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# WEAK COMPACTNESS OF FRÉCHET-DERIVATIVES: APPLICATION TO COMPOSITION OPERATORS

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**Abstract**

We prove a result on compactness properties of Fréchet-derivatives which implies that the Fréchet-derivative of a weakly compact map between Banach spaces is weakly compact. This result is applied to characterize certain weakly compact composition operators on Sobolev spaces which have application in the theory of nonlinear integral equations and in the calculus of variations.

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## 1. Introduction and notation

Let $(E, \mathfrak{R})$ and $(F, \mathfrak{S})$ be topological linear Hausdorff spaces over the field $\mathbf{R}$, and let $\Omega$ be an open subset of $(E, \mathfrak{R})$. The 0-neighbourhood filter of $(E, \mathfrak{R})$ will be denoted by $\mathfrak{U}_0(E, \mathfrak{R})$.

Let $\Phi : (\Omega, \mathfrak{R} \cap \Omega) \to (F, \mathfrak{S})$ be a continuous map. For $x \in \Omega$, $y \in E$, $t \in \mathbf{R} \setminus \{0\}$ such that $x + ty \in \Omega$ we define

$$\Delta\Phi(x, y, t) := \frac{1}{t}(\Phi(x + ty) - \Phi(x)).$$

Let $\bar{x} \in \Omega$ and $U \in \mathfrak{U}_0(E, \mathfrak{R})$ satisfy $\bar{x} + U \subset \Omega$, and let $B$ be a bounded subset of $(E, \mathfrak{R})$ which is balanced (that is, $tB \subset B$ for all $|t| \leqslant 1$). Then since $U$ is absorbing there exists $\delta \in (0, 1)$ such that $\Delta\Phi(\bar{x}, y, t)$ is defined for all $0 < |t| < \delta$ and all $y \in B$. Moreover we have

$$\Delta\Phi(\bar{x}, y, t) \in \frac{1}{t}(\Phi(\bar{x} + \delta B) - \Phi(\bar{x}))$$

for all $y \in B$, $0 < |t| < \delta$.



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A map $\Phi : (\Omega, \mathfrak{R} \cap \Omega) \to (F, \mathfrak{S})$ is called *Fréchet-differentiable at* $\bar{x} \in \Omega$ if there is a continuous linear map $A : (E, \mathfrak{R}) \to (F, \mathfrak{S})$ such that

$$\lim_{\substack{t \to 0 \\ t \neq 0}} (\Delta\Phi(\bar{x}, y, t) - A(y)) = 0$$

in $(F, \mathfrak{S})$ uniformly with respect to $y$ on each bounded subset $B$ of $(E, \mathfrak{R})$. In this case the (uniquely determined) linear map $A$ is called the *Fréchet-derivative of* $\Phi$ *at* $\bar{x}$, and it will be denoted by $D\Phi(\bar{x})$ (compare Yamamuro (1974), p. 7, § 1.2).

Definition (1.1). A continuous map $\Phi : (\Omega, \mathfrak{R} \cap \Omega) \to (F, \mathfrak{S})$ is called *compact* (*precompact*, *sequentially compact*, et cetera) *at* $\bar{x} \in \Omega$ if for each bounded subset $B$ of $(E, \mathfrak{R})$ there exists $\delta > 0$ such that $\Phi(\bar{x} + \delta B)$ is relatively compact (precompact, relatively sequentially compact et cetera) in $(F, \mathfrak{S})$. $\Phi$ is called *compact* (*precompact* et cetera) if $\Phi$ is compact (precompact et cetera) at every point of $\Omega$.

This definition differs from the notions of compactness of nonlinear maps one finds elsewhere (compare Batt (1970), p. 5; Schwartz (1965), p. 34, Definition 1.39; Yamamuro (1974), p. 43, (2.1.1); Krasnosel'skiĭ (1964), p. 15, 3). Compactness in the sense of Definition (1.1) is implied by the notions of compactness of the above-mentioned authors. Moreover, if $(E, \mathfrak{R})$ is a normed space, Definition (1.1) coincides with the definitions of Schwartz (1965), p. 34, Definition 1.39, and Yamamuro (1974), p. 43, (2.1.1), and for linear maps (1.1) coincides with the usual definition of compactness :

A continuous linear map $T : (E, \mathfrak{R}) \to (F, \mathfrak{S})$ is called compact (precompact, sequentially compact et cetera) if for every bounded subset $B$ of $(E, \mathfrak{R})$ the image $T(B)$ is relatively compact (precompact, relatively sequentially compact et cetera).

For several special cases the following proposition is well known (compare Vainberg (1964), p. 51, Theorem 4.7; Schwartz (1965), p. 34, Theorem 1.40; Yamamuro (1974), p. 45, line 8 from below).

Proposition (1.2). *Let* $\Phi : (\Omega, \mathfrak{R} \cap \Omega) \to (F, \mathfrak{S})$ *be precompact and Fréchet-differentiable at* $\bar{x} \in \Omega$. *Then* $D\Phi(\bar{x}) : (E, \mathfrak{R}) \to (F, \mathfrak{S})$ *is precompact.*

Proof. Let $B$ be a bounded subset of $(E, \mathfrak{R})$. We may assume that $B$ is balanced. By hypothesis there is $\delta \in (0, 1)$ such that $\bar{x} + \delta B \subset \Omega$ and $\Phi(\bar{x} + \delta B)$ is precompact in $(F, \mathfrak{S})$. Let $V \in \mathfrak{U}_0(F, \mathfrak{S})$ be given. Then there exists $\eta \in (0, \delta]$ such that $\Delta\Phi(\bar{x}, y, t) - D\Phi(\bar{x})(y) \in V$ holds for all $y \in B$ and all $0 < |t| < \eta$. We therefore obtain

$$D\Phi(\bar{x})(B) \subset \Delta\Phi(\bar{x}, B, t) - V$$
$$\subset \frac{1}{t} (\Phi(\bar{x} + \delta B) - \Phi(\bar{x})) - V$$

for all $0 < |t| < \eta$. Thus $D\Phi(\bar{x})(B)$ is precompact in $(F, \mathfrak{S})$. q.e.d.

## 2. Weak compactness of Fréchet-derivatives

We now consider Fréchet-differentiable maps $\Phi : (\Omega, \mathfrak{R} \cap \Omega) \to (F, \mathfrak{T})$ such that $\Phi$ is sequentially compact for a linear Hausdorff topology $\mathfrak{S} \subset \mathfrak{T}$ on $F$. We shall prove that under some mild restrictions the Fréchet-derivative considered as a map from $(E, \mathfrak{R})$ into $(F, \mathfrak{S})$ is also sequentially compact.

The proof of our result is based on the following lemma.

LEMMA (2.1). *Let $X$ be a set and $F$ a linear space. Let $\mathfrak{S}$ and $\mathfrak{T}$ be two linear Hausdorff topologies on $F$ such that*

(a) $\mathfrak{S} \subset \mathfrak{T}$.

(b) *$\mathfrak{T}$ is $\mathfrak{S}$-polar, that is $\mathfrak{T}$ has a 0-neighbourhood base consisting of $\mathfrak{S}$-closed sets.*

(c) *$\mathfrak{T}$ is sequentially complete.*

*Let $f_n : X \to F$ $(n \in \mathbf{N}_0)$ be a sequence of maps such that*

(d) $\lim_{n\to\infty} f_n(x) = f_0(x)$ *$\mathfrak{T}$-uniformly on $X$.*

(e) *$f_n(X)$ is relatively $\mathfrak{S}$-sequentially compact for $n \geqslant 1$.*

*Then $f_0(X)$ is relatively $\mathfrak{S}$-sequentially compact.*

PROOF. Let $(y_k;\ k \in \mathbf{N})$ be a sequence in $f_0(X)$ and let $(x_k;\ k \in \mathbf{N})$ be a sequence in $X$ such that $f_0(x_k) = y_k$ $(k \in \mathbf{N})$. Since for all $n \in \mathbf{N}$ the set $f_n(X)$ is relatively $\mathfrak{S}$-sequentially compact, we find a subsequence $(x_{k(j)};\ j \in \mathbf{N})$ such that for all $n \in \mathbf{N}$ the sequence $(f_n(x_{k(j)});\ j \in \mathbf{N})$ $\mathfrak{S}$-converges to some point $z_n \in F$. We claim that $(z_n;\ n \in \mathbf{N})$ is a $\mathfrak{T}$-Cauchy-sequence. Let $V \in \mathfrak{U}_0(F, \mathfrak{T})$ be given. Since $\mathfrak{T}$ is $\mathfrak{S}$-polar we may suppose that $V$ is $\mathfrak{S}$-closed. $(f_n;\ n \in \mathbf{N})$ converges uniformly on $X$ with respect to $\mathfrak{T}$. Thus there is $n_0 \in \mathbf{N}$ such that $f_n(x_{k(j)}) - f_m(x_{k(j)}) \in V$ holds for all $j \in \mathbf{N}$ and all $m, n \geqslant n_0$. This implies $z_n - z_m \in V$ for all $m, n \geqslant n_0$ since $V$ is $\mathfrak{S}$-closed.

By hypothesis $(F, \mathfrak{T})$ is sequentially complete. Thus $(z_n;\ n \in \mathbf{N})$ $\mathfrak{T}$-converges to some $z \in F$. We show that $(f_0(x_{k(j)});\ j \in \mathbf{N})$ $\mathfrak{S}$-converges to $z$.

Let $V \in \mathfrak{U}_0(F, \mathfrak{S})$ be given and choose $W \in \mathfrak{U}_0(F, \mathfrak{S})$ such that $W + W + W \subset V$ holds. Since $(f_n;\ n \in \mathbf{N})$ converges to $f_0$ uniformly on $X$ with respect to $\mathfrak{T}$ we find $n_0 \in \mathbf{N}$ such that $f_0(x) - f_n(x) \in W$ holds for all $x \in X$, $n \geqslant n_0$. By the $\mathfrak{T}$-convergence of $(z_n;\ n \in \mathbf{N})$ to $z$ we find $m \geqslant n_0$ such that $z_n - z \in W$ holds for all $n \geqslant m$. Finally $(f_m(x_{k(j)});\ j \in \mathbf{N})$ $\mathfrak{S}$-converges to $z_m$. Thus there is $j_0 \in \mathbf{N}$ such that $f_m(x_{k(j)}) - z_m \in W$ holds for all $j \geqslant j_0$. Taking these three estimates together we obtain

$$\begin{aligned} f_0(x_{k(j)}) - z &= (f_0(x_{k(j)}) - f_m(x_{k(j)})) + (f_m(x_{k(j)}) - z_m) + (z_m - z) \\ &\in W + W + W \subset V \end{aligned}$$

for all $j \geqslant j_0$. Thus $f_0(X)$ is relatively $\mathfrak{S}$-sequentially compact. q.e.d.

PROPOSITION (2.2). *Let $\mathfrak{S} \subset \mathfrak{T}$ be two linear Hausdorff topologies on F such that $\mathfrak{T}$ is $\mathfrak{S}$-polar and sequentially complete. Let $\Phi : (\Omega, \mathfrak{R} \cap \Omega) \to (F, \mathfrak{S})$ be sequentially compact at $\bar{x} \in \Omega$, and let $\Phi : (\Omega, \mathfrak{R} \cap \Omega) \to (F, \mathfrak{T})$ be Fréchet-differentiable at $\bar{x}$. Then $D\Phi(\bar{x}) : (E, \mathfrak{R}) \to (F, \mathfrak{S})$ is sequentially compact.*

PROOF. Let $B$ be a bounded subset of $(E, \mathfrak{R})$. We may assume that $B$ is balanced. By hypothesis there exists $\delta \in (0,1)$ such that $\bar{x} + tB \subset \Omega$ and $\Phi(\bar{x} + tB)$ is relatively $\mathfrak{S}$-sequentially compact for all $|t| \leqslant \delta$. Now we choose a sequence $(t_n;\ n \in \mathbf{N})$ in $\mathbf{R} \setminus \{0\}$, $|t_n| < \delta (n \in \mathbf{N})$, $t_n \to 0 (n \to \infty)$ and apply Lemma (2.1) with $X := B, f_n(y) := \Delta\Phi(\bar{x}, y, t_n)$ $(n \in \mathbf{N},\ y \in \mathbf{B})$ to obtain that $D\Phi(\bar{x})(B)$ is relatively $\mathfrak{S}$-sequentially compact. q.e.d.

In particular Proposition (2.2) applies to every sequentially complete locally convex Hausdorff space $(F, \mathfrak{T})$ if we take for $\mathfrak{S}$ the weak topology $\sigma(F, F')$.

For a large class of locally convex Hausdorff spaces $(F, \mathfrak{T})$ the relatively $\sigma(F, F')$-sequentially compact sets and the relatively $\sigma(F, F')$-compact sets coincide. For instance, all locally convex Hausdorff spaces $(F, \mathfrak{T})$ which admit a metrizable topology coarser than $\mathfrak{T}$ share this property (Floret (1978), p. 30 and p. 39 (2)). We only note one special case.

COROLLARY (2.3). *Let $(F, \mathfrak{T})$ be a Fréchet space, and let $\Phi : (\Omega, \mathfrak{R} \cap \Omega) \to (F, \mathfrak{T})$ be Fréchet-differentiable at $\bar{x} \in \Omega$. If $\Phi$ is weakly compact at $\bar{x}$ then $D\Phi(\bar{x})$ is weakly compact.*

## 3. Application to composition operators

Let $I = (a, b) \subset \mathbf{R}$ be a bounded interval, $\bar{I} := [a, b]$. For $m \in \{0, 1, 2\}$ and $p \in [1, \infty)$ we denote by $W^{m,p}(I)$ the usual Sobolev space,

$$W^{m,p}(I) := \{u \in L^p(I);\ u^{(j)} \in L^p(I) \text{ for } 0 \leqslant j \leqslant m\},$$

$$\|u\|_{m,p} := \left( \sum_{j=0}^{m} \|u^{(j)}\|_p^p \right)^{1/p} \quad (u \in W^{m,p}(I))$$

(compare Adams (1975), p. 45). For $u \in C^m(\bar{I})$ we define

$$\|u\|_{m,\infty} := \max \{ |u^{(j)}(x)|;\ x \in \bar{I},\ 0 \leqslant j \leqslant m\}.$$

Let $\Theta : \bar{I} \times \mathbf{R} \to \mathbf{R}$ and $\Psi : \bar{I} \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be continuous. We consider composition operators of the form

$$\Theta_*(\mathrm{u})(\mathrm{x}) := \Theta(x, u(x)) \quad (u \in L^p(I)), \tag{3.1}$$

$$\Psi_*(\mathrm{u})(\mathrm{x}) := \Psi(x, u(x), u'(x)) \quad (u \in W^{1,p}(I)). \tag{3.2}$$

These composition operators play an important role in the theory of nonlinear integral equations (Krasnosel'skiĭ (1964), p. 46; Vainberg (1964), p. 147) and in the calculus of variations (Michlin (1972), p. 33ff.).

If $\Psi$ is sufficiently smooth, we write $\Psi_j$, $\Psi_{jk}$ instead of $\partial_j\Psi, \partial_{jk}\Psi (j,k = 1,2,3)$. By $\Psi_{j*}$ and $\Psi_{jk*}$ we denote the composition operators generated by $\Psi_j$ and $\Psi_{jk}$ according to (3.2). The corresponding notation is used for $\Theta$ instead of $\Psi$.

For convenience of the reader we collect first some elementary statements concerning Sobolev spaces which we use later.

(3.3) *$C^m(\bar{I})$ is dense in $(W^{m,p}(I), \|\cdot\|_{m,p})$* (Adams (1975), p. 54, Theorem 3.18).

(3.4) *For $m = 1, 2$ and $p \in [1, \infty)$, there exist continuous embeddings*

$$j_{m,p} : (W^{m,p}(I), \|\cdot\|_{m,p}) \to (C^{m-1}(\bar{I}), \|\cdot\|_{m-1,\infty}).$$

(3.5) *For $m = 1, 2$ and $p \in [1, \infty)$ the inclusions*

$$i_m : (W^{m,p}(I), \|\cdot\|_{m,p}) \to (W^{m-1,p}(I), \|\cdot\|_{m-1,p})$$

*are compact* (Adams (1975), p. 144, Theorem 6.2, (4) and (6)).

(3.6) *For every open subinterval $J$ in $I$ the restriction map*

$$R_J : (W^{m,p}(I), \|\cdot\|_{m,p}) \to (W^{m,p}(J), \|\cdot\|_{m,p}),\ R_J(u) := u \mid J,$$

*is (defined, continuous and) surjective.*

(3.7) *For $m = 1, 2$ and $p \in [1, \infty)$ the differentiation map*

$$\partial : (W^{m,p}(I), \|\cdot\|_{m,p}) \to (W^{m-1,p}(I), \|\cdot\|_{m-1,p})$$

*is (continuous and) surjective.*

(3.8) *Let $w \in W^{1,p}(I)$ satisfy $|w(x)| \geqslant \delta > 0$ $(x \in \bar{I})$. Then*

$$[w] : (W^{1,p}(I), \|\cdot\|_{1,p}) \to (W^{1,p}(I), \|\cdot\|_{1,p}),\ [w](u) := w \cdot u,$$

*is a topological isomorphism.*

Now we return to the study of the composition operators. Let $H$ be a $\|\cdot\|_{1,\infty}$-bounded subset of $C^1(\bar{I})$; then there are compact intervals $K, K' \subset \mathbf{R}$ such that $(u(x), u'(x)) \in K \times K'$ $(x \in \bar{I},\ u \in H)$ holds. Moreover, the restriction of $\Psi$ to $\bar{I} \times K \times K'$ is uniformly continuous. Using these facts we obtain

(3.9) *Let $H$ be a bounded subset of $(C^1(\bar{I}), \|\cdot\|_{1,\infty})$. Then $\Psi_*(H)$ is bounded in $(C(\bar{I}), \|\cdot\|_\infty)$, and the restriction $\Psi_* \mid H$ of $\Psi_* : (C^1(\bar{I}), \|\cdot\|_{1,\infty}) \to (C(\bar{I}), \|\cdot\|_\infty)$ to $H$ is uniformly continuous.*

A similar statement holds for $\Theta$ instead of $\Psi$.

(3.10) *Let $M$ be a bounded subset of $(C(\bar{I}), \|\cdot\|_\infty)$. Then $\Theta_*(M)$ is bounded in $(C(\bar{I}), \|\cdot\|_\infty)$, and the restriction $\Theta_* \mid M$ of $\Theta_*$: $(C(\bar{I}), \|\cdot\|_\infty) \to (C(\bar{I}), \|\cdot\|_\infty)$ to $M$ is uniformly continuous.*

Using (3.3), (3.4), (3.9), and (3.10) one easily proves the following two statements.

(3.11) *Let $\Theta : \bar{I} \times \mathbf{R} \to \mathbf{R}$ have continuous partial derivatives up to the order two. For every $u \in W^{2,p}(I)$ the functions $\Theta_*(u)$ and $\Theta_*(u)'$ are absolutely continuous, and we have*

$$\Theta_*(u)' = \Theta_{1*}(u) + \Theta_{2*}(u) \cdot u',$$

$$\Theta_*(u)'' = \Theta_{11*}(u) + 2\Theta_{12*}(u) \cdot u' + \Theta_{22*}(u) \cdot (u')^2 + \Theta_{2*}(u) \cdot u''.$$

*Thus in particular $\Theta_*(W^{2,p}(I)) \subset W^{2,p}(I)$, and $\Theta_* : W^{2,p}(I) \to W^{2,p}(I)$ maps $\|\cdot\|_{2,p}$-bounded sets into $\|\cdot\|_{2,p}$-bounded sets.*

(3.12) *Let $\Psi : \bar{I} \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ have continuous partial derivatives up to the order one. For every $u \in W^{2,p}(I)$ the function $\Psi_*(u)$ is absolutely continuous, and we have*

$$\Psi_*(u)' = \Psi_{1*}(u) + \Psi_{2*}(u) \cdot u' + \Psi_{3*}(u) \cdot u''.$$

*In particular $\Psi_*(W^{2,p}(I)) \subset W^{1,p}(I)$, and $\Psi_* : W^{2,p}(I) \to W^{1,p}(I)$ maps $\|\cdot\|_{2,p}$-bounded sets into $\|\cdot\|_{1,p}$-bounded sets.*

There is one more proposition which we do not prove in detail, namely

PROPOSITION (3.13). *Let $\Psi : \bar{I} \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ have continuous partial derivatives up to the order two. Then $\Psi_* : (W^{2,p}(I), \|\cdot\|_{2,p}) \to (W^{1,p}(I), \|\cdot\|_{1,p})$ is Fréchet-differentiable, and we have*

$$(D\Psi_*)(u)(v) = \Psi_{2*}(u) \cdot v + \Psi_{3*}(u) \cdot v',$$

$$(D\Psi_*)(u)(v)' = \Psi_{2*}(u)' \cdot v + \Psi_{2*}(u) \cdot v' + \Psi_{3*}(u)' \cdot v' + \Psi_{3*}(u) \cdot v''$$

$(u, v \in W^{2,p}(I))$.

The proof of (3.13) consists of a straightforward (but tiresome) estimate using (3.4), (3.9), (3.12), and the mean value theorem for $\Psi$ and $\Psi_j$ $(j = 1, 2, 3)$.

It is known that a compact composition operator $\Theta_* : (L^p(I), \|\cdot\|_p) \to (L^p(I), \|\cdot\|_p)$ of the form $\Theta_*(u)(x) = \Theta(x, u(x))$ is necessarily constant, that is, this operator is generated by a function $\Theta : \bar{I} \times \mathbf{R} \to \mathbf{R}$ which does not depend on the second argument (compare for instance Krasnosel'skiĭ (1964), pp. 30–31). We now show that the composition operators of the form $\Psi_*$ behave similarly.

THEOREM (3.14). *Let $\Psi : \bar{I} \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ have continuous partial derivatives up to the order two. For $p \in [1, \infty)$ the following statements are equivalent:*

(a) $\partial_3 \Psi(x, y, z) = 0$ *for all* $(x, y, z) \in \bar{I} \times \mathbf{R} \times \mathbf{R}$.

(b) $\Psi_* : (W^{2,p}(I), \|\cdot\|_{2,p}) \to (W^{1,p}(I), \|\cdot\|_{1,p})$ *is compact.*

*For $p = 1$ the above statements (a) and (b) are equivalent to*

(c) $\Psi_* : (W^{2,1}(I), \|\cdot\|_{2,1}) \to (W^{1,1}(I), \|\cdot\|_{1,1})$ *is weakly compact.*

PROOF. (a) $\Rightarrow$ (b) : We define $\Theta(x, y) := \Psi(x, y, 0)$ $(x \in \bar{I},\ y \in \mathbf{R})$. According to (a) we have $\Theta_*(u) = \Psi_*(u)$ $(u \in W^{2,p}(I))$. Let $\bar{u} \in W^{2,p}(I)$ and let $U$ be a bounded neighbourhood of $\bar{u}$ in $(W^{2,p}(I), \|\cdot\|_{2,p})$. By (3.11) the image $\Theta_*(U)$ is bounded in $(W^{2,p}(I), \|\cdot\|_{2,p})$, thus relatively compact in $(W^{1,p}(I), \|\cdot\|_{1,p})$ by (3.5).

(b) $\Rightarrow$ (a) : This will be proved by contradiction. Assume there is $(\bar{x}, \bar{y}, \bar{z}) \in \bar{I} \times \mathbf{R} \times \mathbf{R}$ such that $\Psi_3(\bar{x}, \bar{y}, \bar{z}) \neq 0$ holds. Since $\Psi_3$ is continuous we may assume $\bar{x} \in I$. We define $\bar{u}(x) := \bar{y} + \bar{z}(x - \bar{x})$ $(x \in \bar{I})$ and obtain $\bar{u} \in W^{2,p}(I)$, $\Psi_{3*}(\bar{u})(\bar{x}) = \Psi_3(\bar{x}, \bar{y}, \bar{z}) \neq 0$. Since $\Psi_{3*}(\bar{u})$ is continuous (compare (3.4)), there is $\delta > 0$ and an open interval $J \subset I$ such that $\bar{x} \in J$ and $|\Psi_{3*}(\bar{u})(x)| \geqslant \delta$ holds for all $x \in J$. Using (3.5) we obtain that $v \mapsto \Psi_{2*}(\bar{u}) \cdot v$, considered as a map from $(W^{2,p}(I), \|\cdot\|_{2,p})$ into $(W^{1,p}(I), \|\cdot\|_{1,p})$, is compact. From (3.6), (3.7), and (3.8) we infer that $v \mapsto R_J(\Psi_{3*}(\bar{u}) \cdot v')$, considered as a map from $(W^{2,p}(I), \|\cdot\|_{2,p})$ into $(W^{1,p}(J), \|\cdot\|_{1,p})$, is continuous and surjective and therefore—by the open mapping theorem—is open. This implies that $v \mapsto \Psi_{3*}(u) \cdot v'$, considered as a map from $(W^{2,p}(I), \|\cdot\|_{2,p})$ into $(W^{1,p}(I), \|\cdot\|_{1,p})$ is not compact. (Alternatively one can construct a bounded sequence $(v_n;\ n \in \mathbf{N})$ in $W^{2,p}(I)$ such that $(\Psi_{3*}(\bar{u}) \cdot v_n';\ n \in \mathbf{N})$ contains no $\|\cdot\|_{1,p}$-convergent subsequence.) Therefore $(D\Psi_*)(\bar{u})$: $(W^{2,p}(I), \|\cdot\|_{2,p}) \to (W^{1,p}(I), \|\cdot\|_{1,p})$,

$$(D\Psi_*)(\bar{u})(v) = \Psi_{2*}(\bar{u}) \cdot v + \Psi_{3*}(\bar{u}) \cdot v',$$

is not compact. This contradicts (b) by Proposition (1.2).

For $p = 1$ it is obviously sufficient to prove (c) $\Rightarrow$ (a). We proceed as above to show that $v \mapsto \Psi_{2*}(\bar{u}) \cdot v$ is compact, and that $v \mapsto R_J(\Psi_{3*}(\bar{u}) \cdot v')$ is open. The latter now implies that $v \mapsto R_J(\Psi_{3*}(\bar{u}) \cdot v')$ cannot be weakly compact, since $(W^{1,1}(J), \|\cdot\|_{1,1})$ is not reflexive. ($(L^1(J), \|\cdot\|_1)$ is a quotient of $(W^{1,1}(I), \|\cdot\|_{1,1})$ by (3.7).) Therefore $(D\Psi_*)(\bar{u})$: $(W^{2,1}(I), \|\cdot\|_{2,1}) \mapsto (W^{1,1}(I), \|\cdot\|_{1,1})$ is not weakly compact. This contradicts (c) by Corollary (2.3). q.e.d.

ADDED IN PROOF: Lemma (2.1) remains true if $\mathfrak{S}$-sequentially compact is replaced by $\mathfrak{S}$-compact.

For the proof of this variant of (2.1) let $\mathfrak{F}$ be a filter on $\overline{f_0(X)}^{\mathfrak{S}}$; then the corresponding minimal filter $\hat{\mathfrak{F}} := \{A + U;\ A \in \mathfrak{F},\ U \in \mathfrak{U}_0(F, \mathfrak{S})\}$ has a trace $\mathfrak{G}$ on $f_0(X)$. Choose an ultrafilter $\mathfrak{H}$ on $X$ containing $\{f_0^{-1}(G);\ G \in \mathfrak{G}\}$. Now the proof of Lemma (2.1) may be adapted to show that $f_0(\mathfrak{H})$ is $\mathfrak{S}$-convergent.]

As a consequence Proposition (2.2) remains true if sequentially compact is replaced by compact.

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