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TRIANGLES IN A COMPLETE CHROMATIC GRAPH

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Abstract

Suppose that in a complete graph on N points, each edge is given arbitrarily either the color red or the color blue, but the total number of blue edges is fixed at T. We find the minimum number of monochromatic triangles in the graph as a function of N and T. The maximum number of monochromatic triangles presents a more difficult problem. Here we propose a reasonable conjecture supported by examples.

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1. Introduction

Let K_N be the complete graph with N vertices (points) and suppose that each edge (line joining two vertices) is assigned exactly one of two possible colors, here always blue or red. After the coloring let B be the number of monochromatic (solid) blue triangles, and let R be the number of solid red triangles. The minimum of $B + R$ for each $N > 0$ was determined by the author [2]. Later Sauvé [4] gave a simpler proof of the same theorem using an ingenious method of assigning weights to each pair of edges at each vertex. The maximum of $B + R$ is trivially $N(N-1)(N-2)/6$, and the associated extremal graph is obtained by painting every edge blue, or painting every edge red.

Suppose that we modify the problem by considering only those colorings of K_N in which the number of blue edges is T where T is a fixed preassigned number. This also fixes the number of red edges U, and of course $U = N(N-1)/2 - T$.

Let $m(N, T)$ be the minimum of $B + R$ for all colorings with T blue edges, and let $M(N, T)$ be the maximum of $B + R$ under the same conditions. In this work we find $m(N, T)$ explicitly for every pair of integers N, T, with $0 \leqslant T \leqslant N(N-1)/2$ and $0 \leqslant N < \infty$.

The problem of finding $M(N, T)$ seems to be more difficult. In Section 4 we discuss the difficulties and we propose a reasonable conjecture for the form of the solution.

2. The basic equations

Following Sauvé [4], we attach the weight 2 to each pair of blue lines issuing from a vertex of the graph. We assign the same weight 2, to each pair of red lines. For a mixed pair, one edge blue and one edge red we assign the weight -1. Let b_j and r_j be the number of blue edges and red edges respectively at the point P_j, $j = 1, 2, \ldots, N$. Then for each j

$$b_j + r_j = N - 1, \tag{2.1}$$

and

$$\sum b_j = 2T, \qquad \sum r_j = 2U, \tag{2.2}$$

where all sums run from 1 to N unless otherwise noted. When we compute the weight of the edges meeting at P_j in accordance with the above assignment of weights, we obtain

$$b_j(b_j - 1) + r_j(r_j - 1) - b_j r_j.$$

Then the total weight of the colored graph is

$$W = \sum b_j(b_j - 1) + \sum r_j(r_j - 1) - \sum b_j r_j. \tag{2.3}$$

On the other hand if we examine the weight of each triangle, the weight of a monochromatic triangle is 6, and the weight of a mixed triangle is $2 - 1 - 1 = 0$. Then for the total weight we also have $B + R = W/6$. Hence for any coloring,

$$B + R = \frac{1}{6} \sum \left[b_j(b_j - 1) + r_j(r_j - 1) - b_j r_j \right]. \tag{2.4}$$

Now $b(b-1) + r(r-1) - br = (b + r)^2 - (b + r) - 3br$. If we use (2.1) and (2.2), then the formula (2.4) for $B + R$ gives

$$\begin{aligned} B + R &= \frac{1}{6} \sum \left[(b_j + r_j)^2 - (b_j + r_j) - 3b_j(N - 1 - b_j) \right] \\ &= \frac{1}{6} \left[N(N-1)^2 - N(N-1) - 6T(N-1) + 3\sum b_j^2 \right], \end{aligned}$$

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or

$$B + R = \frac{N(N-1)(N-2)}{6} - T(N-1) + \frac{1}{2}\sum b_j^2. \tag{2.5}$$

Thus to minimize or maximize $B + R$ with T constant it is sufficient to minimize or maximize $Q \equiv \Sigma b_j^2$.

We observe that the red edges have disappeared in (2.5). Hence the problem of minimizing or maximizing $B + R$ is equivalent to the problem of finding extreme values of Q for an incomplete graph on N vertices with T edges. With this interpretation, R is the number of independent triples; that is, the number of sets of three vertices with no edges joining any pair.

A sequence of integers $(b_1, b_2, \ldots, b_N)$ with

$$N - 1 \geqslant b_1 \geqslant b_2 \geqslant \cdots \geqslant b_N \geqslant 0, \tag{2.6}$$

is called a *graph sequence* (*degree sequence*) if there is a graph on N vertices such that the vertex P_j has degree b_j. Since we can always number the vertices in such a way that (2.6) is satisfied, there is no loss of generality in assuming that the sequence $(b_1, b_2, \ldots, b_N)$ is nonincreasing. A sequence can satisfy (2.6) and still not be a graph sequence. For example $(3, 2, 1, 0)$ is not the sequence of degrees for any graph on 4 vertices. A necessary and sufficient condition for a sequence to be a graph sequence can be found in the text by Behzad and Chartrand [1, p. 12]. For our objective we will need the following necessary conditions:

$$\sum b_j = 2T, \quad \text{an even number;} \tag{2.7}$$

$$b_1 \leqslant \text{the number of nonzero } b_j \text{ in } (b_2, b_3, \ldots, b_N), \tag{2.8}$$

and for each k with $0 < k \leqslant N$

$$\sum_{j=1}^{k} b_j - \sum_{j=k+1}^{N} b_j \leqslant k(k-1). \tag{2.9}$$

3. The minimum number of monochromatic triangles

Suppose that in a sequence (2.6) there are two indices j and k for which $b_j \geqslant b_k + 2$. We replace b_j by $b_j - 1$ and b_k by $b_k + 1$. Since

$$b_j^2 + b_k^2 > (b_j - 1)^2 + (b_k + 1)^2,$$

this replacement will decrease $Q \equiv \Sigma b_j^2$. Thus

LEMMA 1. *The minimum of Q over all sequences of integers* (2.6) *that satisfy* (2.7) *occurs for a sequence in which either all b_j are the same, or no two differ by more than* 1.

To compute this minimum, we set

$$(3.1) \qquad \frac{2T}{N} = q + \frac{\rho}{N}, \qquad 0 \leqslant \rho < N,$$

where q and ρ are integers. Then the minimizing sequence has the form

$$(3.2) \qquad S = (q+1, q+1, \ldots, q+1, q, q, \ldots, q),$$

where there are ρ terms $b_j = q + 1$, and the remaining $b_j = q$. A brief computation with this sequence and (2.5) gives

$$(3.3) \qquad B + R \geqslant \frac{N(N-1)(N-2)}{6} - T(N-q-1) + \frac{\rho(q+1)}{2}.$$

To show that this lower bound is sharp we must produce a graph that has the required sequence as a graph sequence. From (3.1) we see that $2T = Nq + \rho$ must be even. Thus certain combinations of N, q and ρ cannot occur. Of the eight possibilities (with respect to parity) we can *not* have (1) N odd, q odd, ρ even, (2) N odd, q even, ρ odd, (3) N even, q odd, ρ odd, and (4) N even, q even, ρ odd.

Suppose that q is even. We may visualize the vertices arranged in order on a circle. Join each vertex with its nearest $q/2$ neighbors on either side by blue edges. Thus P_jP_k is blue if $|j-k| \leqslant q/2$, where j or k are reduced mod N if necessary. This gives each $b_j = q$. Now color P_jP_k blue for $k = j + 1 + q/2$ and $j = 1, 2, \ldots, \rho$. Then this graph has the sequence (3.2) with respect to blue edges, and $\Sigma b_j = 2T$.

Suppose that q is odd. Then $q + 1$ is even and we proceed as in the first case joining each vertex by a blue edge to its nearest $(q+1)/2$ neighbors on either side. Thus P_jP_k is blue if $|j-k| \leqslant (q+1)/2$. This gives too many blue edges, so we change some to red as follows. If N is even, then ρ is even, and $N - \rho$ is even. Then change from blue to red the edges $P_NP_{N-1}, P_{N-2}P_{N-3}, \ldots$ until $(N-\rho)/2$ edges have been changed.

If N is odd, and q is odd, then ρ is odd and $N - \rho$ is even. We proceed as before to change from blue to red $P_NP_{N-1}, P_{N-2}P_{N-3}, \ldots$ until $(N-\rho)/2$ edges have been changed. Thus in every possible case we have a graph with the sequence (3.2) with respect to blue edges. This completes the proof of

THEOREM 1. *Suppose that the edges of a complete graph on N vertices are colored with two colors, red and blue, coloring T edges blue, and the remaining red. Then minimum $(R + B)$ is given by the right side of* (3.3), *where ρ and q are defined by* (3.1).

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4. The maximum number of monochromatic triangles

The same computation that gave Lemma 1 shows that for a maximum we should make the early b_j as large as possible. Suppose that the sequence is subject only to the conditions (2.6) and (2.7). Define integers q and ρ by $2T = q(N-1) + \rho$ where $q \geqslant 0$ and $0 \leqslant \rho < N-1$. Then the sequence $S = (N-1, \ldots, N-1, \rho, 0, \ldots, 0)$ with $N-1$ repeated q times will make Σb_j^2 a maximum. But S is not a graph sequence except in the trivial cases, all $b_j = 0$ or all $b_j = N-1$.

To find graph sequences with large Σb_j^2, we introduce two special graphs.

First, there are nonnegative integers q and ρ such that

$$T = q(N-1) - \frac{q(q-1)}{2} + \rho, \qquad 0 \leqslant \rho \leqslant N-q-2. \tag{4.1}$$

With these integers let G^* be the graph obtained as follows. Let $P_1, P_2, \ldots, P_q$ be a complete subgraph K_q on q vertices. Join each vertex of K^q with each of the remaining vertices $P_{q+1}, \ldots, P_N$. Finally join P_{q+1} to $P_{q+2}, \ldots, P_{q+\rho+1}$ using ρ edges. Of course this set of vertices is empty if $\rho = 0$. Then the graph sequence for this graph is

$$S^* = (N-1, \ldots, N-1, q+\rho, q+1, \ldots, q+1, q, \ldots, q), \tag{4.2}$$

where $N-1$ occurs q times, $q+1$ occurs ρ times, and q occurs $N-q-\rho-1$ times. An easy computation shows that $\Sigma b_j = 2T$ for S^*. Since S^* is the sequence for G^*, it must satisfy the conditions (2.7), (2.8), and (2.9). Further when $k = q+1$, the sequence S^* gives equality in (2.9). Thus the degrees, $b_1, b_2, \ldots, b_{q+1}$ are as large as possible. This suggests that G^* makes $B + R$ a maximum, when the edges of G^* are colored blue and the remaining edges of K_N are colored red. We will see shortly that this is *not always the case*.

Next we define non negative integers v and λ by

$$\frac{2T}{v(v-1)} = 1 + \frac{2\lambda}{v(v-1)}, \qquad 0 \leqslant \lambda < v. \tag{4.3}$$

Let G_v be the graph defined as follows. The vertices $P_1, P_2, \ldots, P_v$ form a complete subgraph on v vertices. Then join each of $P_1, P_2, \ldots, P_\lambda$ with P_{v+1} by an edge. The graph sequence for G_v is

$$S_v = (v, \ldots, v, v-1, \ldots, v-1, \lambda, 0, \ldots, 0) \tag{4.4}$$

where v occurs λ times and $v-1$ occurs $v-\lambda$ times. An easy computation shows that $\Sigma b_j = 2T$ for S_v. As we will see, when the edges of G_v are colored blue and the remaining edges of K_N are colored red, this graph occasionally gives a maximum for $R + B$ when G^* fails.

We now look at several examples. First suppose that $N = 4$ and $T = 3$. Then

$$S^* = (3, 1, 1, 1), \qquad \sum b_j^2 = 12,$$

and

$$S_v = (2,2,2,0), \qquad \sum b_j^2 = 12.$$

Therefore both graphs give the same maximum. In fact S^* is the complement of S_v so $R + B$ must be the same for both graphs.

Next, let $N = 5$ and $T = 4$. Then

$$S^* = (4,1,1,1,1), \quad \sum b_j^2 = 20,$$
$$S_v = (2,2,2,2,0), \quad \sum b_j^2 = 16.$$

This time S^* provides the maximum for $R + B$.

Finally let $N = 7$ and $T = 10$. Then

$$S^* = (6,5,2,2,2,2,1), \quad \sum b_j^2 = 78,$$
$$S_v = (4,4,4,4,4,0,0), \quad \sum b_j^2 = 80.$$

This time it is S_v that gives a maximum.

For the sequence S^* in (4.2), we find

$$\text{(4.5)} \qquad \sum b_j^2 = q(N-1)^2 + (4q + \rho + 1)\rho + q^2(N-q),$$

an expression which does not seem to simplify.

For S_v in (4.4) we have

$$\text{(4.6)} \qquad \sum b_j^2 = \lambda v^2 + (v - \lambda)(v-1)^2 + \lambda^2 = 2T(v-1) + \lambda(\lambda - 1).$$

CONJECTURE. If a complete graph on N vertices is colored with T blue edges and the rest red edges, then $\max(B + R)$ occurs either for the graph G^* or G_v. If this is the case, then $\max(B + R)$ is given by (2.5) where either (1) q and ρ are defined by (4.1) and Σb_j^2 is given by (4.5), or (2) v and λ are defined by (4.3) and Σb_j^2 is given by (4.6).

There is a relation between S^* and S_v which we now develop. First observe that the integers N, T, q, and ρ determine S^* uniquely and conversely S^* determines N, T, q, and ρ uniquely. Similarly there is a one-to-one correspondence between the integers N, T, v, and λ and the graph sequences S_v. Now consider $\tilde{G}$ the complement of G. If G is an incomplete graph then $\tilde{G}$ has an edge P_iP_j if and only if P_iP_j is *not* an edge of G. Or if G is regarded as a complete graph with two colors, then the colors are interchanged in $\tilde{G}$. Let $\tilde{S}^*$ be the graph sequence obtained from S^* when we replace b_j by $N - 1 - b_j$ for $j = 1, 2, \ldots, N$ and then reverse the order. With this notation we have

THEOREM 2. *The sequence $\tilde{S}^*$ is the graph sequence S_v for the graph $\tilde{G}^*$.*

PROOF. If S^* is the sequence (4.2) then

$$\tilde{S}^* = (N-1-q,\ldots,N-1-q, N-2-q,\ldots,$$
$$N-2-q, N-1-q-\rho, 0,\ldots,0),$$

where $N-1-q$ occurs $N-q-\rho-1$ times, $N-2-q$ occurs ρ times and 0 occurs q times. We set $v = N-1-q$ and $\lambda = N-q-\rho-1$. Then

$$\tilde{S}^* = (v,\ldots,v, v-1,\ldots,v-1, \lambda, 0,\ldots,0),$$

where v occurs $v-\rho = \lambda$ times, $v-1$ occurs $\rho = v-\lambda$ times and 0 occurs $q = N-v-1$ times, (see equation (4.4)). Then $\tilde{S}^*$ is the sequence S_v for $\tilde{G}^*$.

COROLLARY. *If G^* gives the maximum $R+B$ for all two-color graphs on N points with T blue edges, then G_v gives the maximum $R+B$ for all two-color graphs on N points with $U = N(N-1)/2 - T$ blue edges.*

5. Bounds for other combinations

In a coloring of K_N with blue and red, let BBR be the number of triangles with two blue edges and one red edge. Similarly let BRR be the number of triangles with one blue edge and two red edges.

Let us attach the weight x to each pair of blue edges, the weight y to each mixed pair, and the weight z to each pair of red edges. Again we compute the weight of K_N in two different ways. Following the notation and method of section 2, with this more general system of weights, we obtain

$$\begin{aligned} &3xB + (x+2y)BBR + (z+2y)BRR + 3zR \\ &\quad = \sum\left[\frac{x}{2}b_j(b_j-1) + \frac{z}{2}r_j(r_j-1) + yb_jr_j\right]. \end{aligned} \tag{5.1}$$

We replace r_j by $N-1-b_j$ and use $\Sigma b_j + 2T$. Then (5.1) becomes

$$\begin{aligned} &3xB + (x+2y)BBR + (z+2y)BRR + 3zR \\ &\quad = \frac{z}{2}N(N-1)(N-2) + [2y(N-1) - x - z(2N-3)]T \\ &\qquad + \left(\frac{x+z}{2} - y\right)\sum b_j^2. \end{aligned} \tag{5.2}$$

Various selections of weights (x, y, z) in (5.2) give special relations. The selection $(2, -1, 2)$ gives equation (2.5). The selection $(0, 1, 0)$ gives

$$2(BBR + BRR) = 2(N-1)T - \sum b_j^2. \tag{5.3}$$

Equation (5.3) also follows from (2.5) and the fact that the total number of triangles is $N(N-1)(N-2)/6$, (try $(x, y, z) = (1, 1, 1)$).

Let $(x, y, z) = (1, 0, 0)$. Then (5.2) yields

$$3B + BBR = -T + \frac{1}{2}\sum b_j^2. \tag{5.4}$$

Further $(x, y, z) = (2, -1, 0)$ gives

$$3B - BRR = -NT + \sum b_j^2. \tag{5.5}$$

Finally $(x, y, z) = (0, 1, 1)$ gives

$$2BBR + 3BRR + 3R = \frac{1}{2}N(N-1)(N-2) + T - \frac{1}{2}\sum b_j^2. \tag{5.6}$$

These equations are not independent. Since the vectors $(1, 1, 1)$, $(0, 1, 0)$, and $(1, 0, 0)$ form a basis for E^3, it follows that any equation that can be derived from (5.2) can also be obtained from (5.3), (5.4), and the fact that the total number of triangles is $\binom{N}{3}$.

The methods that gave Theorem 1, also yield

THEOREM 3. *Suppose that the edges of K_N are colored with two colors red and blue and that T edges are colored blue. Define ρ and q by equation* (3.1). *Then we have the following sharp bounds*:

$$BBR + BRR \leqslant T(N - q - 1) - \frac{\rho(q+1)}{2}; \tag{5.7}$$

$$3B + BBR \geqslant T(q - 1) + \frac{\rho(q+1)}{2}; \tag{5.8}$$

$$3B - BRR \geqslant T(2q - N) + \rho(q + 1); \tag{5.9}$$

and

$$2BBR + 3BRR + 3R \leqslant \frac{1}{2}N(N-1)(N-2) - T(q-1) - \frac{\rho(q+1)}{2}. \tag{5.10}$$

Of course (5.7) follows directly from the bound (3.3).

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