N}$,

$$\mu\left(\left\{x \in X : \left| \sum_{i=0}^{n-1} f(T^i x) \right| \leq M_\delta \right\}\right) > 1 - \delta. \tag{2.1}$$

This condition will be used in §5 to show for any measurable function f that is essentially non-zero, then the class of ergodic invertible measure-preserving transformations T such that $f = g - g \circ T$ has a measurable solution g is meager (first category). Anosov

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[4, Theorem 1] demonstrated that there are no measurable solutions g in the case that f is integrable and $\int_X f\, d\mu \neq 0$. However, our category results apply in the situation that $\int_X f\, d\mu = 0$.

2.2. *Non-measurable solutions.* Using the axiom of choice, we can always obtain a solution g. Partition X into orbits (mod measure zero). For each orbit $\mathcal{O}$, choose a single point $x_0 \in \mathcal{O}$. The coboundary equation leads to the following telescoping series, for $n > 0$,

$$g(T^n x) = g(x) - \sum_{i=0}^{n-1} f(T^i x),$$

and for backward iterates,

$$g(T^{-n} x) = g(x) + \sum_{i=1}^{n} f(T^{-i} x).$$

If we define $g(x_0) = 0$, then the recursion formulas above uniquely determine g at all points along the orbit of x_0. It is easily checked that $f(y) = g(y) - g(Ty)$ on the orbit of x_0. By invoking the Axiom of Choice to select a point on each orbit, g is defined at a.e. $x \in X$. However, the result of Anosov implies this g is not measurable when f has a non-zero integral.

Here is another case where this construction clearly leads to a non-measurable solution. Suppose α is irrational and $0 < \alpha < 1$. Define f on [0, 1] by

$$f(x) = \begin{cases} \alpha, & \text{if } x \leq \dfrac{1}{1+\alpha}, \\ -1 & \text{if } x > \dfrac{1}{1+\alpha}. \end{cases}$$

The integral of f is zero. Since $g(x) = 0$ for a single point in each orbit, then the space X equals the following disjoint union (modulo measure zero sets),

$$\bigcup_{i=-\infty}^{\infty} T^i(\{x \in X : g(x) = 0\}).$$

Since T is measure preserving, the set $\{x \in X : g(x) = 0\}$ is not measurable and consequently, g is not measurable.

There are cases where it is known that the coboundary equation has no measurable solution g. It was pointed out in [25] that if f is a non-trivial mean-zero step function taking on two values, then the transformation T must have a non-trivial eigenvalue. Thus, if T is weakly mixing and f is a two-step function, there is no measurable solution g. This implies for a two-step mean-zero non-zero function, the ergodic invertible measure preserving transformation obtained in Theorem 1.1 is never weakly mixing.

In [24], given an irrational rotation T, the authors give necessary and sufficient conditions for a step function ϕ and $t \in \mathbb{R}/\mathbb{Z}$ for there to exist measurable solutions to the multiplicative cohomological equation:

$$e^{2\pi i \phi} = e^{2\pi i t} \frac{f}{f \circ T}.$$

This result is critical in the study of eigenvalues of interval exchange transformations.

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2.3. *Bounded coboundaries.* This raises the question of when do solutions exist for classes of measurable functions f, when T is allowed to range over $\mathcal{E}$. In [2] (Theorem 11.2), it is shown that any finite step, mean-zero function is a coboundary for some ergodic invertible measure-preserving transformation with a bounded transfer function g. In particular, T may be chosen in one of the following categories:

(1) T is a transformation with a discrete spectrum;
(2) T is a product of rotations;
(3) T is a finite extension of a product of rotations.

Also, Theorems 11.2 and 12.1 in [2] show the existence of solutions is extended to mean-zero bounded functions. The case of general L_0^p functions is more subtle and addressed in this paper.

The paper [32] partially addresses the case of bounded coboundaries. However, the arguments given in [32] are viewed as containing a gap, and the main theorem does not apply in general beyond the case of continuous functions f.

2.4. *Operator viewpoint.* The coboundary equation has been viewed from the perspective of operator theory. Note that the coboundary equation may be written as

$$f = (I - U_T)g,$$

where U_T is the Koopman operator defined by $U_T(g) = g \circ T$, and I represents the identity operator. Study of the operator $(I - T)$ when T is a linear operator (and not necessarily unitary) goes back to the 19^{th} century [35]. Similar to the case of real or complex numbers, for an operator T with norm $|T| < 1$, then $I - T$ has an inverse and

$$(I - T)^{-1} = \sum_{i=0}^{\infty} T^k.$$

However, for measure-preserving transformations, $|U_T| = 1$, and solving $f = (I - U_T)g$ becomes more complicated. Browder [8] provided the following equivalent condition for a given contraction T on a reflexive Banach space E. The function $f \in (I - T)E$, if and only if

$$\sup_n \left\| \sum_{i=0}^{n-1} T^i f \right\| < \infty.$$

A two-dimensional version of Browder's result was proved by Cohen and Lin in [10].

Iterative techniques were given in [13–15, 23] as an aid for solving the coboundary equation in this setting. The paper [33] of Lin and Sine shows that for a given T, when a solution exists, it may be obtained in closed form as the following point-wise limit almost everywhere:

$$g(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \sum_{i=0}^{n-1} f(T^i x).$$

Also, the authors extend their results from the classical Poisson equation, $f = (I - U_T)g$ to the case of fractional coboundaries [12]. Their main results produce equivalent conditions for solutions to occur for fixed T. Also, in the case of a unitary operator U on a Hilbert space H, a result from [31] shows that $f \in (I - U)H$, if and only if

$$\sup_N \frac{1}{N} \sum_{n=1}^{N} \left\| \sum_{k=0}^{n-1} U^k f \right\|^2 < \infty.$$

Also, of note, the authors prove in [16] that the following condition is equivalent to T being weak mixing:

$$\frac{1}{N} \sum_{n=1}^{N} \left\| \sum_{k=0}^{n-1} (\mathbb{1}_A - \mu(A)) \circ T^k \right\|_2 \longrightarrow \infty \quad \text{for all } A \in \mathcal{B} \text{ with } \mu(A) > 0.$$

Our main results can be recast in terms of operators in the following way.

COROLLARY 2.1. (Operator theoretic statement of Theorems 1.1 and 1.2) *Let $(X, \mathcal{B}, \mu)$ be a standard probability space and $\mathcal{E}$ be the set of all ergodic invertible measure-preserving transformations on $(X, \mathcal{B}, \mu)$. Then Theorems 1.1 and 1.2 are equivalent to the following statements respectively:*

$$L_0^p \subset \bigcup_{T \in \mathcal{E}} (I - U_T)(L^{p-1})$$

and

$$L_0^p \cap \bigcup_{T \in \mathcal{E}} \bigcup_{q > p-1} (I - U_T)(L^q) \textit{ is meager in } L_0^p.$$

2.5. *Ergodic averages.* One of the main applications of coboundary solutions is to find functions for which the ergodic averages are controlled and converge rapidly. Kachurovskii observed in [26] that the cohomology equation has a solution if and only if the rate of convergence in the Birkhoff theorem [7] is the highest possible rate and convergence is uniform. We observe, in the case where f is a coboundary for T with integrable transfer function g, all moving averages (v_n, L_n) converge pointwise for an increasing sequence $L_n \in \mathbb{N}$,

$$\frac{1}{L_n} \sum_{i=1}^{L_n} f(T^{v_n+i} x) \to 0.$$

A proof of this fact is given in Appendix B. Other results [44] characterize the rate of convergence in measure of ergodic averages of L^∞ functions using uniform approximation by coboundaries where the transfer function lands in a specific L^p space. For $p \geq 1$, the rate is of the order of n^{-p}. For stationary processes exhibiting randomness (e.g., positive entropy, random fields), there is a technique for decomposing the process into coboundary and martingale components. See [18–20, 29, 43] and the references contained therein for background on this technique and its applications. This has made it possible to establish common statistical laws (central limit theorem, weak invariance principle) in these cases.

2.6. *Non-singular transformations.* There is also extensive research on the connections of coboundaries to non-singular transformations. We do not discuss this in detail, but

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encourage the interested reader to check [1, 11] for its connections, including the existence of equivalent finite or sigma-finite invariant measures. Techniques developed from the operator theoretic viewpoint can also be applied to non-singular transformations including [33], and [3], which develops a different approach for addressing L^p integrability constraints on solutions.

3. *Coboundary existence theorem*

In this section, we prove Theorem 1.1, although it is restated here in an equivalent form. We will also show later that this is generally the best possible result.

THEOREM 3.1. *Let $p \in \mathbb{R}$ be such that $p \geq 1$. Suppose $(X, \mathcal{B}, \mu)$ is a standard probability space and $f \in L_0^p(X, \mu)$. There exist an ergodic invertible measure-preserving transformation T and a function $g \in L^{p-1}(X, \mu)$ such that $f(x) = g(Tx) - g(x)$ for almost every $x \in X$.*

For the case of L^∞, this theorem follows from Theorems 11.2 and 12.1 in [2]. However, [2] did not handle unbounded functions.

To prove the existence result for unbounded functions, we will give a construction which reduces the problem to bounded functions. A geometric tower of sets is constructed based loosely on the method of cutting and stacking. (The proof of Theorem 1.2 is not based on cutting and stacking, although it is constructive in nature.) See [17] for a guide to the cutting and stacking method in ergodic theory. Also, we will use a connection between coboundaries for an induced transformation and the coboundary for the full transformation. In addition to looking at induced transformations, we obtain our limiting transformation by iteratively gluing together an ensemble of transformations defined on subsets (towers) of the full space.

3.1. *Induced transformations and coboundaries.*

In this section, we show how to extend a coboundary for an induced transformation to a coboundary for the full transformation. Let $T : X \to X$ be an ergodic measure-preserving transformation. Let $A \subset X$ be a set of positive measure. Suppose

$$T_A(x) = T^{n_A(x)}(x), \, x \in A,$$

where $n_A(x)$ is the smallest positive integer n such that $T^n x \in A$. Note, $n_A(x)$ is defined (and finite) for a.e. $x \in A$ since the measure μ is not atomic. This defines the so-called induced transformation on A. See [17, p. 9, Theorem 1.18] or [36] for further details on induced transformations. Given measurable function $f : X \to \mathbb{R}$ and $x \in A$, define

$$f_A(x) = \sum_{i=0}^{n_A(x)-1} f(T^i x).$$

We have the following lemma which will be used to prove Theorem 3.1.

LEMMA 3.2. *Let $f : X \to \mathbb{R}$ be a measurable function and $T : X \to X$ be an ergodic measure-preserving invertible transformation. Suppose f_A is a coboundary for induced transformation $T_A : A \to A$ with transfer function g_A such that $f_A = g_A \circ T_A - g_A$.*

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Given $z \in X \setminus A$, define $j_z = \inf\{j \in \mathbb{N} : T^{-j}z \in A\}$ and for $z \in A$, $j_z = 0$. Since T is ergodic and invertible, then $j_z < \infty$ for a.e. $z \in X$ and in particular, $0 \le j_z \le n_A(T^{-j_z}z)$ almost everywhere. Then f is a coboundary for transformation T with transfer function g defined such that for $z \in A$, $g(z) = g_A(z)$ and for a.e. $z \in X \setminus A$,

$$g(z) = g_A(T^{-j_z}z) + \sum_{i=0}^{j_z-1} f(T^{i-j_z}z).$$

Proof. For a.e. $z \in X$, let $x = T^{-j_z}z$. If $0 \le j_z < n_A(x) - 1$, then

$$g(Tz) - g(z) = g_A(x) + \sum_{i=0}^{j_z} f(T^i x) - g_A(x) - \sum_{i=0}^{j_z-1} f(T^i x) = f(T^{j_z}x) = f(z).$$

Now suppose $j_z = n_A(x) - 1$. Then

$$g(Tz) - g(z) = g_A(T_A x) - \left(g_A(x) + \sum_{i=0}^{n_A(x)-2} f(T^i x)\right) \tag{3.1}$$

$$= f_A(x) - \sum_{i=0}^{n_A(x)-2} f(T^i x) \tag{3.2}$$

$$= \sum_{i=0}^{n_A(x)-1} f(T^i x) - \sum_{i=0}^{n_A(x)-2} f(T^i x) \tag{3.3}$$

$$= f(T^{n_A(x)-1}x) = f(z). \tag{3.4}$$

This proves that f is a coboundary for T with transfer function g for a.e. $z \in X$. □

3.2. *Tower constructions on subsets of the measure space.* To construct the transformation and transfer function, we will first decompose the measure space into disjoint subsets and then construct towers on each subset. Using the construction from a previous paper [2], we will define the transformation on the top of the towers such that the full transformation is ergodic measure preserving and invertible (that is, it glues all the towers together). This construction will produce a transfer function with the required integrability properties.

Suppose $f : X \to \mathbb{R}$ is a measurable function and $A \subset X$ is a measurable subset. Define $f_{|A} : A \to \mathbb{R}$ as the restriction of f to A:

$$f_{|A}(x) = f(x) \quad \text{for } x \in A.$$

For convenience, we may write $F = f_{|A}$. To emphasize that a transformation is restricted to a subset A, we use the notation $T_{|A} : A \to A$ or, for convenience, $\tau : A \to A$. This is distinct from the notion of induced transformation which is written as T_A. Now we prove the following lemma which is a basic building block for the construction of our full transformation T.

LEMMA 3.3. *Suppose $A \subset X$ has positive measure and $f_{|A} : A \to \mathbb{R}$ takes on two steps with mean zero (that is, $\int_A F\, d\mu = 0$, where $F = f_{|A}$). Given $h \in \mathbb{N}$ and $\epsilon > 0$, there*

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exist $h_1, h_2 > h$*, disjoint* $I_1, I_2 \subseteq A$*, and an invertible measure-preserving map* $\tau : A \to A$ *such that*

$$\mu\Big(\bigcup_{i=0}^{h_1-1} \tau^i I_1 \cup \bigcup_{i=0}^{h_2-1} \tau^i I_2\Big) = \mu(A), \tag{3.5}$$

$$\tau^i I_1, 0 \le i < h_1, \quad \tau^i I_2, 0 \le i < h_2 \quad \textit{are all disjoint}, \tag{3.6}$$

$$\Big|\sum_{i=0}^{k} F(\tau^i x)\Big| \le \|F\|_\infty \quad \textit{for } x \in I_j, k < h_j, j = 1, 2, \tag{3.7}$$

$$\Big|\sum_{i=0}^{h_j-1} F(\tau^i x)\Big| < \epsilon \quad \textit{for } x \in I_j, j = 1, 2, \tag{3.8}$$

$$\sum_{i=0}^{h_j-1} F(\tau^i x) = \sum_{i=0}^{h_j-1} F(\tau^i y) \quad \textit{for } x, y \in I_j, j = 1, 2, \tag{3.9}$$

$$1 - \epsilon < \frac{h_1}{h_2} < 1 + \epsilon. \tag{3.10}$$

Proof. Without loss of generality, assume $A \subset X = [0, 1]$. Suppose $F = bI_B - cI_C$ is mean zero for $b, c > 0$ and disjoint B, C such that $B \cup C = A$. The case where b/c is rational is straightforward, so we assume b/c is irrational. There exist δ_1, δ_2 of the same sign, and p_1, q_1, p_2, q_2 such that $|\delta_2| < |\delta_1| < \epsilon$, $p_1 < \epsilon p_2$, $q_1 < \epsilon q_2$, $p_2 + q_2 - p_1 - q_1 > h$, $p_1 b - q_1 c = \delta_1$, and $p_2 b - q_2 c = \delta_2$. Without loss of generality, assume $0 < \delta_2 < \delta_1 < \epsilon$. The case where δ_1, δ_2 are negative follows similarly. Let $p_3 = p_2 - p_1$ and $q_3 = q_2 - q_1$. Note,

$$p_3 b - q_3 c = \delta_2 - \delta_1 < 0.$$

Let $\delta_3 = \delta_1 - \delta_2$. Split B into two disjoint sets B_1, B_2 such that

$$\mu(B_1) = \frac{p_2\delta_3}{(p_2+q_2)\delta_3 + (p_3+q_3)\delta_2} \quad \text{and} \quad \mu(B_2) = \frac{p_3\delta_2}{(p_2+q_2)\delta_3 + (p_3+q_3)\delta_2}. \tag{3.11}$$

Note,

$$\mu(B_1) + \mu(B_2) = \frac{p_2\delta_3 + p_3\delta_2}{(p_2+q_2)\delta_3 + (p_3+q_3)\delta_2} \tag{3.12}$$

$$= \frac{p_2(q_3c - p_3b) + p_3(p_2b - q_2c)}{(p_2+q_2)(q_3c - p_3b) + (p_3+q_3)(p_2b - q_2c)} \tag{3.13}$$

$$= \frac{(p_2q_3 - p_3q_2)c}{(p_2q_3 - p_3q_2)b + (p_2q_3 - p_3q_2)c} \tag{3.14}$$

$$= \frac{c}{b+c} = \mu(B). \tag{3.15}$$

Similarly, split $C = C_1 \cup C_2$ such that

$$\mu(C_1) = \frac{q_2\delta_3}{(p_2+q_2)\delta_3 + (p_3+q_3)\delta_2} \quad \text{and} \quad \mu(C_2) = \frac{q_3\delta_2}{(p_2+q_2)\delta_3 + (p_3+q_3)\delta_2}. \tag{3.16}$$

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Divide B_1 into p_2 disjoint equimeasurable sets $B_{1,j}$ for $j \in \{1, 2, \ldots, p_2\}$. Divide C_1 into q_2 disjoint equimeasurable sets $C_{1,j}$ for $j \in \{1, 2, \ldots, q_2\}$. Divide B_2 into p_3 disjoint equimeasurable sets $B_{2,j}$ for $j \in \{1, 2, \ldots, p_3\}$. Divide C_2 into q_3 disjoint equimeasurable sets $C_{2,j}$ for $j \in \{1, 2, \ldots, q_3\}$. Thus, $\mu(C_{1,j}) = \mu(B_{1,k})$ for $j \in \{1, 2, \ldots, p_2\}$ and $k \in \{1, 2, \ldots, q_2\}$. Also, $\mu(C_{2,j}) = \mu(B_{2,k})$ for $j \in \{1, 2, \ldots, p_3\}$ and $k \in \{1, 2, \ldots, q_3\}$. Let $I_1 = B_{1,1}$ and $I_2 = B_{2,1}$. Stack the sets $B_{1,j}$ and $C_{1,k}$ such that whenever the sum of the values is negative, place a B next, and otherwise place a C set next. Stack the sets $B_{2,j}$ and $C_{2,k}$ such that whenever the sum of the values is negative, place a B next, and otherwise place a C set next. As long as $\delta_1 < \min\{c, b\}/2$, then we have the precise number of level sets B and C to complete the two towers. In this case, let $h_1 = p_2 + q_2$ and $h_2 = p_3 + q_3$. Thus, the condition $h_2 = p_3 + q_3 = p_2 - p_1 + q_2 - q_1 > h$ implies $h_1 = p_2 + q_2 > h$.

For $x \in I_1$,

$$\left| \sum_{i=0}^{h_1-1} F(\tau^i x) \right| = |p_2 b - q_2 c| = \delta_2 < \epsilon,$$

and for $x \in I_2$,

$$\left| \sum_{i=0}^{h_2-1} F(\tau^i x) \right| = |p_3 b - q_3 c| = \delta_1 - \delta_2 < \epsilon.$$

Equation (3.7) holds due to the greedy stacking algorithm used. The other conditions in the lemma hold by construction. □

LEMMA 3.4. *Suppose $A \subset X$ is a set of positive measure and $F : A \to \mathbb{R}$ is a mean-zero, non-zero finite step function. Explicitly, let $F = \sum_{i=1}^{m} a_i \mathbb{1}_{I_i}$, where $A = \bigcup_{i=1}^{m} I_i$ is a disjoint union of sets I_i of positive measure, and a_i are distinct real numbers for $1 \le i \le m$ and $m \ge 2$. There exist disjoint measurable sets $J_1, J_2, \ldots, J_{m-1}$ such that F takes on at most two values almost everywhere on J_i and $\int_{J_i} F\, d\mu = 0$ for $1 \le i \le m-1$.*

Proof. We prove this by induction on m. Clearly, this is true for $m = 2$. Suppose it is true for $m = n$ and all finite measure spaces. Let $m = n + 1$. Choose j such that for $1 \le i \le n + 1$,

$$\int_{I_j} |F|\, d\mu = |a_j| \mu(I_j) \le \int_{I_i} |F|\, d\mu = |a_i| \mu(I_i).$$

If $a_j \le 0$, choose $k \ne j$ such that $a_k \ge 0$, otherwise choose k such that $a_k \le 0$. Choose $I' \subset I_k$ such that

$$a_j \mu(I_j) + a_k \mu(I') = 0.$$

Define $J_n = I_j \cup I'$. Thus, F takes on at most n steps on the subset $A \setminus J_n$. By induction, there exist $J_1, J_2, \ldots, J_{n-1}$ such that F takes on at most two steps on J_i. Therefore, our lemma is proved by induction. □

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The next lemma uses the notion of a TUB tower which was defined in [2]. For completeness, we present the definition here. The proof of the following Lemma 3.5 uses Lemma A.2 which is stated and proved in Appendix A.2.

Let A be a measurable subset of X and $f : A \to \mathbb{R}$ a bounded, mean-zero function. Given a finite measurable partition Q, $h \in \mathbb{N}$, and $\epsilon > 0$, an ϵ-balanced and uniform tower for f is a set of disjoint measurable sets $I_i \subset A$ for $i = 1, 2, \ldots, h$ and an invertible measure-preserving map $T : I_i \to I_{i+1}$ for $i = 1, 2, \ldots, h - 1$, such that

$$\mu\bigg(\bigcup_{i=1}^{h} I_i\bigg) > \mu(A) - \epsilon, \tag{3.17}$$

$$|f(x) - f(y)| < \epsilon \quad \text{for } x, y \in I_i,\ 1 \le i < h, \tag{3.18}$$

$$\bigg|\sum_{i=0}^{k} f(T^i x)\bigg| < \|f\|_\infty + \epsilon \quad \text{for } x \in I_1,\ k < h, \tag{3.19}$$

$$\sum_{i=1}^{h} \int_{I_i} f\, d\mu = \int_A f\, d\mu, \tag{3.20}$$

$$\bigg|\sum_{i=0}^{h-1} f(T^i x)\bigg| < \epsilon \quad \text{for } x \in I_1, \tag{3.21}$$

$$\text{for each } q \in Q, \text{ there exists } \mathcal{I} \subset \{1, \ldots, h\} \text{ such that } \mu\bigg(q \triangle \bigg(\bigcup_{i \in \mathcal{I}} I_i\bigg)\bigg) < \epsilon. \tag{3.22}$$

We refer to this type of tower as a TUB (ϵ, h, Q) tower for $f_{|A}$.

LEMMA 3.5. *Suppose $A \subset X$ is a set of positive measure and $F : A \to \mathbb{R}$ is bounded and mean zero. Given $\epsilon_i > 0$ for $i \in \mathbb{N}$ such that $\lim_{i\to\infty} \epsilon_i = 0$, there exist an invertible measure-preserving map T, disjoint sets $I_i \subset A$ and natural numbers h_i such that:*

- $A = \bigcup_{i=1}^{\infty} \bigcup_{j=0}^{h_i-1} T^j I_i$ *is a disjoint union;*
- $|\sum_{j=0}^{h_i-1} F(T^j x)| < \epsilon_i$ *for* $x \in I_i$*; and*
- $|\sum_{j=0}^{k} F(T^j x)| < \|F\|_\infty + \epsilon_i$ *for* $x \in I_i$ *and* $0 \le k < h_i$.

Proof. If F is a finite step function, then the lemma follows by applying Lemmas 3.4 and 3.3 with a finite number of sets J_i. If F is not a finite step function, then we apply Lemma A.2, iteratively and potentially infinitely many times, to construct a sequence of TUB towers that satisfy this lemma. By Lemma A.2, there exist $I_1 \subset A$, $h_1 \in \mathbb{N}$ and T which is invertible and measure preserving such that $T^i(I_1) \subset A$ are disjoint for $i = 0, 1, \ldots, h_1 - 1$ and $\mu(\bigcup_{i=0}^{h_1-1} T^i(I_1)) > \mu(A) - \epsilon_1$. Also, using Lemma A.2, we can obtain the last two conditions of Lemma 3.5. Let $A_1 = A \setminus \bigcup_{i=0}^{h_1-1} T^i(I_1)$. In a similar fashion, apply Lemma A.2 to A_1 using parameter ϵ_2. Thus, in general,

$$A_k = A \setminus \bigcup_{j=1}^{k} \bigcup_{i=0}^{h_j-1} T^i(I_j)$$

with $\mu(A_k) < \epsilon_k \to 0$ as $k \to \infty$ and our lemma is satisfied. □

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Remark. The transformation T constructed in Lemma 3.5 does not need to be defined on all of A. It may not be defined on the top of the towers, and instead,

$$T : A \setminus \left(\bigcup_{i=1}^{\infty} T^{h_i - 1} I_i \right) \to A \setminus \left(\bigcup_{i=1}^{\infty} I_i \right).$$

The following proposition was previously proved in [2]. A direct proof is given in Appendix A.2.

PROPOSITION 3.6. *Let $(X, \mathcal{B}, \mu)$ be a standard probability space and $A \subset X$ a set of positive measure. Suppose $f : A \to \mathbb{R}$ is measurable, mean zero, and bounded. There exist an ergodic measure-preserving transformation $T : A \to A$ and bounded function g such that $f = g - g \circ T$ almost everywhere. Moreover, the transformation T and transfer function g may be constructed such that for any $\delta > 0$, $\|g\|_\infty < \|f\|_\infty + \delta$.*

3.3. *Proof of the main positive result.* Now we are ready to proceed with the proof of Theorem 3.1.

Proof. Without loss of generality, we prove this theorem for the case $X = [0, 1)$ and μ equal to Lebesgue measure. Also, we may assume $f \notin L^\infty$, since this case was handled previously in [2], Proposition 3.6, and also by Ber *et al.* [6]. If f does not take on essentially infinitely many bounded values on a compact set, then first apply Lemma 3.3 to generate countable towers and transformation such that the sums are bounded, that is, less than ϵ_i for the ith tower. Let k be the minimum positive integer such that $\mu(\{x : 0 < f(x) \leq k\}) > 0$, and similarly let ℓ be the minimum positive integer such that $\mu(\{x : 0 > f(x) \geq -\ell\}) > 0$. If no such k and no such ℓ exist, then f must equal zero almost everywhere, which contradicts our assumption. Let $X_1 = \{x : k - 1 < f(x) \leq k\}$ and $Y_1 = \{x : 1 - \ell > f(x) \geq -\ell\}$. If $\int_{X_1} f\, d\mu + \int_{Y_1} f\, d\mu \leq 0$, define $Y_1' \subseteq Y_1$ such that

$$\int_{X_1} f\, d\mu + \int_{Y_1'} f\, d\mu = 0.$$

In this case, let $X_1' = X_1$. Otherwise, choose $X_1' \subset X_1$ such that

$$\int_{X_1'} f\, d\mu + \int_{Y_1} f\, d\mu = 0.$$

In this case, set $Y_1' = Y_1$. Also, define $k_1 = k$, $\ell_1 = \ell$, and $X_0 = \{x : f(x) = 0\}$. We may continue this procedure inductively to choose disjoint sets X_n', Y_n' for $n = 1, 2, \ldots,$ and sequences of positive integers k_n, ℓ_n such that:

(1) $k_n - 1 < f(x) \leq k_n$ for $x \in X_n'$;
(2) $1 - \ell_n > f(x) \geq -\ell_n$ for $x \in Y_n'$;
(3) $k_{n+1} \geq k_n$, $\ell_{n+1} \geq \ell_n$;
(4) $\lim_{n\to\infty} k_n + \ell_n = \infty$;
(5) $\int_{X_n'} f\, d\mu + \int_{Y_n'} f\, d\mu = 0$;
(6) $\mu(\bigcup_{n=1}^{\infty}(X_n' \cup Y_n')) = \mu(X \setminus X_0)$.

Item (4) above is true, since $f \notin L^\infty$. Either $k_n \to \infty$ or $\ell_n \to \infty$. Assume without loss of generality that $k_n \to \infty$ and $k_n > \ell_n$ for infinitely many $n \in \mathbb{N}$. If $\ell_n = 1$ for all $n \in \mathbb{N}$, then we need to be careful about the choice of Y'_n. Let $\gamma_0 = 0$. We can choose a sequence of non-decreasing real numbers γ_n, $n \in \mathbb{N}$ such that $0 < \gamma_n \leq 1$ and

$$Y'_n \subseteq \{x : -\gamma_n \leq f(x) \leq -\gamma_{n-1}\}.$$

If ℓ_n is eventually greater than 1, then we can set $\gamma_n = \ell_n$ for $n \in \mathbb{N}$. In either case, for $x \in Y'_n$,

$$|f(x)| \leq \gamma_n.$$

Let $\epsilon > 0$. For each $n \in \mathbb{N}$, apply Lemma 3.5 to f defined on $Z_n = X'_n \cup Y'_n$ to obtain a decomposition into potentially infinitely many towers satisfying the conditions of Lemma 3.5. Thus, there exists an invertible measure-preserving T_n defined on a subset of Z_n such that

$$Z_n = \bigcup_{i=1}^{\infty} \bigcup_{j=0}^{h_{n,i}-1} T_n^j I_{n,i}.$$

Using Lemma 3.5, we can require

$$\left| \sum_{j=0}^{h_{n,i}-1} f(T_n^j x) \right| < \epsilon,$$

and for $0 \leq k < h_{n,i}$,

$$\left| \sum_{j=0}^{k} f(T_n^j x) \right| < \|f_{|Z_n}\|_\infty + \epsilon.$$

Since Z_n are disjoint, we can use T to represent the ensemble of T_n. Let

$$A = \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{\infty} I_{n,i}$$

and define the function f_A by

$$f_A(x) = \sum_{j=0}^{h_{n,i}-1} f(T^j x) \quad \text{for } x \in I_{n,i}.$$

Thus, $|f_A(x)| < \epsilon$ for $x \in A$. Apply Proposition 3.6 to construct an ergodic invertible measure-preserving transformation $\tau : A \to A$ along with a bounded measurable function g_A such that f_A is a coboundary for τ with transfer function g_A. Moreover, we can require that $|g_A(x)| < \|f_A\|_\infty + \epsilon < 2\epsilon$ for $x \in A$. Then apply Lemma 3.2 to show that the ergodic measure-preserving transformation T on X satisfying $T_A = \tau$ has coboundary f with a measurable transfer function g. By Lemma 3.2, g may be bounded such that for

$z \in Z_n$,

$$|g(z)| \leq |g_A(T^{-j_z}z)| + \left|\sum_{i=0}^{j_z-1} f(T^{i-j_z}z)\right| \tag{3.23}$$

$$< 2\epsilon + \|f_{|Z_n}\|_\infty + \epsilon \tag{3.24}$$

$$< \max\{k_n, \ell_n\} + 3\epsilon. \tag{3.25}$$

There exists j such that $k_j \geq \ell_j$, and for $n > j$, $k_n + 3\epsilon \leq 2(k_n - 1)$, and also for $x \in Y'_n$,

$$-\gamma_n \leq f(x) \leq -\gamma_{n-1} < 0.$$

Thus,

$$\int_X |g|^{p-1}\, d\mu = \sum_{n=1}^{\infty} \int_{Z_n} |g|^{p-1}\, d\mu \tag{3.26}$$

$$= \sum_{n=1}^{j} \int_{Z_n} |g|^{p-1}\, d\mu + \sum_{n=j+1}^{\infty} \int_{Z_n} |g|^{p-1}\, d\mu. \tag{3.27}$$

Since $|g|^{p-1}$ is bounded on Z_n for $n \leq j$, then $\sum_{n=1}^{j} \int_{Z_n} |g|^{p-1}\, d\mu < \infty$. For each $n \in \mathbb{N}$, let $m_n = \max\{k_n, \gamma_n\}$. If $m_n = k_n$, let $V_n = X'_n$ and $W_n = Y'_n$. Otherwise, let $V_n = Y'_n$ and $W_n = X'_n$. Thus, for $n > j$,

$$\int_{V_n} |f|\, d\mu \leq \int_{V_n} (m_n)\, d\mu \tag{3.28}$$

$$= (m_n)\mu(V_n) \tag{3.29}$$

$$= \frac{(m_n)\mu(V_n)}{(\gamma_j)\mu(W_n)} (\gamma_j)\mu(W_n) \tag{3.30}$$

$$\leq \frac{(m_n)\mu(V_n)}{(\gamma_j)\mu(W_n)} \int_{W_n} |f|\, d\mu. \tag{3.31}$$

This implies

$$\frac{\mu(W_n)}{m_n} \leq \frac{\mu(V_n)}{\gamma_j}.$$

Thus,

$$\begin{aligned}
\sum_{n=j+1}^{\infty} \int_{Z_n} |g|^{p-1}\, d\mu &= \sum_{n=j+1}^{\infty} \int_{V_n} |g|^{p-1}\, d\mu + \sum_{n=j+1}^{\infty} \int_{W_n} |g|^{p-1}\, d\mu \\
&\leq \sum_{n=j+1}^{\infty} \int_{V_n} (m_n + 3\epsilon)^{p-1}\, d\mu + \sum_{n=j+1}^{\infty} \int_{W_n} (m_n + 3\epsilon)^{p-1}\, d\mu \\
&\leq \sum_{n=j+1}^{\infty} \int_{V_n} (m_n + 3\epsilon)^{p-1}\, d\mu + \sum_{n=j+1}^{\infty} \int_{V_n} (m_n + 3\epsilon)^{p} \frac{1}{\gamma_j}\, d\mu \\
&= \sum_{n=j+1}^{\infty} \int_{V_n} (m_n - 1)^{p-1} \frac{(m_n + 3\epsilon)^{p-1}}{(m_n - 1)^{p-1}}\, d\mu
\end{aligned}$$

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$$+\frac{1}{\gamma_j}\sum_{n=j+1}^{\infty}\int_{V_n}(m_n-1)^p\frac{(m_n+3\epsilon)^p}{(m_n-1)^p}\,d\mu$$

$$\leq 2^{p-1}\sum_{n=j+1}^{\infty}\int_{V_n}(m_n-1)^{p-1}\,d\mu+\frac{2^p}{\gamma_j}\sum_{n=j+1}^{\infty}\int_{V_n}(m_n-1)^p\,d\mu$$

$$\leq 2^{p-1}\sum_{n=j+1}^{\infty}\int_{V_n}|f|^{p-1}\,d\mu+\frac{2^p}{\gamma_j}\sum_{n=j+1}^{\infty}\int_{V_n}|f|^p\,d\mu$$

$$\leq 2^{p-1}\|f\|_{p-1}^{p-1}+\frac{2^p}{\gamma_j}\|f\|_p^p<\infty.$$

This completes the proof that $g \in L^{p-1}(X)$. □

4. *Non-existence of L^q-coboundaries*

In [30], Kornfeld shows that given $T \in \mathcal{E}$, which is a homeomorphism on a compact space X, there exists a continuous and bounded coboundary f such that its associated transfer function is measurable, but not integrable. Also, it is pointed out that given T, f may be constructed such that the transfer function g is in L^p for specified $p \geq 1$, but not contained in L^q for $q > p$. However, if the function $f \in L_0^p$ is specified first, Kornfeld conjectured that there always exist an ergodic invertible measure-preserving transformation T and $g \in L^p$ such that $f = g - g \circ T$ almost everywhere. (Kornfeld conveyed this conjecture to the second author verbally or through email.) In this section, we disprove this conjecture. Furthermore, we prove a strong non-existence result showing that for generic $f \in L_0^p$, there are no $T \in \mathcal{E}$ and $g \in L^q$ for $q > p - 1$ such that $f = g - g \circ T$ almost everywhere. This is the statement of Theorem 1.2, and shows that generic L_0^p functions lead to 'wild' transfer functions (as termed in [30]), universally for all $T \in \mathcal{E}$. Remark 2 in [32] provides an argument for the existence of L_0^p functions f for $p \geq 2$ which are not coboundaries for any ergodic measure-preserving transformation T with transfer function $g \in L^p$. The argument in [32] can be extended to show there are functions $f \in L_0^p$ which are not coboundaries for any ergodic measure-preserving transformation T with transfer function $g \in L^q$ for $q > p - 1$. This is proved in the following section. Then, in §4.2, we show this situation is generic for $f \in L_0^p$.

4.1. *Extension of the Kwapień argument for the non-existence of L^q-coboundaries.*

The following proposition establishes the existence of L^p functions f with no transfer function in L^q for $q > p - 1$. The argument is due to Kwapień [32].

PROPOSITION 4.1. (Remark 2 in [32]) *Given $p \in \mathbb{R}$ such that $p \geq 2$, there exists $f \in L^p$ such that for any solution pair T and g to the equation $f = g - g \circ T$, where T is an ergodic invertible measure-preserving transformation, then $g \notin L^q$ for $q > p - 1$.*

Proof. Let $f \in L^p$ be such that $\int f\,d\mu = 0$, $f(x) \geq -1$ for a.e. x, and for $r > p - 1$,

$$\limsup_{n\to\infty}\left(n\int_{f>n}|f-n|^r\right)d\mu=\infty.$$

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We refer to this as the Kwapień condition. To obtain examples f that satisfy the Kwapień condition, suppose $p \geq 2$, $r > p - 1$, and let $N_k \in \mathbb{N}$ be such that

$$\sum_{k=1}^{\infty} N_k^{-(1+r-p)/2(r+1)} < \frac{1}{2^{p+1}}. \tag{4.1}$$

Let

$$\delta = \frac{1+r-p}{2(r+1)}.$$

By (4.1),

$$\lim_{k\to\infty} N_k^{(1+r-p)/2} = \lim_{k\to\infty} N_k^{\delta(r+1)} = \infty.$$

Let E_k be disjoint sets for $k \in \mathbb{N}$ such that $\mu(E_k) = 1/N_k^p$. Thus,

$$\sum_{k=1}^{\infty} N_k^{-p(1+r-p)/2(r+1)} < 1/2^{p+1}.$$

Define f^+ such that

$$f^+ = \sum_{k=1}^{\infty} 2N_k^{1-\delta} I_{E_k}.$$

We have shown that $\int f^+ \, d\mu < 1/2$. Let E_0 be a subset disjoint from $\bigcup_{k=1}^{\infty} E_k$ such that $\mu(E_0) = \int f^+ \, d\mu$. Define $f = f^+ - I_{E_0}$. Thus, $\int f \, d\mu = 0$ and $\|f\|_p < \infty$.

Let $L_k = n_k \int_{f > n_k} (f - n_k)^r \, d\mu$, where $n_k = N_k^{1-\delta}$. This n_k is not a whole number probably, but we are going to ignore that. Then $L_k \geq N_k^{1-\delta} \int_{E_k} (N_{k_o}^{1-\delta})^r \, d\mu$. We get

$$L_k \geq N_k^{(r+1)(1-\delta)}/N_{k_o}^p = N_k^{(1+r+p)/2}/N_k^p = N_k^{(r+1-p)/2}.$$

Since $\lim_{k\to\infty} L_k = \infty$ and f satisfies the Kwapień condition.

Now we prove that f is not a coboundary with a transfer function in L^r for any $r > p - 1$. Since $f \geq -1$ almost everywhere, then for a.e. x,

$$\left| \sum_{i=0}^{n} f(T^i x) \right| \geq \sum_{i=0}^{n} (f(T^i x) - n)\mathbb{I}\{f(T^i x) > n\}.$$

Each term in the sum on the right-hand side of the inequality is non-negative and therefore,

$$\left\| \sum_{i=0}^{n} f(T^i x) \right\|_r^r \geq \sum_{i=0}^{n} \int_{f\circ T^i > n} (f(T^i x) - n)^r \, d\mu \tag{4.2}$$

$$= (n+1) \int_{f>n} (f-n)^r \, d\mu. \tag{4.3}$$

Therefore, $\sum_{i=0}^{n} f(T^i x) = g(x) - g(T^{n+1}x)$ is unbounded in L^r. □

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4.2. *Genericity of the strong non-existence result.* A principal obstacle to solving the coboundary equation is imbalance between the positive and negative parts of a typical function $f \in L_0^p$. Suppose $a_i \in \mathbb{R}$ for $i \in \mathbb{N}$ is an increasing sequence of real numbers such that $\lim_{i\to\infty} a_i = \infty$, and for all real $\alpha > 0$,

$$\lim_{i\to\infty} \frac{a_i}{a_{i+1}^{\alpha}} = 0. \tag{4.4}$$

An example of a_i satisfying (4.4) is

$$a_i = 2^{i^i}.$$

Given $f \in L^p$ and $i \in \mathbb{N}$, let

$$u_i(f) = \{x \in X : f(x) < -a_i\} \quad \text{and} \quad v_i(f) = \{x \in X : f(x) > a_i\}.$$

We are ready to define our generic class of L_0^p functions. Given $n \in \mathbb{N}$, define

$$\mathcal{G}_n^p = \left\{ f \in L_0^p : \text{there exists } i > n \mid \mu(v_i(f)) > \frac{1}{a_i^p i^2} \text{ and } \mu(u_{i-1}(f)) < \frac{1}{a_{i+1}^p i^2} \right\}.$$

Below we prove that $\mathcal{G}_n^p$ is both open and dense, and $f \in \bigcap_{n=1}^{\infty} \mathcal{G}_n^p$ satisfies the required property. The key property of the sequence a_n is the fast growth rate. The following lemma will be used to guarantee that coboundaries $f \in \bigcap_{n=1}^{\infty} \mathcal{G}_n^p$ do not have transfer functions in L^q for $q > p - 1$.

LEMMA 4.2. *For any $\alpha > 0$,*

$$\lim_{n\to\infty} \frac{a_{n+1}^{\alpha}}{a_n n^2} = \infty.$$

Proof. Let $\alpha > 0$. Define $\beta = \min\{\alpha/2, 1/2\}$. By condition (4.4), for sufficiently large n, $a_{n+1}^{\beta} > 2a_n > 2a_n^{\beta}$. Thus, $a_{n+k}^{\beta} > 2^k$ for sufficiently large n and $k \in \mathbb{N}$. Hence,

$$\lim_{n\to\infty} \frac{a_{n+1}^{\alpha/2}}{n^2} = \infty.$$

Therefore,

$$\lim_{n\to\infty} \frac{a_{n+1}^{\alpha}}{n^2 a_n} = \lim_{n\to\infty} \left(\frac{a_{n+1}^{\alpha/2}}{n^2}\right)\left(\frac{a_{n+1}^{\alpha/2}}{a_n}\right) = \infty.$$

□

Now we prove that $\mathcal{G}_n^p$ is dense in L_p for each $p \geq 1$ and $n \in \mathbb{N}$.

LEMMA 4.3. *For each $n \in \mathbb{N}$, the set $\mathcal{G}_n^p$ is dense in L_0^p.*

Proof. Let $f \in L_0^p$, $\epsilon > 0$, and $n \in \mathbb{N}$. Since bounded measurable functions are dense in L_0^p, we can choose a bounded mean zero $f_0 \in L^p$ such that

$$\|f - f_0\|_p < \frac{\epsilon}{3}.$$

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Choose $i_1 \geq n$ such that $a_{i_1} > \|f_0\|_\infty$ and

$$4 \cdot \frac{2^{1/p}}{i_1^{2/p}} < \frac{\epsilon}{3}.$$

Choose a subset $Y \subset X$ such that

$$\mu(Y) = \frac{2}{a_{i_1}^p i_1^2} + \frac{4}{a_{i_1-1} a_{i_1}^{p-1} i_1^2}$$

and $\int_Y f_0 \, d\mu = 0$. To see that Y exists, note that our measure space (X, μ) is isomorphic to the unit circle S^1 with normalized Lebesgue measure. Let $\mu_0 = 2/a_{i_1}^p i_1^2 + 4/a_{i_1-1} a_{i_1}^{p-1} i_1^2$. By Fubini's theorem,

$$\int_0^1 \bigg(\int_0^{\mu_0} f_0(e^{2\pi i(t+x)}) \, dt \bigg) \, dx = \int_0^{\mu_0} \bigg(\int_0^1 f_0(e^{2\pi i(t+x)}) \, dx \bigg) \, dt = 0.$$

Thus, by a change of variable and the intermediate value theorem applied to the continuous function $x \mapsto \int_x^{x+\mu_0} f_0(e^{2\pi it}) \, dt$, there exists $x_0 \in [0, 1]$ such that

$$\int_{x_0}^{x_0+\mu_0} f_0(e^{2\pi it}) \, dt = \int_0^{\mu_0} f_0(e^{2\pi i(t+x_0)}) \, dt = 0.$$

Since a set Y with measure μ_0 exists in this case, by using a measure-space isomorphism, it exists for a general Lebesgue space.

Let $V \subset Y$ be such that $\mu(V) = 2(a_{i_1}^p i_1^2)^{-1}$ and define $U = Y \setminus V$. Define f_1 as a modification of f_0 in the following manner:

$$f_1(x) = \begin{cases} 2a_{i_1} & \text{if } x \in V, \\ -a_{i_1-1} & \text{if } x \in U, \\ f_0(x) & \text{if } x \in X \setminus Y. \end{cases}$$

Thus,

$$\begin{aligned} \|f - f_1\|_p &\leq \|f - f_0\|_p + \|f_0 - f_1\|_p \\ &< \frac{\epsilon}{3} + 3a_{i_1} \mu(V)^{1/p} + 2a_{i_1-1} \mu(U)^{1/p} \\ &\leq \frac{\epsilon}{3} + 3 \cdot \frac{2^{1/p}}{i_1^{2/p}} + 2 \cdot \frac{4^{1/p}}{i_1^{2/p}} \\ &< \epsilon. \end{aligned}$$

Also, $f_1 \in \mathcal{G}_n^p$, which completes the proof. □

LEMMA 4.4. *For each $n \in \mathbb{N}$, the set $\mathcal{G}_n^p$ is open in L_0^p.*

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Proof. Suppose $f \in \mathcal{G}_n^p$. Then there exists $i \geq n$ such that

$$\mu_i = \mu(v_i(f)) > \frac{1}{a_i^p i^2},$$
$$\nu_i = \mu(u_{i-1}(f)) < \frac{1}{a_{i+1}^p i^2}.$$

Thus, by right continuity of $f > t$, there exist $a' > a_i$ and $a'' > a_{i-1}$, and μ', ν' such that

$$\mu(\{x : f(x) > a'\}) > \mu' > \frac{1}{a_i^p i^2},$$
$$\mu(\{x : f(x) < -a''\}) < \nu' < \frac{1}{a_{i+1}^p i^2}.$$

Define $\epsilon > 0$ as

$$\epsilon = \min\bigg\{\bigg(\mu' - \frac{1}{a_i^p i^2}\bigg)^{1/p}(a' - a_i), \bigg(\frac{1}{a_{i+1}^p i^2} - \nu'\bigg)^{1/p}(a'' - a_{i-1})\bigg\}.$$

It is not difficult to see that the ϵ-ball centered at $f \in L_0^p$ is contained in $\mathcal{G}_n^p$. □

Let

$$\mathcal{G}_p = \bigcap_{n=1}^{\infty} \mathcal{G}_n^p.$$

Note, for $f \in \mathcal{G}_n^p$, also $f \circ T \in \mathcal{G}_n^p$ for any T measure preserving. The same principle applies to $\mathcal{G}_p$; $f \in \mathcal{G}_p$ implies $f \circ T \in \mathcal{G}_p$.

We have the following core result of this paper.

PROPOSITION 4.5. *Suppose $f \in \mathcal{G}_p$, $T \in \mathcal{E}$, and g is a measurable function. If the coboundary equation $f = g - g \circ T$ is satisfied almost everywhere, then $g \notin L^q$ for $q > p - 1$.*

Prior to proving Proposition 4.5, we prove the following basic lemma used in the proposition.

LEMMA 4.6. *Let T be an ergodic invertible measure-preserving transformation on a standard probability space $(X, \mathcal{B}, \mu)$. Suppose $B \subset X$ is a set of positive measure and $K \in \mathbb{N}$. Let $\ell : B \to \mathbb{N}$ be a measurable map such that $\ell(x) \leq K$ for $x \in B$. Define*

$$B(x) = \{T^i x : 0 \leq i < \ell(x)\}.$$

There exists a measurable set J such that for $x \neq y$, $x, y \in J$,

$$B(x) \cap B(y) = \emptyset$$

and

$$\mu\bigg(B \cap \bigcup_{x \in J} B(x)\bigg) > \frac{1}{2}\mu(B).$$

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Proof. Choose $H \in \mathbb{N}$ such that

$$\frac{K}{H} < \frac{1}{4}\mu(B).$$

Let $\overline{B}$ be a set of positive measure, and $\overline{B}, T\overline{B}, T^2\overline{B}, \ldots, T^{H-1}\overline{B}$ a Rokhlin tower of height H such that

$$\mu\bigg(\bigcup_{i=0}^{H-1} T^i\overline{B}\bigg) > 1 - \frac{K}{H}.$$

Partition $\overline{B}$ into finitely many sets of positive measure such that $\overline{B} = \bigcup_{j=0}^{H'} \overline{B}_j$ and for each i, $0 \le i < H$, either $T^i\overline{B}_j \subset B$ or $T^i\overline{B}_j \subset B^c$. These are B-pure subcolumns ($T^i\overline{B}_j$, $0 \le i < H-1$). For each j, let i_j be the minimum i such that $T^i\overline{B}_j \subset B$. If there is no minimum, we can discard that subcolumn. Partition $T^{i_j}\overline{B}_j$ into finitely many sets of positive measure with equal values of $\ell(x)$. Call these sets $\overline{B}_{j,\ell}$. Insert $\overline{B}_{j,\ell}$ in J. Consider $T^\ell\overline{B}_{j,\ell}$ and let $i_{j,\ell}$ be the minimum i such that $T^{\ell+i}\overline{B}_{j,\ell} \subset B$. Then partition $T^{\ell+i_{j,\ell}}\overline{B}_{j,\ell}$ into subsets with the same value $\ell(x)$. Continue this process until each subcolumn is swept out up to at least $H-K$ levels of the Rokhlin tower. Each time, a disjoint set is inserted into J. Since B is covered except for a subset of the top K levels, then

$$\mu\bigg(B \cap \bigcup_{x\in J} B(x)\bigg) > \frac{1}{2}\mu(B).$$

□

Proof of Proposition 4.5. Let $p \ge 1$, $f \in \mathcal{G}_p$ and $q > p-1$. Choose integer $k > 1$ such that $kq > p$ and $N \in \mathbb{N}$ such that for $n \ge N$,

$$\frac{a_{n+1}^p}{a_n^{k+p}} > 4.$$

Let sgn be the standard sign function defined as $\operatorname{sgn}(i) = -1$ if $i < 0$, $\operatorname{sgn}(i) = 0$ if $i = 0$, and $\operatorname{sgn}(i) = 1$ if $i > 0$. For $i \in \mathbb{Z}$, let $[i] = \{j \in \mathbb{Z} : i \le j < 0\}$ if $i < 0$, and $[i] = \{j \in \mathbb{Z} : 0 \le j < i\}$ if $i \ge 0$. Note, for $i \in \mathbb{Z}$, the coboundary equation expands to the following:

$$g(T^i x) = g(x) - \operatorname{sgn}(i) \sum_{j\in[i]} f(T^j x).$$

Define our specialized sign function $\rho_n : X \to \{-1, 1\}$ based on the following:

(1) if $g(x) \le a_n/2$, let $\rho_n(x) = 1$;

(2) otherwise if $g(x) > a_n/2$, then let $\rho_n(x) = -1$.

For $n \in \mathbb{N}$, let $c_n = a_n/a_{n-1}$. Let $A_n = u_{n-1}(f)$ and $B_n = v_n(f)$. For $x \in B_n$, let

$$\ell_x = \min\bigg\{\ell : \lceil (c_n)^h \rceil \le \ell < \lceil (c_n)^{h+1} \rceil,\ 1 \le h < k,\ |g(T^{\rho_n(x)\ell}x)| < \frac{a_n}{4}(c_n)^{h-1}\bigg\},$$

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if the min exists, otherwise let $\ell_x = \lceil (c_n)^k \rceil$. Given $x \in X$, define the set $L_n(x) = [\rho_n(x)\ell_x]$. Thus, for infinitely many $n \geq N$,

$$\mu\bigg(\bigcup_{j=-c_n^k}^{c_n^k} T^j(A_n)\bigg) < \frac{2a_n^k}{a_{n+1}^p n^2} < \frac{1}{2}\frac{1}{a_n^p n^2} < \frac{1}{2}\mu(B_n).$$

Let

$$B_n' = B_n \setminus \bigg(\bigcup_{j=-c_n^k}^{c_n^k} T^j(A_n)\bigg).$$

Hence, $\mu(B_n') > \frac{1}{2}\mu(B_n)$ for $n \geq N$. We break the proof down into four separate cases and handle each separately.

(1) $B_{n,1} = \{x \in B_n' : g(x) < a_n/2, \ell_x < \lceil (c_n)^k \rceil\}$.
(2) $B_{n,2} = \{x \in B_n' : g(x) < a_n/2, \ell_x \geq \lceil (c_n)^k \rceil\}$.
(3) $B_{n,3} = \{x \in B_n' : g(x) \geq a_n/2, \ell_x < \lceil (c_n)^k \rceil\}$.
(4) $B_{n,4} = \{x \in B_n' : g(x) \geq a_n/2, \ell_x \geq \lceil (c_n)^k \rceil\}$.

At least one of the $B_{n,m}$ satisfies $\mu(B_{n,m}) \geq (1/8)\mu(B_n)$ for $m = 1, 2, 3, 4$. We handle the case $\mu(B_{n,1}) \geq (1/8)\mu(B_n)$ first. We create tiles in the following way. For $x \in B_{n,1}$, let

$$B_{n,1}(x) = \{T^i x : i \in L_n(x)\}.$$

By Lemma 4.6 with $K = \lceil (c_n)^k \rceil$, there exists $J = J_{n,1}$ such that for $x \neq y$, $x, y \in J$,

$$B_{n,1}(x) \cap B_{n,1}(y) = \emptyset$$

and

$$\mu\bigg(B_{n,1} \cap \bigcup_{x \in J} B_{n,1}(x)\bigg) > \frac{1}{2}\mu(B_{n,1}).$$

The L^q-norm of the transfer function g will blow up on the set J. Before completing the general proof, it is helpful to see how the argument goes in a special case. Suppose $\ell_x = a_n/a_{n-1}$ for $x \in J$. This implies for $x \in J$, $T^i(x) \notin B_n$ of the order of a_n/a_{n-1} times. Also, for this special case, $T^i(x)$ cannot fall in B_n for $0 < i < \ell_x$. Note that $T^i x$, $0 \leq i < \ell_x$, does not fall in A_n by the previous choice of J. However, for $x \in J$, the transfer function at $T^i x$ will be of the order of the sum, so that $g(T^i x)$ will be of the order of a_n (or $a_n/4$). This implies

$$\int_X |g(x)|^q \, d\mu \approx \bigg(\frac{a_n}{4}\bigg)^q \bigg(\frac{a_n}{a_{n-1}}\bigg)\mu(B_n) \tag{4.5}$$

$$= \frac{1}{4^q}\frac{a_n^{q+1}}{a_{n-1}a_n^p n^2} \tag{4.6}$$

$$= \frac{1}{4^q}\frac{a_n^{q+1-p}}{a_{n-1}n^2}. \tag{4.7}$$

However, the last term tends to infinity as $n \to \infty$ by the definition of a_n and Lemma 4.2.

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General proof for case 1: First we prove the following lemma.

LEMMA 4.7. *Let* $1 \leq h < k$, *given in the definition of* ℓ_x, *satisfy* $\lceil (c_n)^h \rceil \leq \ell_x < \lceil (c_n)^{h+1} \rceil$. *If*

$$\ell_0 = \#\{i \in L_n(x) : T^i x \in B_n\},$$

then

$$\ell_x > \tfrac{1}{2} c_n \ell_0.$$

Proof of lemma. Suppose the lemma is not true. Then

$$\begin{aligned}
\bigg| g(x) - \operatorname{sgn}(\rho_n(x)) \sum_{i \in L_n(x)} f(T^i x) \bigg| &\geq \bigg(\sum_{i \in L_n(x)} f(T^i x) - g(x) \bigg) && (4.8)\\
&\geq \ell_0 a_n - (\ell_x - \ell_0) a_{n-1} - \frac{a_n}{2} && (4.9)\\
&= \big(\ell_0 - \tfrac{1}{2}\big) a_n - \ell_x a_{n-1} + \ell_0 a_{n-1} && (4.10)\\
&\geq \frac{\ell_0}{2} a_n - \ell_x a_{n-1} + \ell_0 a_{n-1} && (4.11)\\
&\geq a_{n-1} \ell_x + \frac{2 a_{n-1}^2}{a_n} \ell_x && (4.12)\\
&\geq \frac{a_n^h}{a_{n-1}^{h-1}} + 2 \frac{a_n^{h-1}}{a_{n-1}^{h-2}} > a_n (c_n)^{h-1}. && (4.13)
\end{aligned}$$

This contradicts the definition of ℓ_x. □

Resume proof of proposition: The measurable set J was constructed such that truncated orbits of points in J are disjoint. In particular, for $x, y \in J$, $x \neq y$, then $\{x, Tx, \ldots, T^{\ell_x - 1} x\} \cap \{y, Ty, \ldots, T^{\ell_y - 1} y\} = \emptyset$. Thus,

$$\int_X \bigg| g(x)|^q \, d\mu \geq \int_J \sum_{i \in L_n(x)} \bigg| g(T^i x)|^q \, d\mu. \tag{4.14}$$

Also, since $x \in B_n'$, then $f(x) > a_n$. As noted previously, $T^i x \notin A_n$ for $0 \leq i < c_n^k$. Thus, by the definition of ℓ_x,

$$|g(T^i x)| \geq \frac{a_n}{4} \quad \text{for } i \in L_n(x).$$

Hence, we have the following:

$$\begin{aligned}
\int_X |g(x)|^q \, d\mu &\geq \int_J \sum_{i \in L_n(x)} |g(T^i x)|^q \, d\mu && (4.15)\\
&= \int_J \sum_{i \in L_n(x)} \bigg| g(x) - \sum_{j \in [i]} f(T^j x) \bigg|^q \, d\mu && (4.16)\\
&\geq \int_J \sum_{i \in L_n(x)} \bigg| \frac{a_n}{4} \bigg|^q \, d\mu && (4.17)
\end{aligned}$$

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$$= \left|\frac{a_n}{4}\right|^q \int_J \ell_x \, d\mu \tag{4.18}$$

$$> \left|\frac{a_n}{4}\right|^q \int_J \frac{1}{2}\left(\frac{a_n}{a_{n-1}}\right) \sum_{i\in L_n(x)} I_{B_n}(T^i x)\, d\mu \tag{4.19}$$

$$> \left|\frac{a_n}{4}\right|^q \frac{1}{2}\left(\frac{a_n}{a_{n-1}}\right)\left(\frac{1}{2}\mu(B_{n,1})\right) \tag{4.20}$$

$$> \left(\frac{1}{32}\right)\left|\frac{a_n}{4}\right|^q \left(\frac{a_n}{a_{n-1}}\right)\mu(B_n) \tag{4.21}$$

$$\geq \frac{a_n^{q+1}}{32(4^q)a_{n-1}a_n^p n^2} \tag{4.22}$$

$$= \frac{a_n^{q+1-p}}{32(4^q)a_{n-1}n^2}. \tag{4.23}$$

The proof for this case is complete, since, by Lemma 4.2,

$$\lim_{n\to\infty} \frac{a_n^{q+1-p}}{32(4^q)a_{n-1}n^2} = \infty.$$

Proof for case 2:

$$\int_{B_n} |g(x)|^q \, d\mu \geq \int_J \sum_{i\in L_n(x)} |g(T^i x)|^q I_{B_n}(T^i x)\, d\mu \tag{4.24}$$

$$= \int_J \sum_{i\in L_n(x)} \left| g(x) - \operatorname{sgn}(x) \sum_{j\in[-i]} f(T^j x)\right|^q I_{B_n}(T^i x)\, d\mu \tag{4.25}$$

$$\geq \int_J \sum_{i=c_n^{k-1}}^{c_n^k} \left| g(x) - \operatorname{sgn}(x) \sum_{j\in[-i]} f(T^j x)\right|^q I_{B_n}(T^{-i} x)\, d\mu \tag{4.26}$$

$$\geq \sum_{i=c_n^{k-1}}^{c_n^k} \int_J \left(\left(\frac{a_n}{4}\right)\left(\frac{a_n}{a_{n-1}}\right)^{k-1}\right)^q I_{B_n}(T^i x)\, d\mu \tag{4.27}$$

$$\geq \left(\frac{c_n^k - c_n^{k-1}}{c_n^k}\right)\left(\frac{1}{16}\right)\mu(B_n)\left(\left(\frac{a_n}{4}\right)\left(\frac{a_n}{a_{n-1}}\right)^{k-1}\right)^q \tag{4.28}$$

$$> \left(\frac{1}{32}\right)\left(\frac{a_n^{kq}}{a_n^p n^2 4^q a_{n-1}^{q(k-1)}}\right) \tag{4.29}$$

$$= \left(\frac{1}{32}\right)\left(\frac{a_n^{kq-p}}{n^2 4^q a_{n-1}^{q(k-1)}}\right). \tag{4.30}$$

Since

$$\lim_{n\to\infty} \left(\frac{1}{32}\right)\left(\frac{a_n^{kq-p}}{n^2 4^q a_{n-1}^{q(k-1)}}\right) = \infty,$$

then our result follows for case 2.

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Case 3 would be handled in a similar manner as case 1, except we would base our estimate of $g(x)$ on the inverse of T. Thus, we have the following:

$$\int_X |g(x)|^q \, d\mu \geq \int_J \sum_{i\in L_n(x)} |g(T^i x)|^q \, d\mu \tag{4.31}$$

$$= \int_J \sum_{i\in L_n(x)} \bigg| g(x) + \sum_{j\in[i]} f(T^j x)\bigg|^q \, d\mu \tag{4.32}$$

$$\geq \int_J \sum_{i\in L_n(x)} \bigg|\frac{a_n}{4}\bigg|^q \, d\mu. \tag{4.33}$$

The next steps continue in a similar manner as case 1. Also, case 4 follows in a similar manner as case 2, except by using T^{-1} instead of T. □

Proof of Theorem 1.2. Define

$$\mathcal{G}_p = \bigcap_{n=1}^{\infty} \mathcal{G}_n^p.$$

By Lemmas 4.3 and 4.4, the set $\mathcal{G}_p$ is a dense G_δ subset of L_0^p. Also, by Proposition 4.5, $f \in \mathcal{G}_p$ satisfies the conditions of Theorem 1.2. □

4.3. *Not a moment.* Let $\phi : \mathbb{R} \to \mathbb{R}$ be a measurable function such that

$$\lim_{x\to\infty} \phi(x) = \infty.$$

For $i \in \mathbb{N}$, let A, B_i be disjoint sets in X, and $b_i > 0$. Define f as

$$f = I_A - \sum_{i=1}^{\infty} b_i I_{B_i}.$$

We will give conditions on the fast growth rate of b_i as well as conditions on the sets A, B_i to guarantee that f is contained in L^1, but such that $\phi \circ |g|$ is not in L^1 for any transfer function g of an ergodic invertible measure-preserving transformation T. Let $A \subset X$ have measure $\mu(A) = 1/2$. Choose $b_i > 0$ for $i \in \mathbb{N}$ such that $\lim_{i\to\infty} b_i = \infty$, and such that for all real $\alpha > 0$,

$$\lim_{i\to\infty} \frac{b_i}{b_{i+1}^{\alpha}} = 0, \tag{4.34}$$

and also for $y \geq b_i/4$,

$$\frac{\phi(y)}{2^i} \geq i. \tag{4.35}$$

It is possible to satisfy condition (4.35) by taking a faster growing subsequence for b_i. Choose disjoint sets $B_i \subset A^c$ such that

$$\mu(B_i) = \frac{1}{b_i 2^{i+1}}. \tag{4.36}$$

Observe that $f \in L^1$ is mean zero.

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PROPOSITION 4.8. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a measurable function satisfying $\lim_{x\to\infty} \phi(x) = \infty$. Suppose the mean-zero function $f = I_A - \sum_{i=1}^{\infty} b_i I_{B_i}$ satisfies the conditions above, including (4.34), (4.35), and (4.36). If T is an ergodic invertible measure-preserving transformation $T : X \to X$ and g is a transfer function satisfying $f(x) = g(Tx) - g(x)$ for a.e. $x \in X$, then*

$$\int_X \phi(|g|)\, d\mu = \infty.$$

Proof. Let sgn be the standard sign function defined as $\mathrm{sgn}(i) = -1$ if $i < 0$, $\mathrm{sgn}(i) = 0$ if $i = 0$, and $\mathrm{sgn}(i) = 1$ if $i > 0$. For $i \in \mathbb{Z}$, let $[i] = \{j \in \mathbb{Z} : i \le j < 0\}$ if $i < 0$, and $[i] = \{j \in \mathbb{Z} : 0 \le j < i\}$ if $i \ge 0$. Note, for $i \in \mathbb{Z}$, the coboundary equation expands to the following:

$$g(T^i x) = g(x) + \mathrm{sgn}(i) \sum_{j\in[i]} f(T^j x).$$

Define our specialized sign function $\rho : X \to \{-1, 1\}$ based on the following:

(1) if $g(x) \le b_n/2$, let $\rho(x) = 1$;

(2) otherwise if $g(x) > b_n/2$, then let $\rho(x) = -1$.

Assume $f \in L^1$. For $x \in B_n$, let

$$\ell_x = \min \left\{ \ell : \ell > 0, |g(T^{\rho(x)\ell} x)| < \frac{b_n}{4}(b_n)^{h-1}, \lceil (b_n)^h \rceil \le \ell < \lceil (b_n)^{h+1} \rceil \right\}.$$

Note, $\ell_x < \infty$ for a.e. $x \in X$, otherwise our result follows directly. Thus, exclude points $x \in X$ where $\ell_x = \infty$. Choose $k_n \in \mathbb{N}$ such that

$$\mu(\{x \in B_n : \ell_x < b_n^{k_n+1}\}) > \tfrac{1}{2}\mu(B_n).$$

Given $x \in X$, define the set $K_n(x) = [\frac{1}{2}\rho(x)\ell_x]$. We do not need to consider all of the cases as in Proposition 4.5, due to the special nature of the counterexamples f in this result. We create tiles in the following way. For $x \in B_n$, let

$$B_n(x) = \{T^i x : i \in K_n(x)\}.$$

There exists J_n such that for $x \ne y$, $x, y \in J_n$,

$$B_n(x) \cap B_n(y) = \emptyset$$

and

$$\mu\left(B_n \cap \bigcup_{x\in J_n} B_n(x) \right) > \frac{1}{4}\mu(B_n).$$

First we prove the following lemma.

LEMMA 4.9. *Suppose $\lceil (b_n)^h \rceil \le \ell_x < \lceil (b_n)^{h+1} \rceil$ for $1 \le h < k_n + 1$. If*

$$\ell_0 = \#\{i \in [\rho(x)\ell_x] : T^i x \in B_n\},$$

then

$$\ell_x > \tfrac{1}{2} b_n \ell_0.$$

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Proof of lemma. Suppose the lemma is not true. Then

$$\left|g(x) + \operatorname{sgn}(x) \sum_{i\in[\rho(x)\ell_x]} f(T^i x)\right| \geq \ell_0 b_n - (\ell_x - \ell_0) = \ell_0 b_n - \ell_x + \ell_0 \tag{4.37}$$

$$\geq \ell_x + \frac{2}{b_n}\ell_x \tag{4.38}$$

$$\geq b_n^h + 2b_n^{h-1} > b_n^h. \tag{4.39}$$

This contradicts the definition of ℓ_x.

Resume proof of proposition: Thus, we have the following:

$$\int_X \phi(|g(x)|)\,d\mu \geq \int_{J_n} \sum_{i\in K_n(x)} \phi(|g(T^i x)|)\,d\mu \tag{4.40}$$

$$= \int_{J_n} \sum_{i\in K_n(x)} \phi\bigg(\bigg|g(x) + \sum_{j\in[i]} f(T^j x)\bigg|\bigg)\,d\mu \tag{4.41}$$

$$\geq \int_{J_n} n2^n \ell_x\,d\mu \tag{4.42}$$

$$> n2^n \int_{J_n} \frac{1}{2}(b_n) \sum_{i\in K_n(x)} I_{B_n}(T^i x)\,d\mu \tag{4.43}$$

$$> n2^n \tfrac{1}{2}(b_n)(\tfrac{1}{4}\mu(B_n)) \tag{4.44}$$

$$> (\tfrac{1}{8}) n2^n (b_n)\mu(B_n) \tag{4.45}$$

$$= \frac{n2^n}{8(2^{n+1})} \to \infty \quad \text{as } n \to \infty. \tag{4.46}$$

□

5. *Category of transformation solutions*

This section contains two propositions. The first proposition gives a general condition for the existence of solutions to the coboundary equation when f is not integrable. The second proposition shows the class of transformations with a measurable solution is meager for any measurable function f.

PROPOSITION 5.1. *Suppose $f : X \to \mathbb{R}$ is a measurable function. The coboundary equation $f = g - g \circ T$ has solutions $T \in \mathcal{E}$, $g \in L^0$, if and only if,*

$$\int_{f>0} f\,d\mu = \int_{f<0} (-f)\,d\mu \quad (\infty \textit{ or finite}).$$

Proof. The case where both $\int_{f>0} f\,d\mu$ and $\int_{f<0}(-f)\,d\mu$ are finite and unequal is already covered by Anosov's result [4]. If the integrals are finite and equal, it follows from Theorem 1.1.

Next, we prove the case where one integral is finite and the other is infinite. Without loss of generality, assume $\int_{f>0} f\,d\mu = \infty$ and $\int_{f<0}(-f)\,d\mu < \infty$. Choose a measurable

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subset $A \subset \{f > 0\}$ such that

$$\int_A f \, d\mu + \int_{f<0} f \, d\mu = 1.$$

Let

$$f_0(x) = \begin{cases} f(x) & \text{if } x \in A \cup \{f < 0\}, \\ 0 & \text{if } x \in \{f > 0\} \setminus A. \end{cases}$$

Thus, $f_0 \in L^1$ and $\int_X f_0 \, d\mu = 1$. Given $T \in \mathcal{E}$, by the mean ergodic theorem,

$$\lim_{n\to\infty} \int_X \left| \frac{1}{n} \sum_{i=0}^{n-1} f_0(T^i x) - \int_X f_0 \, d\mu \right| d\mu = 0.$$

Let $\delta > 0$. Then $\mu\{x \in X : |1/n \sum_{i=0}^{n-1} f_0(T^i x) - 1| < \delta\} \to 1$ as $n \to \infty$. Hence,

$$\lim_{n\to\infty} \mu\left\{x \in X : \sum_{i=0}^{n-1} f_0(T^i x) > n(1-\delta)\right\} = 1.$$

Since $\sum_{i=0}^{n-1} f(T^i x) \geq \sum_{i=0}^{n-1} f_0(T^i x)$ for a.e. $x \in X$, then

$$\lim_{n\to\infty} \mu\left\{x \in X : \sum_{i=0}^{n-1} f(T^i x) > n(1-\delta)\right\} = 1.$$

Since the Halász–Schmidt condition (2.1) does not hold, there is no measurable solution g.

The final case to prove is where $\int_{f>0} f \, d\mu = \int_{f<0} (-f) \, d\mu = \infty$. It is proved using a construction similar to the one used in Theorem 1.1. Choose disjoint measurable sets $X_n \subset X$ for $n \in \mathbb{N}$ such that f is bounded on X_n, $\int_{X_n} f \, d\mu = 0$, and

$$\mu\left(\bigcup_{i=0}^{\infty} X_i\right) = 1.$$

Let $\epsilon_{n,i} > 0$ for $n, i \in \mathbb{N}$ be such that $\sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \epsilon_{n,i} < \infty$. By Lemma 3.5, for each $n \in \mathbb{N}$, there exist invertible measure-preserving T_n, disjoint $I_{n,i} \subset X_n$, and $h_{n,i} \in \mathbb{N}$, such that:

- $X_n = \bigcup_{i=1}^{\infty} \bigcup_{j=0}^{h_{n,i}-1} T_n^j I_{n,i}$ is a disjoint union;
- $|\sum_{j=0}^{h_{n,i}-1} f(T_n^j x)| < \epsilon_{n,i}$;
- $|\sum_{j=0}^{h_{n,i}-1} f(T_n^j x)| < \|f\|_\infty + \epsilon_{n,i}$.

Note each T_n is defined on X_n except for the top levels of each tower. Let T be the join of all T_n:

$$Tx = T_n x \quad \text{for } x \in X_n \setminus \left(\bigcup_{i=1}^{\infty} T_n^{h_{n,i}-1} I_{n,i}\right).$$

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Define the set $A \subset X$ as

$$A = \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{\infty} I_{n,i}.$$

Define $f_A(x) = \sum_{j=0}^{h_{n,i}-1} f(T^j x)$ for $x \in I_{n,i}$. Note that f_A is mean zero, since

$$\int_A f_A \, d\mu = \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \int_{I_{n,i}} \sum_{j=0}^{h_{n,i}-1} f(T^j x) \, d\mu = \sum_{n=1}^{\infty} \int_{X_n} f \, d\mu = 0. \tag{5.1}$$

Since f_A is bounded and mean zero, by Proposition 3.6, there exist $\tau : A \to A$ and g_A such that $f_A = g_A \circ \tau - g_A$ almost everywhere. There exists a unique extension of $T : X \to X$ (up to a set of measure zero) such that the induced transformation $T_A = \tau$ almost everywhere. By Lemma 3.2,

$$f(x) = g(Tx) - g(x) \quad \text{for a.e. } x \in X.$$

Also, the explicit transfer function g, defined by Lemma 3.2, is measurable, since g_A is measurable. □

Now we are ready to prove the class of transformations with a measurable solution is a first category set (meager).

PROPOSITION 5.2. *Let f be a measurable function such that $\mu(\{x : f(x) \neq 0\}) > 0$. Let $\mathcal{T}$ be the set of ergodic invertible measure-preserving transformations T such that $f = g - g \circ T$ has a measurable solution g. The set $\mathcal{T}$ is a set of first category (meager).*

Proof. Let $\eta \in \mathbb{R}$ such that $0 < \eta < 1/10$. For each $n \in \mathbb{N}$, define

$$D_n = \bigg\{ T \in \mathcal{E} : \text{there exists } k > n \text{ such that } \mu\bigg(\bigg\{x : \bigg|\sum_{i=0}^{k-1} f(T^i x)\bigg| > n\bigg\}\bigg) > \eta \bigg\}.$$

For each $n \in \mathbb{N}$, the set D_n is both open and dense. Establishing open-ness is straightforward. Let $T \in D_n$. There exists $\delta_0 > 0$ such that

$$\mu\bigg(\bigg\{x : \bigg|\sum_{i=0}^{k-1} f(T^i x)\bigg| > n + \delta_0\bigg\}\bigg) > \eta.$$

Thus, if $\eta_0 = \mu(\{x : |\sum_{i=0}^{k-1} f(T^i x)| > n + \delta_0\})$, then

$$\bigg\{ S \in \mathcal{E} : \int_X |f \circ S^i - f \circ T^i| \, d\mu < \frac{\delta_0(\eta_0 - \eta)}{k} \bigg\}$$

is an open neighborhood containing T and contained in D_n.

To establish that D_n is dense, it can be accomplished by an application of the ergodic theorem. Let $S \in \mathcal{E}$ and $\epsilon \in \mathbb{R}$ be such that $1/20 > \epsilon > 0$. If $S \in D_n$, then we set $T = S$. Otherwise, assume $S \notin D_n$. Choose $\alpha > 0$ such that the set $A = \{x \in X : f(x) > \alpha\}$ has positive measure. Similarly, choose $\beta > 0$ such that the set $B = \{x \in X : f(x) < -\beta\}$ has

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positive measure. Let $\gamma \in \mathbb{N}$ be such that

$$\gamma \geq \max\left\{\frac{2n}{\alpha}, \frac{2n}{\beta}\right\}.$$

Choose $\ell_0 > n$ such that for $\ell \geq \ell_0$,

$$\mu\left(\left\{x \in X : \sum_{i=0}^{\ell-1} I_A(S^i x) > \gamma\right\}\right) > 1 - \epsilon$$

and

$$\mu\left(\{x \in X : \sum_{i=0}^{\ell-1} I_B(S^i x) > \gamma\}\right) > 1 - \epsilon.$$

Choose $h > \ell_0$ such that

$$\frac{\ell_0}{h} < \frac{\epsilon}{4}.$$

There is a Rohklin tower of height $4h$ with base I such that

$$\mu\left(\bigcup_{i=0}^{4h-1} S^i I\right) > 1 - \frac{\epsilon}{4h}.$$

There exist disjoint sets $I_1, I_2 \subset I$ such that for each $x \in I_1$ and $y \in I_2$, there exist $j(x), j(y)$ such that $h \leq j(x) < 2h$, $h \leq j(y) < 2h$, and

$$\sum_{i=0}^{\ell_0-1} I_A(S^{i+j(x)}x) > \gamma$$

and

$$\sum_{i=0}^{\ell_0-1} I_B(S^{i+j(y)}y) > \gamma.$$

By the choice of $\epsilon < 1/20$, then I_1, I_2 may be chosen such that

$$\mu(I_1) = \mu(I_2) > \tfrac{1}{4}\mu(I).$$

For each $x \in I_1$, let $i_1(x), i_2(x), \ldots i_\gamma(x)$, be increasing such that

$$S^{i_j(x)}(x) \in A$$

and similarly, for each $y \in I_2$, let $i_1(y), i_2(y), \ldots i_\gamma(y)$, be such that

$$S^{i_j(y)}(y) \in B$$

and $h \leq i_j(x) < 2h - 1$, $i_\gamma(x) < i_1(x) + \ell_0$, and $h \leq i_j(y) < 2h - 1$, $i_\gamma(y) < i_1(y) + \ell_0$. Let $\phi : I_1 \to I_2$ be an invertible measure-preserving map. The transformation T will be defined in the following manner: for $x \in I_1$, let $y = \phi(x) \in I_2$,

$$T^{i_j(x)}(x) = S^{i_j(y)}(y)$$

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and

$$T^{i_j(y)}(y) = S^{i_j(x)}(x).$$

Otherwise, define T to be identical to S everywhere else on X. Consider

$$\sum_{i=0}^{3h-1} f(T^i x)$$

for $x \in \bigcup_{i=0}^{h-1} T^i(I_1 \cup I_2)$. Note for such x,

$$\left| \sum_{i=0}^{3h-1} f(T^i x) - \sum_{i=0}^{3h-1} f(S^i x) \right| > 2n.$$

Since

$$\mu\bigg(\bigcup_{i=0}^{h-1} T^i(I_1 \cup I_2) \bigg) > \frac{1}{5},$$

then

$$\mu\bigg(\bigg\{x \in X : \bigg| \sum_{i=0}^{3h-1} f(T^i x) \bigg| > n \bigg\}\bigg) > \eta.$$

This implies $T \in D_n$ and $\|T - S\| < \epsilon$. Thus, $\mathcal{T} = \bigcap_{n=1}^{\infty} D_n$ is a dense G_δ set. If $T \in \mathcal{T}$, then the Halász–Schmidt condition (2.1) for a measurable transfer function does not hold, and our result follows. □

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A. *Appendix. Coboundary existence for bounded measurable functions*

In this section, we prove Proposition 3.6 which is our main tool used in §3. Also, we prove Lemma A.2 which is used in §3 as well.

A.1. *Balanced partitions.* Let A be a measurable subset of X and $f : A \to \mathbb{R}$ in $L_1(A, \mu_A)$. Let $\epsilon > 0$. We say a finite partition Π of A is ϵ-balanced and uniform, if there exists $E \in \Pi$ such that:

(1) $\mu(E) < \epsilon\mu(A)$;
(2) $\int_{A\setminus E} f \, d\mu = \mu(A \setminus E)/\mu(A) \int_A f \, d\mu$;
(3) $|f(x) - f(y)| < \epsilon$ for $x, y \in a$ and $a \in \Pi \setminus \{E\}$;
(4) $\mu(c) = \mu(d)$ for $c, d \in \Pi \setminus \{E\}$.

We refer to this type of partition as a PUB(ϵ) partition for $f_{|A}$. The set E is referred to as the exceptional set of the PUB.

LEMMA A.1. *Suppose $A \subset X$ is measurable and $f : A \to \mathbb{R}$ is integrable with mean zero and takes on essentially infinitely many values. Given $\epsilon > 0$, there exists a PUB(ϵ) partition such that f takes on essentially infinitely many values on both its exceptional set E and its complement $A \setminus E$.*

Proof. Without loss of generality, it is sufficient to prove the lemma where $0 < \|f\|_\infty < 1$ and $\epsilon < 1$. Let $N \in \mathbb{N}$. Choose $m \in \mathbb{N}$ such that

$$\frac{2}{m} < \epsilon. \tag{A.1}$$

For $i = 0, 1, 2, \ldots, 2m - 1$, let

$$A_i = \left\{ x \in A : -1 + \frac{i}{m} \le f(x) < -1 + \frac{i+1}{m} \right\}. \tag{A.2}$$

Let $\alpha = \min\{\mu(A_i) : \mu(A_i) > 0\}$. There exists i_0 such that f takes on infinitely many values on A_{i_0}. By the definition of A_{i_0}, there exist disjoint subsets E_0 and E_1 of A_{i_0} with equal measure and such that

$$\frac{1}{\mu(E_0)} \int_{E_0} f \, d\mu < \frac{1}{\mu(A_{i_0})} \int_{A_{i_0}} f \, d\mu, \tag{A.3}$$

$$\frac{1}{\mu(E_1)} \int_{E_1} f \, d\mu > \frac{1}{\mu(A_{i_0})} \int_{A_{i_0}} f \, d\mu, \tag{A.4}$$

and f takes on infinitely many values on the set $A_{i_0} \setminus (E_0 \cup E_1)$ and on the set $E_0 \cup E_1$. Let

$$d = \min\left\{ \left| \frac{1}{\mu(E_i)} \int_{E_i} f \, d\mu - \frac{1}{\mu(A_{i_0})} \int_{A_{i_0}} f \, d\mu \right| : i = 0, 1 \right\}.$$

By simultaneous Diophantine approximation [9, p. 14, Theorem VII], there exist $q \in \mathbb{N}$ and $p_i \in \mathbb{N}$ such that

$$q > \max\left\{ \frac{2N}{(1-\epsilon)\mu(A)}, \frac{2\mu(A)}{d\mu(E_1)} \right\}, \tag{A.5}$$

and for $i = 0, 1, \ldots, 2m - 1$,

$$|q\mu(A_i) - p_i| < q^{-1/2m}, \tag{A.6}$$

$$2mq^{-1/2m} < \epsilon, \tag{A.7}$$

$$2mq^{-1/2m} < d\left(\frac{2\alpha}{3} - q^{-1/2m} \right). \tag{A.8}$$

Let $n = q + 1$. Thus,

$$\left| \mu(A_i) - \left(\frac{p_i}{n} + \frac{\mu(A_i)}{n} \right) \right| < n^{-1} q^{-1/2m}. \tag{A.9}$$

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For $i = 0, 1, \ldots, 2m-1$, we can choose subsets $B_i \subset A_i$ such that

$$\mu(B_i) = \mu(A_i) - \frac{p_i}{n}, \tag{A.10}$$

$$\frac{1}{\mu(B_i)} \int_{B_i} f \, d\mu = \frac{1}{\mu(A_i)} \int_{A_i} f \, d\mu. \tag{A.11}$$

Thus,

$$\left| \sum_{i=0}^{2m-1} \int_{B_i} f \, d\mu \right| = \left| \sum_{i=0}^{2m-1} \frac{\mu(B_i)}{\mu(A_i)} \int_{A_i} f \, d\mu \right| = \left| \sum_{i=0}^{2m-1} \left(\frac{\mu(B_i)}{\mu(A_i)} - \frac{1}{n} \right) \int_{A_i} f \, d\mu \right| \tag{A.12}$$

$$\leq \sum_{i=0}^{2m-1} \left| \mu(B_i) - \frac{\mu(A_i)}{n} \right| = \sum_{i=0}^{2m-1} \left| \mu(A_i) - \frac{p_i + \mu(A_i)}{n} \right| \tag{A.13}$$

$$< 2mn^{-1} q^{-1/2m} < \frac{d}{n} \left(\frac{2\alpha}{3} - q^{-1/2m} \right). \tag{A.14}$$

This implies we can choose B_i such that

$$\sum_{i=0}^{2m-1} \int_{B_i} f \, d\mu = 0. \tag{A.15}$$

Let $E = \bigcup_{i=0}^{2m-1} B_i$ and partition each set $A_i \setminus B_i$ into p_i subsets of measure $1/n$ to form Π. Therefore, $\mu(E) < \epsilon \mu(A)$ and our lemma is proven. □

A.2. *Balanced uniform towers.* Let A be a measurable subset of X and $f : A \to \mathbb{R}$ a bounded, mean-zero function. Given finite measurable partition Q, $h \in \mathbb{N}$ and $\epsilon > 0$, an ϵ-balanced and uniform tower for f is a set of disjoint measurable sets $I_i \subset A$ for $i = 1, 2, \ldots, h$ and an invertible measure-preserving map $T : I_i \to I_{i+1}$ for $i = 1, 2, \ldots, h-1$, such that:

$$\mu\left(\bigcup_{i=1}^{h} I_i \right) > \mu(A) - \epsilon, \tag{A.16}$$

$$|f(x) - f(y)| < \epsilon \quad \text{for } x, y \in I_i, 1 \leq i < h, \tag{A.17}$$

$$\left| \sum_{i=0}^{k} f(T^i x) \right| < \|f\|_\infty + \epsilon \quad \text{for } x \in I_1, k < h, \tag{A.18}$$

$$\sum_{i=1}^{h} \int_{I_i} f \, d\mu = \int_A f \, d\mu \; = \; 0, \tag{A.19}$$

$$\left| \sum_{i=0}^{h-1} f(T^i x) \right| < \epsilon \quad \text{for } x \in I_1, \tag{A.20}$$

$$\text{for each } q \in Q, \text{ there exists } \mathcal{I} \subset \{1, \ldots, h\} \text{ such that } \mu\left(q \triangle \left(\bigcup_{i \in \mathcal{I}} I_i \right) \right) < \epsilon. \tag{A.21}$$

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We refer to this type of tower as a TUB (ϵ, h, Q) tower for $f_{|A}$.

LEMMA A.2. *Let* $(X, \mathcal{B}, \mu)$ *be a standard probability space and* A *a measurable subset of* X. *Suppose* $f : A \to \mathbb{R}$, $f \in L_0^\infty$, *takes on essentially infinitely many values. Given* $N \in \mathbb{N}$, $\epsilon > 0$ *and finite measurable partition* Q, *there exists* $h > N$ *such that* f *has a TUB* (ϵ, h, Q) *tower.*

Proof. From the construction of PUB$(\epsilon/3)$ in the previous lemma, partition $A_i \setminus B_i$ into a disjoint union of sets $A_i(j)$ for $j = 1, 2, \ldots, p_i$, such that

$$\mu(A_i(j)) = \frac{1}{n}. \tag{A.22}$$

A.2.1. Greedy stacking. Now we give an inductive procedure for stacking the sets $A_i(j)$. Choose arbitrary $A_i(j)$ and label the set I_1. Given $I_1, I_2, \ldots, I_{k-1}$, let

$$\sigma_{k-1} = \sum_{i=1}^{k-1} \int_{I_i} f\, d\mu. \tag{A.23}$$

If $k = h$, then we are done. If $\sigma_{k-1} \leq 0$, choose

$$I_k = A_i(j) \not\subset \bigcup_{i=1}^{k-1} I_i$$

such that $\int_{I_k} f\, d\mu \geq 0$. This is possible, since $k < h$ and $\sigma_h = \sum_{i=0}^{2m-1} \int_{A_i \setminus B_i} f\, d\mu = 0$. Otherwise, if $\sigma_k > 0$, then by the construction of $A_i(j)$, there exists $I_k \not\subset \bigcup_{i=1}^{k-1} I_i$ such that $\int_{I_k} f\, d\mu < 0$. This procedure produces a sequence of sets I_i for $i = 1, 2, \ldots, h$ with the property:

$$\sum_{i=1}^{h} \int_{I_i} f\, d\mu = \sum_{i=0}^{2m-1} \int_{A_i \setminus B_i} f\, d\mu \tag{A.24}$$

$$= \sum_{i=0}^{2m-1} \int_{B_i} f\, d\mu = 0. \tag{A.25}$$

A.2.2. Level refinement. Our transformation τ will map I_i onto I_{i+1} for $i = 1, 2, \ldots, h-1$. Choose k such that $k > 3h/\epsilon$. Partition the range such that for $\ell = -k, -k+1, \ldots, -1, 0, 1, \ldots, k-1$,

$$B_{i,\ell} = \left\{ x \in I_i : \frac{\ell}{k} \leq f(x) < \frac{\ell+1}{k} \right\}.$$

Via a measure-space isomorphism, we can take $X = [0, 1]$ (that is, an ordered set). Let τ_0 be any invertible measure-preserving map such that $\tau_0 : I_i \to I_{i+1}$ for $i \in \{1, 2, \ldots, h-1\}$. Define the quantized function $f_i(x) = \ell/k$ if $x \in B_{i,\ell}$. Define an invertible measure-preserving map $\psi_i : I_i \to I_i$ such that $f_i \circ \psi_i$ is non-decreasing

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for $i \geq 2$. Let ψ_1 be the identity map on I_1. Define an invertible measure-preserving map $\phi_1 : I_1 \to I_1$ such that $f_1 \circ \phi_1$ is non-increasing. The map $g_2 = f_1 \circ \phi_1 + f_2 \circ \psi_2 \circ \tau_0$ is a step function on I_1. Thus, there exists $\phi_2 : I_1 \to I_1$ such that $g_2 \circ \phi_2$ is non-increasing. Let $g_3 = g_2 \circ \phi_2 + f_3 \circ \psi_3 \circ \tau_0^2$. Continue this process until we have defined g_h. In particular, by induction, g_{h-1} will be a step function on I_1. Thus, we can define $\phi_{h-1} : I_1 \to I_1$ such that $g_{h-1} \circ \phi_{h-1}$ is non-increasing. Let $g_h = g_{h-1} \circ \phi_{h-1} + f_h \circ \psi_h \circ \tau_0^{h-1}$. A formula for g_h is

$$g_h = \sum_{\ell=1}^{h} f_\ell \psi_\ell \tau_0^{\ell-1} \Pi_{j=\ell}^{h-1} \phi_j.$$

For $2 \leq \ell \leq h$, define

$$\tau_\ell = \psi_\ell \tau_0^{\ell-1} \Pi_{j=\ell}^{h-1} \phi_j.$$

Let τ_1 be the identity map. Each τ_ℓ is an invertible measure-preserving mapping from $I_1 \to I_\ell$. Define the final mapping τ as $\tau(x) = \tau_{\ell+1} \circ \tau_\ell^{-1}(x)$ for $x \in I_\ell$. Because of the greedy algorithm of sorting at each stage and re-ordering so that the next level has f_ℓ monotonic in the opposite direction, then the quantized functions f_ℓ do not exhibit much variation as points are iterated through the TUB under τ.

CLAIM A.3. *For $m \in \mathbb{N}$, $m < h$ and a.e. $x, y \in I_1$,*

$$\left| \sum_{\ell=1}^{m} f_\ell(\tau^{\ell-1}x) - \sum_{\ell=1}^{m} f_\ell(\tau^{\ell-1}y) \right| < \frac{\epsilon}{3},$$

that is, there does not exist disjoint subsets D_1, D_2 of I_1 with equal positive measure such that for $x \in D_1$ and $y \in D_2$,

$$\left| \sum_{\ell=1}^{m} f_\ell(\tau^{\ell-1}x) - \sum_{\ell=1}^{m} f_\ell(\tau^{\ell-1}y) \right| \geq \frac{\epsilon}{3}.$$

Proof. It is sufficient to prove that for $m \in \mathbb{N}$, $m < h$ and a.e. $x, y \in I_1$,

$$\left(\sum_{\ell=1}^{m} f_\ell(\tau^{\ell-1}x) - \sum_{\ell=1}^{m} f_\ell(\tau^{\ell-1}y) \right) < \frac{\epsilon}{3}.$$

By applying the invertible measure-preserving isomorphism, $\phi_{h-1}^{-1}\phi_{h-2}^{-1} \cdots \phi_m^{-1}$, it is sufficient to prove for $m \in \mathbb{N}$, $m < h$ and a.e. $x, y \in I_1$,

$$\left(\sum_{\ell=1}^{m} f_\ell \psi_\ell \tau_0^{\ell-1} \Pi_{i=\ell}^{m-1} \phi_i(x) - \sum_{\ell=1}^{m} f_\ell \psi_\ell \tau_0^{\ell-1} \Pi_{i=\ell}^{m-1} \phi_i(y) \right) < \frac{\epsilon}{3}.$$

We can prove the claim inductively on m. Clearly, it is true for $m = 1$ (by applying the PUB condition on I_1). Suppose it is true for $m < h$. Let x_0 and y_0 be distinct points in I_1. Let $x_1 = \phi_m(x_0)$ and $y_1 = \phi_m(y_0)$. Consider first the case:

$$0 < \sum_{\ell=1}^{m} f_\ell \psi_\ell \tau_0^{\ell-1} \Pi_{i=\ell}^{m-1} \phi_i(x_1) - \sum_{\ell=1}^{m} f_\ell \psi_\ell \tau_0^{\ell-1} \Pi_{i=\ell}^{m-1} \phi_i(y_1) < \frac{\epsilon}{3}.$$

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By the construction of ϕ_m, $x_0 < y_0$. This is because the following function is non-increasing in x:

$$\sum_{\ell=1}^{m} f_\ell \psi_\ell \tau_0^{\ell-1} \Pi_{i=\ell}^{m} \phi_i(x).$$

Since the function

$$f_{m+1}\psi_{m+1}\tau_0^{m}(x)$$

is non-decreasing in x, then

$$f_{m+1}\psi_{m+1}\tau_0^{m}(x_0) \leq f_{m+1}\psi_{m+1}\tau_0^{m}(y_0).$$

By combining terms,

$$\sum_{\ell=1}^{m+1} f_\ell \psi_\ell \tau_0^{\ell-1} \Pi_{i=\ell}^{m} \phi_i(x_0) - \sum_{\ell=1}^{m+1} f_\ell \psi_\ell \tau_0^{\ell-1} \Pi_{i=\ell}^{m} \phi_i(y_0) < \frac{\epsilon}{3}.$$

The case where

$$0 < \sum_{\ell=1}^{m} f_\ell \psi_\ell \tau_0^{\ell-1} \Pi_{i=\ell}^{m-1} \phi_i(y_1) - \sum_{\ell=1}^{m} f_\ell \psi_\ell \tau_0^{\ell-1} \Pi_{i=\ell}^{m-1} \phi_i(x_1) < \frac{\epsilon}{3}$$

may be handled in a similar fashion. This completes the proof of the claim. □

Now we complete the proof of the lemma. The function f was quantized to f_ℓ in such a way that for $x \in I_1$ and $m \in \{1, 2, \ldots, h\}$,

$$\left| \sum_{\ell=1}^{m} f(\tau^{\ell-1}x) - \sum_{\ell=1}^{m} f_\ell(\tau^{\ell-1}x) \right| \leq \sum_{\ell=1}^{m} |f(\tau^{\ell-1}x) - f_\ell(\tau^{\ell-1}x)| \tag{A.26}$$

$$< h\left(\frac{\epsilon}{3h}\right) = \frac{\epsilon}{3}. \tag{A.27}$$

Hence,

$$\left| \sum_{\ell=1}^{m} f(\tau^{\ell-1}x) - \sum_{\ell=1}^{m} f(\tau^{\ell-1}y) \right| \leq \left| \sum_{\ell=1}^{m} f(\tau^{\ell-1}x) - \sum_{\ell=1}^{m} f_\ell(\tau^{\ell-1}x) \right| \tag{A.28}$$

$$+ \left| \sum_{\ell=1}^{m} f_\ell(\tau^{\ell-1}x) - \sum_{\ell=1}^{m} f_\ell(\tau^{\ell-1}y) \right| \tag{A.29}$$

$$+ \left| \sum_{\ell=1}^{m} f_\ell(\tau^{\ell-1}y) - \sum_{\ell=1}^{m} f_\ell(\tau^{\ell-1}y) \right| \tag{A.30}$$

$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \tag{A.31}$$

Therefore, this proves (A.20) of our lemma. Claim (A.18) follows in a similar manner. □

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Proof of Proposition 3.6. If $f = \sum_{i=1}^{m} a_i \mathbb{1}_{A_i}$ is a finite step function, a solution is given in [2]. The transfer function g is bounded, since, by [33],

$$g(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \sum_{i=0}^{k-1} f(T^i x) \le \sum_{i=1}^{m} |a_i|.$$

Otherwise, f takes on essentially infinitely many values. Let $\delta_i > 0$ be such that $\sum_{i=1}^{\infty} \delta_i < \infty$ and Q_i for $i \in \mathbb{N}$ a refining sequence of partitions which generate the sigma algebra $\mathcal{B}$. Let $\epsilon_1 = \delta_1$. Use Lemma A.2 to construct a TUB (ϵ_1, h_1, Q_1) tower with decomposition into sets A_1, B_1, and measure-preserving map T_1. Also, assume A_1 is made of levels $I_{1,i}$ for $1 \le i \le h_1$. Let $S_1 = T_1$. Define $f_1 : B_1 \cup I_{1,1} \to \mathbb{R}$ by

$$f_1(x) = \begin{cases} \sum_{i=0}^{h_1-1} f(S_1^i x) & \text{if } x \in I_{1,1}, \\ f(x) & \text{if } x \in B_1. \end{cases}$$

Let $\epsilon_2 = \delta_2 \mu(I_{1,1})$. Since $\int_{B_1 \cup I_{1,1}} f_1 \, d\mu = 0$, then we can apply Lemma A.2 to f_1 to obtain a TUB (ϵ_2, h_2, Q_2) tower and decompose $B_1 \cup I_{1,1}$ into $A_2 = \bigcup_{i=1}^{h_2} I_{2,i}$ and B_2 such that there exists measure-preserving $T_2 : I_{2,i} \to I_{2,i+1}$ for $i = 1, \ldots, h_2 - 1$. Define S_2 as

$$S_2(x) = \begin{cases} S_1(x) & \text{if } x \in S_1^i I_{2,j} \subset S_1^i I_{1,1}, \text{ for } 0 \le i \le h_1 - 1 \text{ and } 1 \le j \le h_2, \\ T_2(S_1^{1-h_1}(x)) & \text{if } x \in S_1^{h_1-1} I_{2,j} \subset S_1^{h_1-1} I_{1,1}, \text{ for } 1 \le j < h_2, \\ T_2(x) & \text{if } x \in B_1 \cap A_2 \setminus I_{2,h_2}. \end{cases}$$

Suppose T_n and S_n have been defined. Proceed in a similar manner to define S_{n+1}. In particular, for a.e. $y \in X \setminus B_n$, there exist a unique $x \in I_{n,1}$ and $j_y \ge 0$ such that $y = S_n^{j_y} x$. For a.e. $x \in I_{n,1}$, there exists a minimum $k_{n,x} \ge 0$ such that $S_n^{k_{n,x}} x \in I_{n,h_n}$. Define $f_n : B_n \cup I_{n,1} \to \mathbb{R}$ such that

$$f_n(x) = \begin{cases} \sum_{i=0}^{k_{n,x}} f(S_n^i x) & \text{if } x \in I_{n,1}, \\ f(x) & \text{if } x \in B_n. \end{cases}$$

Let $\epsilon_{n+1} = \delta_{n+1} \mu(I_{n,1})$. Since $\int_{B_n \cup I_{n,1}} f_n \, d\mu = 0$, then we can apply Lemma A.2 to f_n to obtain a TUB $(\epsilon_{n+1}, h_{n+1}, Q_{n+1})$ tower and decompose $B_n \cup I_{n,1}$ into $A_{n+1} = \bigcup_{i=1}^{h_{n+1}} I_{n+1,i}$ and B_{n+1} such that there exists measure-preserving $T_{n+1} : I_{n+1,i} \to I_{n+1,i+1}$ for $i = 1, \ldots, h_{n+1} - 1$. Define S_{n+1} as

$$S_{n+1}(x) = \begin{cases} S_n(x) & \text{if } x \in S_n^i I_{n+1,j} \subset S_n^i I_{n,1}, \\ & \quad \text{for } 0 \le i \le h_n - 1 \text{ and } 1 \le j \le h_{n+1}, \\ T_{n+1}(S_n^{1-h_n}(x)) & \text{if } x \in S_n^{h_n-1} I_{n+1,j} \subset S_n^{h_n-1} I_{n,1}, \text{ for } 1 \le j < h_{n+1}, \\ T_{n+1}(x) & \text{if } x \in B_n \cap A_{n+1} \setminus I_{n+1,h_{n+1}}. \end{cases}$$

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Note that $S_{n+1}(x) = S_n(x)$ except for x in a set of measure less than or equal to

$$\beta_n = \mu(\{S_n^{k_{n,x}}(\omega) : \omega \in I_{n,1}\} \cup \{S_{n+1}^{k_{n+1,x}}(\omega) : \omega \in I_{n+1,1}\} \cup B_n \cup B_{n+1}).$$

Since $\sum_{n=1}^{\infty} \beta_n < \infty$, then $S(x) = \lim_{n\to\infty} S_n(x)$ exists almost everywhere. In particular, $S_n(x)$ is eventually constant for a.e. $x \in X$. Thus, since each S_n is invertible and measure preserving, then S is invertible and measure preserving. By a careful choice of Q_i, S will be ergodic. □

B. *Appendix. Universal moving averages*

Below we prove that if f is a coboundary with an L^1-transfer function g, then all moving averages converge pointwise.

THEOREM B.1. *Suppose T is an ergodic invertible measure-preserving transformation on $(X, \mathcal{B}, \mu)$. If f is a coboundary with integrable transfer function g, then for all strictly increasing $L_n \in \mathbb{N}$, $v_n \in \mathbb{Z}$, and a.e. $x \in X$,*

$$\lim_{n\to\infty} \frac{1}{L_n} \sum_{i=1}^{L_n} f(T^{v_n+i}x) = 0.$$

Proof. Suppose $f(x) = g(x) - g(Tx)$ for a.e. $x \in X$. Then

$$\sum_{i=1}^{L_n} f(T^{v_n+i}x) = g(T^{v_n+1}x) - g(T^{v_n+L_n+1}x).$$

Since $L_n \geq n$ is strictly increasing, it is sufficient to show each of the following:

- for a.e. x, $\lim_{n\to\infty} g(T^{v_n+1}x)/n = 0$;
- for a.e. x, $\lim_{n\to\infty} g(T^{v_n+L_n+1}x)/n = 0$.

Here, we show the first term converges to zero. A similar argument will show the second term converges.

For $n, k \in \mathbb{N}$, define

$$E_{n,k} = \{x \in X : n - 1 \leq |g(T^{v_k+1}x)| < n\}.$$

Since $g \in L^1(\mu)$, then

$$\sum_{n=1}^{\infty} (n-1)\mu(E_{n,1}) \leq \int_X |g(T^{v_1+1}x)|\, d\mu < \infty.$$

There exists non-decreasing $K_n \in \mathbb{N}$ such that $\lim_{n\to\infty} n/K_n = 0$ and

$$\sum_{n=1}^{\infty} K_n \mu(E_{n,1}) < \infty.$$

Define

$$E = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \bigcup_{k=1}^{K_n} E_{n,k}.$$

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First, since T is measure preserving, we have

$$\mu(E) \leq \sum_{n=m}^{\infty}\sum_{k=1}^{K_n}\mu(E_{n,k}) = \sum_{n=m}^{\infty}\sum_{k=1}^{K_n}\mu(E_{n,1})$$
$$= \sum_{n=m}^{\infty} K_n\mu(E_{n,1}) \to 0 \quad \text{as } m \to \infty.$$

Thus, $\mu(E) = 0$. Second, we show for $x \notin E$, we have almost everywhere convergence. If $x \notin E$, there exists m sufficiently large such that

$$x \notin \bigcup_{k=1}^{K_n} E_{n,k} \quad \text{for } n \geq m.$$

Hence, if $n > |g(T^{v_k+1}x)| \geq n-1$, then $k > K_n$. Therefore,

$$\frac{|g(T^{v_k+1}x)|}{k} < \frac{n}{K_n} \to 0 \quad \text{as } n \to \infty. \qquad \square$$

COROLLARY B.2. *Suppose $(X, \mathcal{B}, \mu)$ is a standard probability space and $f \in L_0^2(\mu)$. There exists an ergodic invertible measure-preserving transformation T on $(X, \mathcal{B}, \mu)$ such that all moving averages converge for f. In particular, for all strictly increasing $L_n \in \mathbb{N}$, $v_n \in \mathbb{Z}$, and a.e. $x \in X$,*

$$\lim_{n\to\infty}\frac{1}{L_n}\sum_{i=1}^{L_n} f(T^{v_n+i}x) = 0.$$

Proof. By Theorem 1.1, there exists a solution pair T and $g \in L^1(\mu)$ such that $f = g - g \circ T$ almost everywhere. Therefore, by Theorem B.1, all moving averages converge to zero for f and T. $\square$

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