DISPLACEMENT OF THE LAGRANGE EQUILIBRIUM POINTS IN THE RESTRICTED THREE BODY PROBLEM WITH RIGID BODY SATELLITE

W.J. Robinson
University of Bradford

Abstract

In the restricted problem of three point masses, the positions of the equilibrium points are well known and are tabulated. When the satellite is a rigid body, these values no longer correspond to the equilibrium points. This paper seeks to determine the magnitudes of the discrepancies.

1. Introduction

In the restricted problem of three point masses the positions of the three collinear equilibrium points have been extensively calculated (Szebehely, 1969). It has always been an attractive feature that the other two points make equilateral triangles with the primaries.

In this paper the possibility is considered that the satellite may be a rigid body rather than a point mass. Since the satellite is always regarded as extremely small, this assumption is unlikely to have noticeable effects on the overall picture. However, being small, it may have an appreciable effect on the positions of the equilibrium points as viewed from the satellite.

It is assumed that the centre of mass of the satellite is situated at an equilibrium point and that the attitude of the satellite is one of stable equilibrium as described in an earlier paper by the author (Robinson, 1974).

2. Description of the system

The primaries are point bodies with masses m_1 and m_2 located at points A_1 and A_2. They rotate with constant angular velocity $\underline{\omega}$ about their common mass centre O under the action of their mutual gravitational attractions. The distance A_1A_2 is constant. Choosing the units of mass, length and time so that $m_1 + m_2 = 1$, $A_1A_2 = 1$ and $|\underline{\omega}| = \omega = 1$ respectively, the gravitational constant also takes the value 1.

A rectangular coordinate frame OXYZ is chosen so that OZ is the axis of rotation and A_1 has coordinates $(m_2, 0, 0)$.

V. Szebehely (ed.), Dynamics of Planets and Satellites and Theories of Their Motion, 305-314.

The satellite has mass m, its centre of mass is located at G and

$$\overrightarrow{A_1G} = \underline{r}_1, \qquad \overrightarrow{OG} = \underline{r}, \qquad \overrightarrow{A_2G} = \underline{r}_2.$$

The mass m is assumed to be so small in comparison with the masses of the primaries that their motion is unaffected by its presence.

When the satellite is a point mass, its equation of motion is

$$\ddot{\underline{r}} + 2\underline{\omega} \times \dot{\underline{r}} + \underline{\omega} \times (\underline{\omega} \times \underline{r}) = - \Lambda \frac{m_1}{r_1^2} \hat{\underline{r}}_1. \tag{2.1}$$

In this equation, $\dot{\underline{r}}$ and $\ddot{\underline{r}}$ are the time derivatives relative to the rotating frame OXYZ. $\hat{\underline{r}}_1$ is the unit vector in the direction of $\underline{r}_1$ and $\underline{r}_1 = r_1\hat{\underline{r}}_1$. The symbol Λ is always followed by an expression referring to m_1. The symbol means that a similar expression referring to m_2 has been omitted. Thus the symmetry of the expressions in terms of m_1 and m_2 is used to shorten the written equations. In the simplest case $\Lambda m_1 = 1$.

In the case where the satellite is a rigid body, let P be the position of an element of the body of mass μ. With

$$\overrightarrow{A_1P} = \underline{\rho}_1, \qquad \overrightarrow{OP} = \underline{\rho}, \qquad \overrightarrow{A_2P} = \underline{\rho}_2 \text{ and } \overrightarrow{GP} = \underline{\sigma}$$

it follows that

$$\Sigma\, \mu\underline{\rho}_1 = m\underline{r}_1, \qquad \Sigma\, \mu\underline{\rho} = m\underline{r} \quad\text{and}\quad \Sigma\, \mu\underline{\sigma} = \underline{0}$$

and the equation of motion of the satellite is

$$m\{\ddot{\underline{r}} + 2\underline{\omega} \times \dot{\underline{r}} + \underline{\omega} \times (\underline{\omega} \times \underline{r})\} = - \Lambda m_1 \sum \frac{\mu\hat{\underline{\rho}}_1}{\rho_1^2} \tag{2.2}$$

where Σ indicates summation over all the elements of the satellite.

Since $\underline{\rho}_1 = \underline{r}_1 + \underline{\sigma}$ and $\frac{\sigma}{r_1}$ is usually an extremely small quantity, a polynomial $Q_{n,m}(\alpha_1)$ may be defined for non-negative integral values of n and m by the equation

$$\frac{1}{\rho_1^n} = \frac{1}{r_1^n} \sum_{m=0}^{\infty} \left(- \frac{\sigma}{r_1}\right)^m Q_{n,m}(\alpha_1)$$

where $\alpha_1 = (\hat{\underline{\sigma}} \cdot \hat{\underline{r}}_1)$ is the cosine of an angle. When $m < 0$, $Q_{n,m}(\alpha)$ is defined as having the value 0. Some of the properties of $Q_{n,m}(\alpha)$ are listed in Appendix 1.

Equation (2.2) can now be expressed in the form

$$m\{\ddot{\underline{r}} + 2\underline{\omega} \times \dot{\underline{r}} + \underline{\omega} \times (\underline{\omega} \times \underline{r})\}$$
$$= - \Lambda \frac{m_1}{r_1^2} \sum_{s=0}^{\infty} \left(- \frac{1}{r_1}\right)^s \left\{\hat{\underline{r}}_1 S_s(\alpha_1) - \underline{T}_s(\alpha_1)\right\} \tag{2.3}$$

where $S_s(\alpha_1)$ and $\underline{T}_s(\alpha_1)$ are scalar and vector moments of the satellite about G which are defined by

$$S_s(\alpha_1) = \Sigma\, \mu\sigma^s\, Q_{3,s}(\alpha_1) \tag{2.4}$$

$$\underline{T}_s(\alpha_1) = \Sigma\, \mu\underline{\sigma}\, \sigma^{s-1}\, Q_{3,s-1}(\alpha_1). \tag{2.5}$$

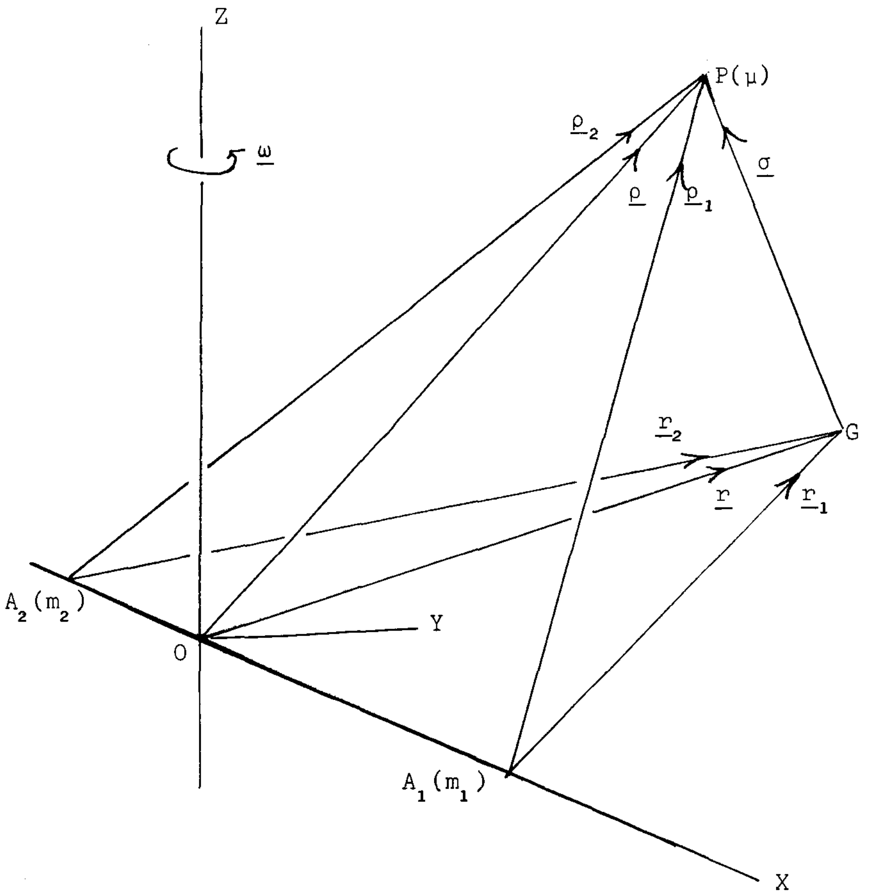

Diagram illustrating the notation of Section 2.

It can be seen that if a is a length commensurate with the linear dimensions of the satellite then $S_s(\alpha_1)$ and $\underline{T}_s(\alpha_1)$ have magnitudes of the order of ma^s. Some of the lower order moments are calculated in Appendices 2 and 3.

3. The equilibrium points

The positions of the equilibrium points are found by placing $\ddot{\underline{r}} = \dot{\underline{r}} = \underline{0}$ in the equation of motion and then solving the resulting equations.

In the case of the point mass satellite the five equilibrium points are usually denoted by L_1, L_2, L_3, L_4 and L_5, the first three being the collinear points the remaining two being the equilateral points.

If L_0 is one of these points and

$$\overrightarrow{A_1 L_0} = \underline{r}_{01}, \qquad \overrightarrow{OL_0} = \underline{r}_0$$

it follows that

$$\underline{\omega} \times (\underline{\omega} \times \underline{r}_0) = -\Lambda \frac{m_1}{r_{01}^2} \hat{\underline{r}}_{01}. \tag{3.1}$$

The corresponding equation for a rigid body satellite is

$$m\,\underline{\omega} \times (\underline{\omega} \times \underline{r}) = -m\Lambda \frac{m_1}{r_1^2} \hat{\underline{r}}_1$$

$$-\Lambda \frac{m_1}{r_1^2} \sum_{s=2}^{\infty} \left(-\frac{1}{r_1}\right)^s \left\{\hat{\underline{r}}_1 S_s(\alpha_1) - \underline{T}_s(\alpha_1)\right\} \tag{3.2}$$

the values of $S_0(\alpha_1)$, $S_1(\alpha_1)$, $\underline{T}_0(\alpha_1)$ and $\underline{T}_1(\alpha_1)$ having been substituted in Equation (2.3).

The convergence of the infinite series depends on the moment of inertia terms rather than on the inverse powers of r_1.

It is clear that Equations (3.1) and (3.2) have the same solutions only when all the moments of the satellite about G vanish. Since the differences between the two equations are very small, it is to be expected that their solutions will differ by similarly small amounts. If G is at the point L, the equilibrium point of Equation (3.2) corresponding to the point L_0, let $\overrightarrow{L_0 L} = \underline{\varepsilon}$. It follows that $\underline{r}_1 = \underline{r}_{01} + \underline{\varepsilon}$ and

$$\frac{1}{r_1^n} = \frac{1}{r_{01}^n} \sum_{m=0}^{\infty} \left(-\frac{\varepsilon}{r_{01}}\right)^m Q_{n,m}(\beta_{01}) \tag{3.3}$$

where $\beta_{01} = \hat{\underline{r}}_{01} \cdot \hat{\underline{\varepsilon}}$, the expansion being possible on the assumption that $\frac{\varepsilon}{r_{01}}$ is sufficiently small.

Regarding $\underline{r}_{01}$ as a known quantity (Szebehely, 1967), the object is to determine the displacement vector $\underline{\varepsilon}$, Equation (3.1) being regarded as a first approximation to Equation (3.2).

Under these circumstances the following expansions are necessary.

$$\alpha_1 = \sum_{s=0}^{\infty} \left(-\frac{\varepsilon}{r_{01}}\right)^s \alpha_{s1} \tag{3.4}$$

https://doi.org/10.1017/S0252921100062400 Published online by Cambridge University Press

$$S_s(\alpha_1) = \sum_{p=0}^{\infty} \left(-\frac{\varepsilon}{r_{01}}\right)^p S_{sp1} \qquad (3.5)$$

$$\underline{T}_s(\alpha_1) = \sum_{p=0}^{\infty} \left(-\frac{\varepsilon}{r_{01}}\right)^p \underline{T}_{sp1} \qquad (3.6)$$

where α_{s1}, S_{sp1} and $\underline{T}_{sp1}$ are functions of ε, $\underline{r}_{01}$ and the moments of inertia. These coefficients are discussed in Appendix 4.

Equations (3.1) and (3.2) now reduce to

$$m\,\underline{\omega} \times (\underline{\omega} \times \underline{\varepsilon}) = -\Lambda \frac{m_1}{r_{01}^2} \sum_{s=1}^{\infty} \left(-\frac{\varepsilon}{r_{01}}\right)^s \underline{V}_{0s01}$$
$$-\Lambda \frac{m_1}{r_{01}^2} \sum_{s=2}^{\infty} \sum_{t=0}^{\infty} \sum_{p=0}^{\infty} \frac{\varepsilon^{t+p}}{(-r_{01})^{s+t+p}} \underline{V}_{stp1} \qquad (3.7)$$

where

$$\underline{V}_{stp1} = \left\{\hat{\underline{r}}_{01} Q_{3+s,t}(\beta_{01}) - \hat{\underline{\varepsilon}}\, Q_{3+s,t-1}(\beta_{01})\right\} S_{sp1}$$
$$- Q_{2+s,t}(\beta_{01})\, \underline{T}_{sp1} \qquad (3.8)$$

The properties of $\underline{V}_{stp1}$ are listed in Appendix 5. The suffices have the following significances. s is the order of the moments contained in $\underline{V}_{stp1}$ and (t + p) is the power of the associated term in ε and s + t + p the power of $\frac{1}{r_{01}}$.

With m and a, a characteristic dimension of the satellite, as the very small parameters which determine the relative magnitudes of the terms in Equation (3.7), the linearized form of the equation becomes

$$m\,\underline{\omega} \times (\underline{\omega} \times \underline{\varepsilon}) - \Lambda \frac{m_1}{r_{01}^3}\, \varepsilon\, \underline{V}_{0101}$$
$$= -\Lambda \frac{m_1}{r_{01}^2} \left(-\frac{1}{r_{01}}\right)^p \underline{V}_{p001} \qquad (3.9)$$

where p is the smallest positive integer exceeding 1 which gives a non-zero solution.

4. The collinear equilibrium points

At the collinear points $\hat{\underline{r}}_{01} = [\lambda_1, 0, 0]$ and $\hat{\underline{r}}_{02} = [\lambda_2, 0, 0]$ where λ_1, λ_2 have the values -1, -1 at L_1; -1, +1 at L_2 and +1, +1 at L_3.

It follows that $\underline{\varepsilon} = \xi\underline{i} + \eta\underline{j} + \zeta\underline{k}$ is given by

https://doi.org/10.1017/S0252921100062400 Published online by Cambridge University Press

$$m\xi\left[1 + 2\Lambda \frac{m_1}{r_{01}^3}\right] = \Lambda \frac{m_1}{r_{01}^{a+2}} (-1)^a \underline{V}_{a001} \cdot \underline{i}$$

$$m\eta\left[1 - \Lambda \frac{m_1}{r_{01}^3}\right] = \Lambda \frac{m_1}{r_{01}^{b+2}} (-1)^b \underline{V}_{b001} \cdot \underline{j}$$

$$m\zeta \Lambda \frac{m_1}{r_{01}^3} = \Lambda \frac{m_1}{r_{01}^{c+2}} (-1)^{c+1} \underline{V}_{c001} \cdot \underline{k}$$

where $\underline{i}$, $\underline{j}$ and $\underline{k}$ are the unit vectors along the axes of OXYZ and a, b and c are the smallest integers greater than 1 which give non-zero terms.

It has been shown (Robinson, 1974), that if the centre of mass is held at a collinear equilibrium point, the principal axes of the satellite align themselves parallel to the axes of OXYZ when the satellite is at relative rest. If A, B and C are the principal moments about OX, OY and OZ respectively the satellite reaches a stable attitude with $C > B > A$. For some bodies there is a second possibility with $B > A > C$.

If the satellite is such that $2A \neq B + C$, then

$$\underline{V}_{2001} = \frac{3}{2} \lambda_1 (-2A + B + C) [1, 0, 0]$$

so that

$$m\xi\left[1 + 2\Lambda \frac{m_1}{r_{01}^3}\right] = \frac{3}{2} (-2A + B + C) \Lambda \frac{m_1 \lambda_1}{r_{01}^4}$$

To obtain η and ζ, the third moments have to be considered. If these do not vanish

$$m\eta\left[1 - \Lambda \frac{m_1}{r_{01}^3}\right] = \frac{3}{2} \Sigma\mu y(4x^2 - y^2 - z^2) \Lambda \frac{m_1}{r_{01}^5}$$

$$m\zeta \Lambda \frac{m_1}{r_{01}^3} = \frac{3}{2} \Sigma\mu z(4x^2 - y^2 - z^2) \Lambda \frac{m_1}{r_{01}^5}$$

If a is a length commensurate with the linear dimensions of the satellite it can be seen that ξ is of the same order at a^2 and η and ζ of the same order as a^3.

5. The equilateral equilibrium points

In this case $r_{01} = r_{02} = 1$ and

https://doi.org/10.1017/S0252921100062400 Published online by Cambridge University Press

$$\hat{\underline{r}}_{01} = \frac{1}{2}[-1, \sqrt{3}, 0] \quad \text{and} \quad \hat{\underline{r}}_{02} = \frac{1}{2}[1, \sqrt{3}, 0].$$

Referring again to the stable stationary position of the satellite (Robinson, 1974) when G is at L , the principal axes Ox and Oy are turned through an angle $-\kappa$ about Oz where $\cos 2\kappa = \frac{1}{N}$ and $N = \sqrt{1 + 12m_0^2}$ with $2m_0 = m_2 - m_1$.

Equation (3.9) now becomes

$$\begin{bmatrix} \xi + \sqrt{12}\, m_0 \eta \\ \sqrt{12}\, m_0 \xi + 3\eta \\ -\frac{4}{3}\zeta \end{bmatrix} = \frac{4}{3m}(-1)^p \Lambda\, m_1 \underline{V}_{p001} \tag{5.1}$$

ξ and η can be determined when p = 2 in some cases, but to find ζ, p must take the value 3 at least.

With these values

$$\xi = \frac{m_0(68 - 11N^2)}{4Nm(4 - N^2)}(A - B)$$

$$\eta = \frac{6 - N^2}{4\sqrt{3}\, Nm(4 - N^2)}[4NC + 3(A-B) - 2N(A-B)] + \frac{N(B-A)}{2\sqrt{3}(4-N^2)m}$$

$$\zeta = \frac{1}{m}\Lambda\, m_1 \underline{k} \cdot \underline{V}_{3001}$$

Again it can be seen that ξ and η are of order a^2 while ζ is of order a^3.

Conclusions

It has been shown that the displacements of the equilibrium points are extremely small, which was to be expected. If a is the length of the satellite, remembering that $A_1A_2 = 1$, then ξ is of the same order as a^2. η is of the same order in the equilateral case, but of order a^3 in the collinear case. The coordinate ζ is, at its greatest, of order a^3 in either case.

It can also be mentioned that since all bodies certainly have one stable attitude at each equilibrium point, some may have two stable attitudes. The conditions for the second case are given in the paper referred to earlier (Robinson, 1974). The outcome is that some bodies may actually have ten equilibrium points, while others have only five, or some intermediate number. Of course, in those cases where the points occur in pairs, the members of such pairs are very near each other.

Appendix 1. The polynomials $Q_{mn}(\alpha)$

From the definition

$$(1 + 2\alpha x + x^2)^{-n/2} = \sum_{s=0}^{\infty} (-x)^s\ Q_{ns}(\alpha)$$

it follows that

$$Q'_{ns}(\alpha) = \frac{dQ_{ns}(\alpha)}{d\alpha} = n\ Q_{n+1,s-1}(\alpha)$$

and

$$Q_{n0}(\alpha) = 1$$
$$Q_{n1}(\alpha) = n\alpha$$
$$Q_{n2}(\alpha) = \tfrac{1}{2}n\{(n + 2)\alpha^2 - 1\}$$
$$Q_{n3}(\alpha) = \tfrac{1}{6}n(n + 2)\ \{(n + 4)\alpha^3 - 3\alpha\}.$$

Note that

$$Q_{1s}(\alpha) = P_s(\alpha)$$

which are the familiar Legendre Polynomials.

Appendix 2. The scalar moments $S_s(\alpha_1)$

$$S_s(\alpha_1) = \Sigma\ \mu\sigma^s Q_{3s}(\alpha_1)$$

where

$$\alpha_1 = \hat{\underline{\sigma}} \cdot \hat{\underline{r}}_1$$

$$S_0(\alpha_1) = \Sigma\mu = m$$
$$S_1(\alpha_1) = 3\Sigma\mu(\underline{\sigma} \cdot \hat{\underline{r}}_1) = 0$$
$$S_2(\alpha_1) = \frac{15}{2}\ \Sigma\mu(\underline{\sigma} \cdot \hat{\underline{r}}_1)^2 - \frac{3}{2}\ \Sigma\ \mu\sigma^2$$
$$S_3(\alpha_1) = \frac{35}{2}\ \Sigma\mu(\underline{\sigma} \cdot \hat{\underline{r}}_1)^3 - \frac{15}{2}\ \Sigma\ \mu\sigma^2(\underline{\sigma} \cdot \hat{\underline{r}}_1)$$

Appendix 3. The vector moments $\underline{T}_s(\alpha_1)$

$$\underline{T}_s(\alpha_1) = \Sigma\ \mu\underline{\sigma}\ \sigma^{s-1} Q_{3s-1}(\alpha_1).$$

Since the above definition holds for $s \geqslant 1$ the additional definition is made

$$\underline{T}_0(\alpha_1) = \underline{0}.$$

From the definition

$$\underline{T}_1(\alpha_1) = \Sigma\ \mu\ \underline{\sigma} = \underline{0}$$
$$\underline{T}_2(\alpha_1) = 3\ \Sigma\ \mu\ \underline{\sigma}(\underline{\sigma} \cdot \hat{\underline{r}}_1)$$
$$\underline{T}_3(\alpha_1) = \frac{15}{2}\ \Sigma\ \mu\ \underline{\sigma}(\underline{\sigma} \cdot \hat{\underline{r}}_1)^2 - \frac{3}{2}\ \Sigma\ \mu\ \underline{\sigma}\ \sigma^2.$$

Appendix 4

Using Equation (3.3) it can be shown that

$$\alpha_1 = \sum_{s=0}^{\infty} \left(-\frac{\varepsilon}{r_{01}}\right)^s \alpha_{s1}$$

where

$$\alpha_{s1} = \alpha_{01} Q_{1,s}(\beta_{01}) - \gamma\, Q_{1,s-1}(\beta_{01}),$$

$$\alpha_{01} = \underline{\hat{r}}_{01} \cdot \underline{\sigma} \quad \text{and} \quad \gamma = \underline{\hat{\varepsilon}} \cdot \underline{\hat{\sigma}}.$$

Making use of the relation

$$Q'_{m,n}(\alpha) = m\, Q_{m+2,n-1}(\alpha)$$

and Taylor's Theorem

$$Q_{m,n}(\alpha_1) = \sum_{s=0}^{n} \frac{2^s}{s!} \frac{\Gamma\left(\frac{m}{2}+s\right)}{\Gamma\left(\frac{m}{2}\right)} \left\{\sum_{t=1}^{\infty} \left(-\frac{\varepsilon}{r_{01}}\right)^t \alpha_{t1}\right\}^s Q_{m+2s,n-s}(\alpha_{01})$$

$$= \sum_{s=0}^{\infty} \left(-\frac{\varepsilon}{r_{01}}\right)^s q_{mns1}.$$

From the definition

$$S_s(\alpha_1) = \sum_{\mu} \mu\, \sigma^s\, Q_{3,s}(\alpha_1)$$

it follows that

$$S_s(\alpha_1) = \sum_{\mu} \mu\, \sigma^s \sum_{t=0}^{\infty} \left(-\frac{\varepsilon}{r_{01}}\right)^t q_{3st1}$$

$$= \sum_{t=0}^{\infty} \left(-\frac{\varepsilon}{r_{01}}\right)^t S_{st1}$$

where

$$S_{st1} = \sum_{\mu} \mu\, \sigma^s\, q_{tmn1}$$

is a scalar moment of the s-th order.

In a similar manner

$$\underline{T}_s(\alpha_1) = \sum_{t=0}^{\infty} \left(-\frac{\varepsilon}{r_{01}}\right)^t \underline{T}_{st1}.$$

Since $S_0(\alpha_1) = m$, $S_1(\alpha_1) = 0$, $\underline{T}_0 = \underline{T}_1 = \underline{0}$

$$S_{001} = m, \quad S_{0t1} = 0 \ (t \neq 0), \quad S_{1t1} = 0 \text{ for all } t.$$

$$\underline{T}_{0t1} = \underline{T}_{1t1} = \underline{0} \text{ for all } t.$$

Also $S_s(\alpha_{01}) = S_{s01}$, $\underline{T}_s(\alpha_{01}) = \underline{T}_{s01}$.

Appendix 5. The function $\underline{V}_{stp1}$

$$\underline{V}_{stp1} = \{\underline{\hat{r}}_{01} Q_{3+s,t}(\beta_{01}) - \underline{\hat{\varepsilon}}\, Q_{3+s,t-1}(\beta_{01})\}\, S_{sp1}$$
$$- Q_{2+s,t}(\alpha_{01})\, \underline{T}_{sp1}.$$

If s, t and p are non-zero positive integers it follows that

$$\underline{V}_{0001} = m\,\hat{\underline{r}}_{01}$$

$$\underline{V}_{s001} = \hat{\underline{r}}_{01} S_s(\alpha_{01}) - \underline{T}_s(\alpha_{01})$$

$$\underline{V}_{0t01} = m\{\hat{\underline{r}}_{01} Q_{3,t}(\beta_{01}) - \hat{\underline{\varepsilon}}\, Q_{3,t-1}(\beta_{01})\}$$

$$\underline{V}_{00p1} = \underline{0}$$

$$\underline{V}_{0tp1} = \underline{0}$$

$$\underline{V}_{s0p1} = \hat{\underline{r}}_{01} S_{sp1} - \underline{T}_{sp1}$$

$$\underline{V}_{st01} = \{\hat{\underline{r}}_{01} Q_{3+s,t}(\beta_{01}) - \hat{\underline{\varepsilon}}\, Q_{3+s,t-1}(\beta_{01})\}\, S_s(\alpha_{01}) - Q_{2+s,t}(\beta_{01})\, \underline{T}_s(\alpha_{01})$$

$$\underline{V}_{1tp1} = \underline{0}\,.$$

References

Robinson, W.J.: 1974, Celes. Mech. 10, 17.

Szebehely, V.: 1967, Theory of Orbits, Academic Press, New York and London.

https://doi.org/10.1017/S0252921100062400 Published online by Cambridge University Press