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STIRRING OUR WAY TO SHARKOVSKY'S THEOREM

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The periodic-point or cycle structure of a continuous map of a topological space has been a subject of great interest since A.N. Sharkovsky completely explained the hierarchy of periodic orders of a continuous map $f : R \to R$, where R is the real line. In this paper the topological idea of "stirring" is invoked in an effort to obtain a transparent proof of a generalisation of Sharkovsky's Theorem to continuous functions $f : I \to I$, where I is any interval. The stirring approach avoids all graph-theoretical and symbolic abstraction of the problem in favour of a more concrete intermediate-value-theorem-oriented analysis of cycles inside an interval.

1. INTRODUCTION

Let X be a topological space, $f : X \to X$ a continuous map of X, and denote the n^{th} iteration of f by f^n. Let $k > 1$. A point $p \in X$ is a point of period n for f in X if $f^m(p) \neq p$, for $1 \leqslant m \leqslant n-1$ and $f^n(p) = p$. The points of the forward orbit of $p : \{p, f(p), \ldots, f^{n-1}(p)\}$ are said to form an n-cycle. In the early 1960's, Sharkovsky [**7**] elucidated the hierarchy of periodic orders for a continuous map of R, the real line. He discovered the following for $X = R$:

SHARKOVSKY'S THEOREM. *Assume $f : X \to X$ is a continuous function. If p precedes q in the following ordering of the natural numbers, then the existence of a p-cycle for f in X implies the existence of a q-cycle for f in X:*

$$3,\ 5,\ 7,\ \ldots,\ 2\cdot 3,\ 2\cdot 5,\ 2\cdot 7,\ \ldots,\ 2^2\cdot 3,\ 2^2\cdot 5,\ 2^2\cdot 7,\ \ldots,\ 2^2,\ 2,\ 1$$

where $2^j(2m+1)$ precedes $2^k(2n+1)$ if $0 \leqslant j, k, m, n$ and exactly one of the following conditions is satisfied:

(i) $j = k$ *and* $1 \leqslant m < n$;
(ii) $j < k$ *and* $1 \leqslant m, n$;
(iii) $j > k$ *and* $m = n = 0$;
(iv) $m > n = 0$.

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It is well known that the converse of Sharkovsky's Theorem also holds. More precisely, for each natural number n, there exists a continuous function $f_n : J \to J$, where J is a compact interval, such that the leftmost Sharkovsky period for f_n in J is n [**2, 7, 9**]. Since his discovery, Sharkovsky's work has been advanced. Stefan [**9**] translated Sharkovsky's Russian paper into English in 1977. Straffin [**10**] offered a graph-theoretic proof of part of Sharkovsky's Theorem one year later. Block, Guckenheimer, Misiurewicz and Young [**2**] employed a similar Markov-graph representation of the periodic structure of f to obtain a complete proof of the theorem in 1979. A paper of Ho and Morris [**4**] completed Straffin's work using his k-periodic digraph methodology as a means of interpreting the periodic structure of f in 1981. Burkhart [**3**] completed Sraffin's work independently in 1982.

In addition to work on the original statement of Sharkovsky's result, there has been considerable work on generalising this notion of a periodic hierarchy to other spaces X. Block [**1**] succeeded in extending the Theorem to $X = S^1$, the circle, with the additional hypothesis that f has a fixed point. Schirmer [**8**] succeeded in showing that Sharkovsky's Theorem remains true for $X = L$, a linear continuum, with the order topology. A linear continuum is a linear ordered set with more than one point such that

(i) L has the least upper bound property,

(ii) L is order dense, that is, if $x < y$, then there exists z so that $x < z < y$.

Munkres provides an introduction to this idea of linear continua in [**6**]. The generalisation of the original statement of Sharkovsky's Theorem to higher-dimensional Euclidean space is impeded by the need for the order relation of a linear continuum. However, Kloeden [**5**] discovered a generalisation to $X = C$, a compact subset of R^n, with the additional hypothesis that i^{th} component of f depends on the first i independent variables.

2. A stirring proof

We shall show that Sharkovsky's Theorem holds for $X = I$, where I is any interval, by use of the following topological notion of stirring. A point $a \in I$ such that

$$f^{2k}(a) < \cdots < f^4(a) < f^2(a) < f^{2k+1}(a) \leqslant a < f(a) < f^3(a) < \cdots < f^{2k-1}(a)$$
$$[\text{respectively} f^{2k}(a) > \cdots > f^4(a) > f^2(a) > f^{2k+1}(a) \geqslant a > f(a) > f^3(a) > \cdots > f^{2k-1}(a)]$$

will be called a $2k+1$-*stirring point.* If $k = 1$, we simply require $f^2(a) < a < f(a)$ and $f^3(a) \leqslant a$ [respectively $f^2(a) > a > f(a)$ and $f^3(a) \geqslant a$]. If $a < f(a)$ in the above, then a is referred to as an *up* $2k+1$-*stirring point.* If $a > f(a)$ above, then a is referred to as a *down* $2k+1$-*stirring point.* If there is a $2k+1$-stirring point for any k, we say there is stirring for f in I.

We first have the following:

PROPOSITION. *If there is a $2k+1$-stirring point for f in I, then f necessarily has points of all periods except for the odd numbers strictly between 1 and $2k+1$.*

PROOF: Fix an up $2k+1$-stirring point $a \in I$. Assume that $f^{2k+1}(a) < a$. Define $g(x) = f(x) - x$. Note that $g(a) > 0 > g(f(a))$ and there is thus a fixed point in $(a, f(a))$ by the intermediate value theorem. It remains to show

(i) the existence of points of all periods $N \geqslant 2k$, and
(ii) the existence of points of periods $2j$, for $1 \leqslant j \leqslant k-1$.

For convenience, define an index set $K = \{i \mid i \geqslant 2k \text{ or } i = 2j \text{ for } 1 \leqslant j \leqslant k-1\}$. We shall not distinguish between (i) and (ii); instead, we shall just consider the set K.

Let $f^{2k}(a) = p_0$ and $f^{2k-2}(a) = q_0$. Now, for $0 \leqslant j \leqslant k-2$, define:

$$\begin{aligned}
p_{2j+1} &= \sup\Big\{x \in \big[f^{2(k-j)-3}(a), f^{2(k-j)-1}(a)\big) \;\Big|\; f(x) = q_{2j}\Big\},\\
q_{2j+1} &= \inf\Big\{x \in \big(p_{2j+1}, f^{2(k-j)-1}(a)\big] \;\Big|\; f(x) = p_{2j}\Big\},\\
q_{2j+2} &= \inf\Big\{x \in \big(f^{2(k-j-1)}(a), f^{2(k-j-2)}(a)\big] \;\Big|\; f(x) = p_{2j+1}\Big\},\\
p_{2j+2} &= \sup\Big\{x \in \big[f^{2(k-j-1)}(a), q_{2j+2}\big) \;\Big|\; f(x) = q_{2j+1}\Big\}.
\end{aligned}$$

Also let

$$\begin{aligned}
p_{2k-1} &= \sup\{x \in (a, f(a)) \mid f(x) = q_{2k-2}\},\\
q_{2k-1} &= \inf\{x \in (p_{2k-1}, f(a)] \mid f(x) = p_{2k-2}\},\\
q_{2k} &= \inf\{x \in (a, p_{2k-1}) \mid f(x) = p_{2k-1}\},\\
p_{2k} &= \sup\{x \in [a, q_{2k}) \mid f(x) = q_{2k-1}\}.
\end{aligned}$$

Finally, define for $j \geqslant k$:

$$\begin{aligned}
p_{2j+1} &= \sup\{x \in (q_{2j}, p_{2j-1}) \mid f(x) = q_{2j}\},\\
q_{2j+1} &= \inf\{x \in (p_{2j+1}, p_{2j-1}) \mid f(x) = p_{2j}\},\\
q_{2j+2} &= \inf\{x \in (q_{2j}, p_{2j+1}) \mid f(x) = p_{2j+1}\},\\
p_{2j+2} &= \sup\{x \in (q_{2j}, q_{2j+2}) \mid f(x) = q_{2j+1}\}.
\end{aligned}$$

Clearly, covering properties of the stirring structure allow for all of the points defined to exist. Now let $I_i = [p_i, q_i]$, for $0 \leqslant i$. Note that $f(p_j) = q_{j-1}$ and $f(q_j) = p_{j-1}$ for $j \geqslant 1$. Since $f(I_j) = I_{j-1}$, for $1 \leqslant j$, it is clear that $f(\operatorname{int}(I_j)) = \operatorname{int}(I_{j-1})$, where $\operatorname{int}(I_i)$ denotes the interior of I_i. Define $g_i(x) = f^i(x) - x$. Note first that

$f(I_0) \supset [a, f^{2k-1}(a)]$ so that $f^i(I_{i-1}) \supset I_{i-1}$ for $i \in K$. Moreover, for these i, g_i must have a zero in $\operatorname{int}(I_{i-1})$ by the intermediate value theorem, as $g_i(p_{i-1}) < 0$ and $g_i(q_{i-1}) > 0$ for i odd and $g_i(p_{i-1}) > 0$ and $g_i(q_{i-1}) < 0$ for i even. To conclude that i is the period of this zero for f, we simply observe that, by construction, the I_i's have disjoint interiors. In the case that $f^{2k+1}(a) = a$, it may happen that $g_{2k+1}(p_{2k}) = 0$, in which case p_{2k} is a point of period $2k+1$. In the case of a down-stirring point, the proof is symmetric. ∎

Before proceeding, let us consider a cycle $P = \{p_i\}$ with $\#P = N < \infty$. Let $q = \max\{p_i \mid f(p_i) > p_i\}$ and $p = \min\{p_i \mid f(p_i) < p_i\}$.

Lemma 1. *Suppose P, as just described, is a cycle for f. If $p < q$, then there is 3-stirring for f in I.*

Proof: We will use the intermediate value theorem three times. Let c_2 be the rightmost fixed point in the interval (p, q). Let $k = \min\{j \geqslant 1 \mid f^{j+1}(q) < c_2\}$. Find a preimage c_1 of c_2 in the interval $\big(q, f^k(q)\big)$. Last, find a preimage c_0 of c_1 in the interval (c_2, c_1) and note that c_0 is an up 3-stirring point for f in I. ∎

Lemma 2. *Suppose that P is an N-cycle for f and $q < p$ as just described. If there exists an i so that $p_i < f(p_i) \leqslant q$ (respectively, there exists a j so that $p \leqslant f(p_j) < p_j$), then there is a $2k+1$-stirring point for f in I, for some $1 \leqslant k \leqslant \lfloor N/2 \rfloor$.*

Proof: Assume $p_i < f(p_i) \leqslant q$ for some i. Let $a_0 = \min\{p_i \in P \mid p_i < f(b_i) \leqslant q\}$. Now define for $i \geqslant 0$:

$$a_{2i+1} = \min\{p_i \in P \mid f(p_i) \leqslant a_{2i}\}$$
$$a_{2i+2} = \max\{p_i \in P \mid f(p_i) \geqslant a_{2i+1}\}$$

Let $h = \min\{i \geqslant 1 \mid f(a_0) \leqslant a_{2i}\}$. By construction,

$$a_0 < a_2 < \cdots < f(a_0) \leqslant a_{2h} \leqslant q < a_{2h-1} < \cdots < a_3 < a_1.$$

Since $\#P = N$, it is clear that $1 \leqslant h \leqslant \lfloor N/2 \rfloor$. The intermediate value theorem allows us now to define for $i \geqslant 0$, setting $b_0 = a_0$,

$$b_{2i+1} = \sup\{x \leqslant a_{2i+1} \mid f(x) = b_{2i}\}$$
$$b_{2i+2} = \inf\{x \geqslant a_{2i+2} \mid f(x) = b_{2i+1}\}.$$

Note that $q \leqslant b_{2i+1}$ and $p \leqslant b_{2i+2}$ for $i \geqslant 0$. Let $k = \min\{i \geqslant 1 \mid f(a_0) \leqslant b_{2i}\}$. Note that $b_0 < b_2 < \cdots < f(a_0) \leqslant b_{2k} < q < b_{2k-1} < \cdots < b_3 < b_1$. Since $f(a_0) \leqslant a_{2j} \leqslant b_{2j}$, it is clear that $k \leqslant h$, so that b_{2k} is an up $2k+1$-stirring point with $1 \leqslant k \leqslant \lfloor N/2 \rfloor$. If there is a j such that $p \leqslant f(p_j) < p_j$, we obtain down stirring in a symmetric way. ∎

COROLLARY. *Suppose P is a $2n+1$-cycle for f in I. Then there is a $2k+1$-stirring point for f in I for some $1 \leqslant k \leqslant n$.*

PROOF: Assume there is no $2k+1$-stirring for f in I, for $1 \leqslant k \leqslant \lfloor (2n+1)/2 \rfloor = n$. Then all points of P must change sides of $[q,p]$ under f, by Lemma 2. Since q changes sides by definition, we must have that $f^{2k}(q)$ lies to the left of q. But $f\big(f^{2k}(q)\big) = q$, so that $f^{2k}(q)$ does not change sides of $[q,p]$ under f, proving the corollary. ∎

This stirring characterisation of odd cycles will enable us to prove Sharkovsky's hierarchy of periodic orders for f. Our task is two-fold. First, we must establish *the body* of Sharkovsky's ordering: $2^j(2m+1)$ precedes $2^k(2n+1)$ in lexicographic order for $0 \leqslant j,k$ and $i \leqslant m,n$. Second we must elucidate *the tail* of Sharkovsky's ordering: it remains to show that the powers of 2 (i) appear in decreasing numerical order and (ii) follow all the natural numbers with odd divisors.

THE BODY

We must show that the existence of a point of period $2^j(2m+1)$ implies the existence of a point of period $2^k(2n+1)$ if exactly one of the following conditions is satisfied:

(i) $j = k$ and $m < n$;
(ii) $j < k$,

for $0 \leqslant j,k$ and $1 \leqslant m,n$.

To show (i), simply note that a point of period $2^j(2m+1)$ for f is a point of period $2m+1$ for $\mathbf{f}^{2^j}$, which is also a continuous function. Thus the Corollary establishes (i).

To show (ii), again note that a point of period $2^j(2m+1)$ for f is a point of period $2m+1$ for $\mathbf{f}^{2^j}$. By the Corollary, there is stirring for $\mathbf{f}^{2^j}$. Since, by the Proposition, stirring for $\mathbf{f}^{2^j}$ implies all even periods for $\mathbf{f}^{2^j}$, it suffices that 2^{k-j} is an even number, establishing (ii).

THE TAIL

First we will show that the powers of 2 appear in decreasing numerical order. It suffices to show that period 2 implies period 1. Suppose that $\{a,b\}$ is a 2-cycle for f in I so that $a < b$. As in the proof of the Proposition, note that $g(x) = f(x) - x$ has a zero in (a,b). To show that period 2^{k+1} precedes period 2^k for $k \geqslant 0$, note that what is period 2^{k+1} for f is period 2 for $\mathbf{f}^{2^k}$ and what is period 2^k for f is period 1 for $\mathbf{f}^{2^k}$. Since continuity is preserved under composition, we are done.

It remains to show that the powers of two follow all the natural numbers with odd divisors. Suppose there is a $2^k(2n+1)$-cycle for f in I. Then there is a $(2n+1)$-cycle for $\mathbf{f}^{2^k}$, implying there is a $2m+1$-stirring point for $\mathbf{f}^{2^k}$ for some $1 \leqslant m \leqslant n$. Stirring for $\mathbf{f}^{2^k}$ implies all even periods for $\mathbf{f}^{2^k}$. In particular, $\mathbf{f}^{2^k}$ must have points of periods 2^j, for $j > 0$. Thus f must have points of periods 2^{j+k}, for $j > 0$. This takes care of all powers of 2 except 2^i, for $0 \leqslant i < k+1$. But we already proved that the existence of a point of period 2^j implies the existence of a point of period 2^i, for $i < j$.

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