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Principal series representations of direct limit groups

Joseph A. Wolf

Abstract

We combine the geometric realization of principal series representations on partially holomorphic cohomology spaces, with the Bott–Borel–Weil theorem for direct limits of compact Lie groups, obtaining limits of principal series representations for direct limits of real reductive Lie groups. We introduce the notion of weakly parabolic direct limits and relate it to the conditions that the limit representations are norm-preserving representations on a Banach space or unitary representations on a Hilbert space. We specialize the results to diagonal embedding direct limit groups. Finally we discuss the possibilities of extending the results to limits of tempered series other than the principal series.

1. Introduction

Harmonic analysis on a real reductive Lie group G depends on several series of representations, one for each conjugacy class of Cartan subgroups of G. See [HCh66, HCh75, HCh76a, HCh76b] for the case where G is Harish-Chandra class and [Wol74, HW86a, HW86b] for the general case. The simplest of these series is the *principal series*. It consists of representations constructed from representations of compact Lie groups, characters on real vector groups, and the induced representation construction. The other series are somewhat more delicate, replacing Élie Cartan's theory of representations of compact Lie groups by Harish-Chandra's theory of discrete series representations of real reductive Lie groups.

This paper is the first step in a program to extend the construction, analysis and geometry of those series of representations from the finite-dimensional setting to a nontrivial but well behaved family of infinite-dimensional Lie groups, the direct limits of real reductive Lie groups. Here we consider the case of the principal series. The case of the discrete series, and then the general case, will be considered separately in [Wol05a, Wol05b].

The classical Bott–Borel–Weil theorem [Bot57] realizes representations of compact Lie groups as cohomology spaces of holomorphic vector bundles over complex flag manifolds. It has since been extended to direct limits of compact Lie groups and direct limits of complex Lie groups, both in the analytic category [NRW01] and in the algebraic category [DPW02]. With some technical adjustment, the results of [NRW01] replace Cartan's theory of representations of compact Lie groups for construction of direct limit principal series representations. There are, however, a number of technical points, some of them delicate, that have to be addressed and we mention them as we describe the contents of this paper.

Section 2 recalls our class of finite-dimensional real reductive Lie groups and the standard construction of their not-necessarily-unitary principal series representations. Section 3 recalls the

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geometric realization of those representations on partially holomorphic cohomologies of vector bundles over closed orbits in complex flag manifolds. In Section 4 we discuss alignment questions for minimal parabolic subgroups. The alignment is needed in order to define limit principal series representations of our direct limit groups. In effect, this is the first technical issue, and it addresses the question of whether $G = \varinjlim G_i$ can have a meaningful direct limit of principal series representations. For that we need the connecting maps $\phi_{j,i} : G_i \to G_j$ of the direct system to respect the ingredients of the principal series recipe. Initially that must be done for the components M_i, A_i and N_i of minimal parabolic subgroups $P_i = M_i A_i N_i \subset G_i$. That alignment on components is not quite automatic, but it holds (possibly after passing to a cofinal subsystem, which yields the same limit group) for the most interesting cases, the diagonal embedding direct limit groups of Section 9. See Proposition 9.12. Next, it must be done on the level of representations of the M_i. That, of course, is automatic for spherical principal series representations, but more generally we use an appropriate extension of Cartan's highest weight theory. Thus we obtain representations of G that are direct limits of principal series representations of the G_i.

The second issue is to construct good geometric realizations of these 'principal series' representations of the limit groups G. This is the heart of the paper. The method of [Wol76], illustrated in [Wol76, Section 1], gives natural partially holomorphic realizations of principal series representations π_i of G_i. That involves a certain extension of the classical Bott–Borel–Weil theorem [Bot57] which we need for the groups M_i. In order to pass to the limit, we construct and study the appropriate limit flag manifolds, limit of closed orbits, limits of holomorphic arc components, and limit sheaves, in Sections 5 and 6. This is done in such a way that the limit Bott–Borel–Weil theorem of [NRW01] applies over the holomorphic arc components of the closed orbits. That defines the geometric setting for the representations in question. In order to see that the cohomology of the limit sheaf is the limit of the cohomologies, we prove a Mittag–Leffler condition at the end of Section 6. Thus, we have the possibility of obtaining good geometric realizations of limit principal series representations of G directly on cohomology spaces.

We actually construct the geometric realizations in Section 7. Theorem 7.1 is the 0-cohomology result in the style of the Borel–Weil theorem, and Theorem 7.2 is the higher cohomology result in the style of the Bott–Borel–Weil theorem. For the latter it is essential to have the cohomologies all occur in the same degree. That is the third technical issue, and we reduce it to the same question for $M = \varinjlim M_i$, where it was settled in [NRW01].

The fourth issue is whether these principal series representations of G are norm-preserving Banach space representations, or even unitary representations, of G. That is settled in Theorem 8.9. There the key idea is that of a weakly parabolic direct system.

It is very important to have a large number of interesting examples. For that we consider diagonal embedding direct limits of classical real simple Lie groups. We examine their behavior relative to the various general notions studied earlier and see that our constructions work very well for these interesting direct limit groups. This is done in Section 9. These diagonal embedding direct limits have been studied extensively in the context of locally finite Lie algebras. That is a rapidly developing area; see [LN04] and the references therein. A locally finite Lie algebra of countable dimension can be represented as a direct limit $\varinjlim\{\mathfrak{g}_m, d\phi_{n,m}\}_{m,n\in\mathbb{Z}^+}$ of finite-dimensional Lie algebras, and the diagonal embedding direct limits are essentially just those where the group level maps $\phi_{n,m}$ are polynomials of degree 1.

Finally, in Section 10 we discuss the place of the principal series in our program for constructing limit representations corresponding to all tempered series, and indicate some of the problems to be settled in [Wol05a, Wol05b].

The notions of parabolic and weakly parabolic direct systems developed from a conversation with Andrew Sinton.

2. Principal series for general reductive groups

Let G be a reductive real Lie group. In other words, its Lie algebra $\mathfrak{g}$ is reductive in the sense that it is the direct sum of a semisimple Lie algebra $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ and an abelian idea $\mathfrak{z}$ which is the center of $\mathfrak{g}$. As usual, $\mathfrak{g}_{\mathbb{C}}$ denotes the complexification of $\mathfrak{g}$, so $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}'_{\mathbb{C}} \oplus \mathfrak{z}_{\mathbb{C}}$ is the direct sum of the respective complexifications of $\mathfrak{g}'$ and $\mathfrak{z}$.

Definition 2.1. The real reductive Lie group G is a *general reductive group* if it satisfies the following two conditions.

(1) If $g \in G$ then $\mathrm{Ad}(g)$ is an inner automorphism of $\mathfrak{g}_{\mathbb{C}}$.
(2) G has a closed normal abelian subgroup Z such that:
  (i) Z centralizes the identity component G^0 of G;
  (ii) ZG^0 has finite index in G; and
  (iii) $Z \cap Z_{G^0}$ is co-compact in the center Z_{G^0} of G^0.

These are the conditions of [Wol74]. They are inherited by Levi components of cuspidal parabolic subgroups of G, and they lead to a nice Plancherel formula. See [Wol74, HW86a, HW86b]. The first condition says that the standard tempered representation construction yields representations that have an infinitesimal character. It can be formulated: $\mathrm{Ad}(G) \subset \mathrm{Int}(\mathfrak{g}_{\mathbb{C}})$. The famous *Harish-Chandra class* is the case where the semisimple component $(G^0)' = [G^0, G^0]$ of G^0 has finite center and the component group G/G^0 is finite.

From now on we assume that G is a general reductive group in the sense of Definition 2.1.

Note that the kernel of $\mathrm{Ad} : G \to \mathrm{Ad}(G)$ is the centralizer $Z_G(G^0)$ of the identity component and that the image $\mathrm{Ad}(G)$ is a closed subgroup of the complex semisimple group $\mathrm{Int}(\mathfrak{g}_{\mathbb{C}})$ with only finitely many topological components. Thus $\mathrm{Ad}(G)$ has maximal compact subgroups, as usual for semisimple linear groups, and every maximal compact subgroup of $\mathrm{Ad}(G)$ is of the form $K/Z_G(G^0)$ for some closed subgroup $K \subset G$.

Given a maximal compact subgroup $K/Z_G(G^0)$ of $\mathrm{Ad}(G)$, it is known [Wol74, Lemma 4.1.1] that K is the fixed point set of a unique involutive automorphism θ of G. These automorphisms θ are called *Cartan involutions* of G, and they are lifts of the Cartan involutions of the linear group $\mathrm{Ad}(G)$. The groups K are the *maximal compactly embedded* subgroups of G.

One also knows [Wol74, Lemma 4.1.2] that $K \cap G^0$ is the identity component K^0 of K, that K meets every topological component of G, that any two Cartan involutions of G are conjugate by an element of $\mathrm{Ad}(G^0)$, and that every Cartan subgroup of G is stable under some Cartan involution. Here we use the usual definition: Cartan subgroup of G means the centralizer of a Cartan subalgebra of $\mathfrak{g}$.

Let us establish some notation. First, fix a Cartan involution θ of G and the corresponding maximal compactly embedded subgroup $K = G^\theta$ of G. Then $\mathfrak{a}$ denotes a maximal abelian subspace of $\{\xi \in \mathfrak{g} \mid \theta(\xi) = -\xi\}$. If $\xi \in \mathfrak{a}$ then $\mathrm{ad}(\xi)$ is a semisimple linear transformation of $\mathfrak{g}$ with all eigenvalues real. Now, as usual, $\mathfrak{g}$ is the direct sum of the joint eigenspaces (= restricted root spaces) $\mathfrak{g}^\gamma = \{\eta \in \mathfrak{g} \mid [\xi, \eta] = \gamma(\xi)\eta \text{ for every } \xi \in \mathfrak{a}\}$ as γ runs over $\mathfrak{a}^*$. $\Sigma(\mathfrak{g}, \mathfrak{a})$ denotes the $\mathfrak{a}$ root system $\{\gamma \in (\mathfrak{a}^* \setminus \{0\}) \mid \mathfrak{g}^\gamma \neq 0\}$ of $\mathfrak{g}$ and $\Sigma(\mathfrak{g}, \mathfrak{a})^+$ denotes a positive subsystem. In other words, $\Sigma(\mathfrak{g}, \mathfrak{a}) = \Sigma(\mathfrak{g}, \mathfrak{a})^+ \,\dot\cup\, -\Sigma(\mathfrak{g}, \mathfrak{a})^+$, and if $\alpha, \beta \in \Sigma(\mathfrak{g}, \mathfrak{a})^+$ with $\alpha + \beta \in \Sigma(\mathfrak{g}, \mathfrak{a})$ then $\alpha + \beta \in \Sigma(\mathfrak{g}, \mathfrak{a})^+$. Any two positive $\mathfrak{a}$ root systems are conjugate by the normalizer of $\mathfrak{a}$ in K.

A choice of positive $\mathfrak{a}$ root system $\Sigma(\mathfrak{g}, \mathfrak{a}^+)$ specifies a nilpotent subalgebra $\mathfrak{n} = \sum_{\gamma \in \Sigma(\mathfrak{g}, \mathfrak{a})^+} \mathfrak{g}^{-\gamma} \subset \mathfrak{g}$ and a nilpotent subgroup $N = \exp(\mathfrak{n})$. The corresponding *minimal parabolic subalgebra* $\mathfrak{p} \subset \mathfrak{g}$ is the normalizer of $\mathfrak{n}$ in $\mathfrak{g}$, and the corresponding *minimal parabolic subgroup* $P \subset G$ is the normalizer of N in G. Let A denote the analytic subgroup of G for $\mathfrak{a}$ and let M denote the centralizer of A in K.

Then we have the standard

$$\mathfrak{p} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n} \quad \text{and} \quad P = MAN \quad \text{with } MA = M \times A.$$

The corresponding *Iwasawa decompositions* are

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n} \quad \text{and} \quad G = KAN.$$

Both M and MA are general reductive groups as in Definition 2.1. Also, M is compact modulo $Z_M(M^0)$, the centralizer of M^0 in M. We write $\widehat{\ }$ for unitary dual. If $\xi \in \widehat{Z_{M^0}}$ we write $(\widehat{M^0})_\xi$ for the classes $[\eta^0] \in \widehat{M^0}$ such that $\eta^0|_{Z_{M^0}}$ is a multiple of ξ, and we write $(\widehat{Z_M(M^0)})_\xi$ for the classes $[\chi] \in \widehat{Z_M(M^0)}$ such that $\chi|_{Z_{M^0}}$ is a multiple of ξ.

The extension of Cartan's highest weight theory appropriate for M is as follows.

Proposition 2.2 (Cf. [Wol74, Proposition 1.1.3]). *Let G be a general reductive group and retain the notation just described for M, for its Cartan, and for its roots.*

(1) $M = Z_M(M^0)M^0$.

(2) *Every irreducible representation of M is finite dimensional.*

(3) *If $[\eta] \in \widehat{M}$ there exist unique $\xi \in \widehat{Z_{M^0}}$, $[\chi] \in (\widehat{Z_M(M^0)})_\xi$ and $[\eta^0] \in (\widehat{M^0})_\xi$ such that $[\eta] = [\chi \otimes \eta^0]$.*

(4) *Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{m}$, $\Sigma(\mathfrak{m}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})^+$ a positive $\mathfrak{t}_{\mathbb{C}}$-root system on $\mathfrak{m}_{\mathbb{C}}$, and $T^0 = \exp(\mathfrak{t})$, so $\Lambda^+_{\mathfrak{m}} = \{\nu \in \mathbf{i}\mathfrak{t}^* \mid e^\nu$ is well defined on T^0 and $\langle \nu, \gamma\rangle \geqslant 0$ for all $\gamma \in \Sigma(\mathfrak{m}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})^+\}$ is the set of dominant integral weights for M^0. Then there is a bijection $\nu \leftrightarrow [\eta_\nu]$ of $\Lambda^+_{\mathfrak{m}}$ onto $\widehat{M^0}$ given by: ν is the highest weight of η_ν. Furthermore, $[\eta_\nu] \in (\widehat{M^0})_\xi$ where $\xi = e^\nu|_{Z_{M^0}}$.*

(5) *$M = TM^0$ where T is the Cartan subgroup $\{m \in M \mid \mathrm{Ad}(m)\mu = \mu$, every $\mu \in \mathfrak{t}\}$ of M that corresponds to the Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{m}$.*

Define $\mathfrak{h} = \mathfrak{t} + \mathfrak{a}$. It is a maximally split Cartan subalgebra of $\mathfrak{g}$, and any two such Cartan subalgebras are $\mathrm{Ad}(G^0)$-conjugate. The positive root systems $\Sigma(\mathfrak{g}, \mathfrak{a})^+$ and $\Sigma(\mathfrak{m}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})^+$ determine a positive $\mathfrak{h}_{\mathbb{C}}$-root system $\Sigma(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})^+$ for $\mathfrak{g}_{\mathbb{C}}$ as follows. A root $\gamma \in \Sigma(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ is positive if it is nonzero and positive on $\mathfrak{a}$, or if it is zero on $\mathfrak{a}$ and positive on $\mathfrak{t}_{\mathbb{C}}$. In other words,

$$\begin{aligned} &\Sigma(\mathfrak{g}, \mathfrak{a})^+ = \{\gamma|_{\mathfrak{a}} \mid \gamma \in \Sigma(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})^+ \text{ and } \gamma|_{\mathfrak{a}} \neq 0\} \quad \text{and} \\ &\Sigma(\mathfrak{m}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})^+ = \{\gamma|_{\mathfrak{t}} \mid \gamma \in \Sigma(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})^+ \text{ and } \gamma|_{\mathfrak{a}} = 0\}. \end{aligned} \tag{2.3}$$

Now let $[\eta_{\chi,\nu}] = [\chi \otimes \eta_\nu] \in \widehat{M}$ and $\sigma \in \mathfrak{a}^*_{\mathbb{C}}$. That is equivalent to the datum $\eta_{\chi,\nu,\sigma} \in \widehat{P}$ where $\eta_{\chi,\nu,\sigma}$ is defined by

$$\eta_{\chi,\nu,\sigma}(man) = e^\sigma(a)\eta_{\chi,\nu}(m) \quad \text{for } m \in M,\ a \in A \text{ and } n \in N.$$

Here $e^\sigma(\exp(\xi))$ means $e^{\sigma(\xi)}$ for $\xi \in \mathfrak{a}$. In other words $e^\sigma(a)$ means $e^{\sigma(\log a)}$. Also, we will write $V_{\chi,\nu,\sigma}$ for the representation space of $\eta_{\chi,\nu,\sigma}$ although as a vector space it is independent of σ.

The corresponding *principal series* representation of G is

$$\pi_{\chi,\nu,\sigma} = \mathrm{Ind}_P^G(\eta_{\chi,\nu,\sigma}) \quad \text{(induced representation)}. \tag{2.4}$$

Here one must be careful about the category in which one takes the induced representation. For example, if $\mathcal{F}$ is a smoothness class of functions such as $C^k, 0 \leqslant k \leqslant \infty$, C_c^∞ (test functions), $C^{-\infty}$ (distributions), C^ω (analytic) or $C^{-\omega}$ (hyperfunctions), one can take $\pi_{\chi,\nu,\sigma}$ to be the natural representation (by translation of the variable) of G on

$$\begin{aligned} \mathcal{F}(G, P : V_{\chi,\nu,\sigma}) = \{f \in \mathcal{F}(G : V_{\chi,\nu,\sigma}) \mid f(gman) = e^{-\sigma}(a)\eta_{\chi,\nu}(m)^{-1}(f(g)) \\ \text{for all } g \in G, m \in M, a \in A, \text{ and } n \in N\}. \end{aligned} \tag{2.5}$$

The representation is always given by the formula $\pi_{\chi,\nu,\sigma}(g)(f(g')) = f(g^{-1}g')$.

One can also consider the analog of (2.5) using K-finite functions. Those functions are C^ω, and the representations spaces of the resulting K-finite induced representations are the common underlying Harish-Chandra modules for the representation spaces of the various smoothness classes (2.5) of induced representations.

Banach space representations, in particular unitary representations, are more delicate. We have to discuss this because we will have to keep track of how they behave in a direct limit process. The modular function of P is $\Delta_P(man) = e^{-2\rho_{\mathfrak{g},\mathfrak{a}}}(a)$, where $2\rho_{\mathfrak{g},\mathfrak{a}}(\xi)$ is the trace of $\mathrm{ad}(\xi)|_{\mathfrak{n}}$ for $\xi \in \mathfrak{a}$, because we defined $\mathfrak{n}$ to be the sum of the negative $\mathfrak{a}$-root spaces. G is unimodular, so $(\Delta_G/\Delta_P)(man) = \Delta_P^{-1}(man) = e^{2\rho_{\mathfrak{g},\mathfrak{a}}}(a)$. Let ζ be a norm-preserving representation of P on a Banach space V_ζ and let $1 \leqslant p \leqslant \infty$. If $f \in C_c(G, P : \zeta \otimes \Delta_P^{-1/p}) = C_c(G, P : \zeta \otimes e^{(2/p)\rho_{\mathfrak{g},\mathfrak{a}}})$ then $\|f(\cdot)\|_{V_\zeta} \in C_c(G, P; \Delta_P^{-1/p})$, so the global norm

$$\|f\|_p = \begin{cases} \left(\displaystyle\int_{G/P} \|f(gP)\|^p_{V_\zeta}\, d\mu_{G/P}(gP)\right)^{1/p} & \text{for } p < \infty, \\ \|f\|_\infty = \text{ess sup}_{\,G/P} \|f(gP)\|_{V_\zeta} & \text{for } p = \infty, \end{cases} \tag{2.6}$$

is well defined and invariant under the left translation action of G. Let $L_p(G, P : \zeta \otimes e^{(2/p)\rho_{\mathfrak{g},\mathfrak{a}}})$ denote the Banach space completion of $(C_c(G, P : \zeta \otimes e^{(2/p)\rho_{\mathfrak{g},\mathfrak{a}}}), \|\cdot\|_p)$. Each $\pi_{\zeta \otimes e^{(2/p)\rho_{\mathfrak{g},\mathfrak{a}}}}(g)$ extends by continuity from $C_c(G, P : \zeta \otimes e^{(2/p)\rho_{\mathfrak{g},\mathfrak{a}}})$ to a norm-preserving operator on $L_p(G, P : \zeta \otimes e^{(2/p)\rho_{\mathfrak{g},\mathfrak{a}}})$ and that defines a norm-preserving Banach representation of G on $L_p(G, P, \zeta \otimes e^{(2/p)\rho_{\mathfrak{g},\mathfrak{a}}})$. If ζ is unitary then the global inner product

$$\langle f, f' \rangle = \int_{G/P} \langle f(gH), f'(gH) \rangle_{V_\zeta}\, d\mu_{G/P}(gP) \quad \text{for } f, f' \in C_c(G, P : \zeta \otimes e^{\rho_{\mathfrak{g},\mathfrak{a}}}) \tag{2.7}$$

is G-invariant, $L_2(G, P : \zeta \otimes e^{\rho_{\mathfrak{g},\mathfrak{a}}})$ is the Hilbert space completion of $(C_c(G, P : \zeta \otimes e^{\rho_{\mathfrak{g},\mathfrak{a}}}), \langle \cdot, \cdot \rangle)$, and $\pi_{\zeta \otimes e^{\rho_{\mathfrak{g},\mathfrak{a}}}}$ is a unitary representation of G. Those unitary representations form the *unitary principal series* of G. We translate the discussion to our terminology (2.4) for principal series representations as follows.

Proposition 2.8. *The principal series representation $\pi_{\chi,\nu,\sigma}$ extends by continuity from a representation of G on $C_c(G, P : V_{\chi,\nu,\sigma})$ to a norm-preserving representation on $L_p(G, P : V_{\chi,\nu,\sigma})$ if and only if $\sigma \in \mathbf{i}\mathfrak{a}^* + (2/p)\rho_{\mathfrak{g},\mathfrak{a}}$. In particular, it extends by continuity to a unitary representation of G on $L_2(G, P : V_{\chi,\nu,\sigma})$ if and only if $\sigma \in \mathbf{i}\mathfrak{a}^* + \rho_{\mathfrak{g},\mathfrak{a}}$.*

Remark 2.9. The restriction $\pi_{\chi,\nu,\sigma}|_K = \mathrm{Ind}_M^K(\eta_{\chi,\nu})$, independent of σ. Decompose $\sigma = \sigma' + \sigma''$ where $\sigma' \in \mathbf{i}\mathfrak{a}^* + \rho_{\mathfrak{g},\mathfrak{a}}$ and $\sigma'' \in \mathfrak{a}$. Then $\pi_{\chi,\nu,\sigma}|_K = \pi_{\chi,\nu,\sigma'}|_K$, restriction of a unitary representation. In other words, we may always view the underlying Harish-Chandra module of $\pi_{\chi,\nu,\sigma}$ as a pre-Hilbert space. This will be important when we look at direct limit groups.

3. Geometric form of principal series representations

Let $G_{\mathbb{C}}$ be a connected reductive complex Lie group for which G is a real form. In other words there is a homomorphism $\varphi : G \to G_{\mathbb{C}}$ with discrete kernel such that $d\varphi(\mathfrak{g})$ is a real form of $\mathfrak{g}_{\mathbb{C}}$. If Q is a parabolic subgroup of $G_{\mathbb{C}}$, then we can view the complex flag manifold $Z = G_{\mathbb{C}}/Q$ as the set of all $G_{\mathbb{C}}$-conjugates of Q, say $Z \ni z \leftrightarrow Q_z \subset G_{\mathbb{C}}$, because Q is its own normalizer in $G_{\mathbb{C}}$. Now we can view Z as the set of all $\mathrm{Int}(\mathfrak{g}_{\mathbb{C}})$-conjugates of $\mathfrak{q}$ by $Z \ni z \leftrightarrow \mathfrak{q}_z \subset \mathfrak{g}_{\mathbb{C}}$.

The first condition of Definition 2.1 ensures that G acts on Z through φ and conjugation. In other words, G acts on Z through its adjoint action on $\mathfrak{g}_{\mathbb{C}}$. Thus G acts on Z by $g(z) = z'$ where $\mathfrak{q}_{z'} = \mathrm{Ad}(g)(\mathfrak{q}_z)$. This will be important when we construct direct limits of complex flag manifolds.

Let $\Psi \subset \Sigma(\mathfrak{m}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})^+$ be any set of simple roots. That defines a parabolic subalgebra $\mathfrak{r} = \mathfrak{j}_{\mathbb{C}} + \mathfrak{n}_{\mathfrak{m}}$ in $\mathfrak{m}_{\mathbb{C}}$, with nilradical $\mathfrak{n}_{\mathfrak{m}}$ and Levi component $\mathfrak{j}_{\mathbb{C}}$, where the reductive algebra $\mathfrak{j}_{\mathbb{C}}$ contains $\mathfrak{t}_{\mathbb{C}}$ and has simple root system Ψ. The corresponding parabolic subgroup of $M_{\mathbb{C}}$ is $J_{\mathbb{C}} N_{\mathfrak{m}}$, and its φ^{-1}-image is a real form J of $J_{\mathbb{C}}$. Note that $J = TJ^0$ where T is the Cartan subgroup of M corresponding to $\mathfrak{t}$.

Conversely to (2.3) we extend roots of $\mathfrak{m}_{\mathbb{C}}$ to roots of $\mathfrak{m}_{\mathbb{C}} + \mathfrak{a}_{\mathbb{C}}$ by zero on $\mathfrak{a}_{\mathbb{C}}$ and obtain $\Sigma(\mathfrak{m}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})^+ = \Sigma(\mathfrak{m}_{\mathbb{C}} + \mathfrak{a}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})^+ \subset \Sigma(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})^+$. Every $\psi \in \Psi$ remains simple for $\Sigma(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})^+$. Thus Ψ also defines a parabolic subalgebra $\mathfrak{q} = \mathfrak{l}_{\mathbb{C}} + \mathfrak{u}$ in $\mathfrak{g}_{\mathbb{C}}$, with nilradical $\mathfrak{u}$ and Levi component $\mathfrak{l}_{\mathbb{C}}$, where the reductive algebra $\mathfrak{l}_{\mathbb{C}}$ contains $\mathfrak{h}_{\mathbb{C}}$ and has simple root system Ψ. The corresponding parabolic subgroup of $G_{\mathbb{C}}$ is $Q = L_{\mathbb{C}} U$, and its φ^{-1}-image is a real form L of $L_{\mathbb{C}}$. Note that $L = JA = AJ = ATJ^0 = HJ^0$ where $H = T \times A$ is the Cartan subgroup of G corresponding to $\mathfrak{h}$.

As before, Z is the complex flag manifold $G_{\mathbb{C}}/Q$. Let $z_0 = 1Q \in Z$. Then the closed G-orbit in Z is $F = G(z_0) = K(z_0)$. We will realize principal series representations on partially holomorphic vector bundles over F.

Define $S = M(z_0)$. Note that M acts on Z as a compact group. The basic properties of S, from [Wol74, ch. 1], are (1) $S = M_{\mathbb{C}}(z_0)$, so S is a complex flag manifold, $S = M_{\mathbb{C}}/R$ where $R = \{m \in M_{\mathbb{C}} \mid m(z_0) = z_0\}$ is a parabolic subgroup of $M_{\mathbb{C}}$ with Lie algebra $\mathfrak{r}$ as described above, (2) $S \cong M/J$ with J as described above, and (3) if $g, g' \in G$ and gS meets $g'S$ then $gS = g'S$; and $P = \{g \in G \mid gS = S\}$. In fact, in the notation of [Wol69] the gS are the holomorphic arc components of F. Thus we have the following.

Proposition 3.1. *Define $\beta : F \to G/P = \{gS \mid g \in G\}$ by $\beta(gz_0) = gS$. Then $\beta : F \to G/P$ is a well defined C^ω fiber bundle with structure group P. The fiber over gP is gS, which is maximal among complex submanifolds of Z that are contained in F.*

Note that $T = Z_M(M^0)T^0$. For $Z_M(M^0)$ centralizes $\mathfrak{t}$, thus is contained in T, and if $t \in T$ then $\mathrm{Ad}(t)|_{M^0}$ is an inner automorphism of M^0 that fixes every $\mu \in \mathfrak{t}$, thus given by $\mathrm{Ad}(t')|_{M^0}$ for some $t' \in T^0$, so $tT^0 \subset Z_M(M^0)T^0$. As $T \subset J$ and $J \cap M^0 = J^0$ we have the following.

Lemma 3.2 (Cf. [Wol74, Proposition 1.1.3]). *Let $J = TJ^0$, real form of the reductive part of the parabolic subgroup of $M_{\mathbb{C}}$ defined by $\mathfrak{r} = \mathfrak{j}_{\mathbb{C}} + \mathfrak{n}_{\mathfrak{m}}$.*

(1) *$J = Z_M(M^0)J^0$.*
(2) *If $[\zeta] \in \widehat{J}$ there exist unique $\xi \in \widehat{Z_{M^0}}$, $[\chi] \in (\widehat{Z_M(M^0)})_\xi$ and $[\zeta^0] \in (\widehat{M^0})_\xi$ such that $[\zeta] = [\chi \otimes \zeta^0]$.*
(3) *Let $\Lambda_{\mathfrak{j}}^+ = \{\nu \in i\mathfrak{t}^* \mid e^\nu$ is well defined on T^0 and $\langle \nu, \gamma \rangle \geqslant 0$ for all $\gamma \in \Sigma(\mathfrak{j}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})^+\}$, the set of dominant integral weights for J^0. Then there is a bijection $\nu \leftrightarrow [\zeta_\nu^0]$ of $\Lambda_{\mathfrak{j}}^+$ onto $\widehat{J^0}$ given by: ν is the highest weight of ζ_ν^0. Furthermore, $[\zeta_\nu^0] \in (\widehat{J^0})_\xi$ where $\xi = e^\nu|_{Z_{M^0}}$.*

The set of $\mathfrak{m}$-nonsingular dominant integral weights for J^0 is

$$(\Lambda_{\mathfrak{j}}^+)' = \{\nu \in \Lambda_{\mathfrak{j}}^+ \mid \langle \nu + \rho_{\mathfrak{m},\mathfrak{t}}, \gamma \rangle \neq 0 \text{ for all } \gamma \in \Sigma(\mathfrak{j}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})\} \tag{3.3}$$

where $\rho_{\mathfrak{m},\mathfrak{t}}$ is half the sum of the roots in $\Sigma(\mathfrak{m}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})^+$. If $\nu \in (\Lambda_{\mathfrak{j}}^+)'$ there is a unique Weyl group element $w \in W(\mathfrak{m}, \mathfrak{t})$ such that

$$\widetilde{\nu} = w(\nu + \rho_{\mathfrak{m},\mathfrak{t}}) - \rho_{\mathfrak{m},\mathfrak{t}} \in \Lambda_{\mathfrak{m}}^+. \tag{3.4}$$

We write $q(\nu)$ for the length $\ell(w)$ of that Weyl group element.

Let $\nu \in (\Lambda_{\mathfrak{j}}^+)'$. Let ζ_ν denote the irreducible representation of J^0 with highest weight ν as in Lemma 3.2. Denote $\xi = e^\nu|_{Z_{M^0}}$ and choose $[\chi] \in \widehat{Z_M(M^0)}_\xi$. Then $[\zeta_{\chi,\nu}] = [\chi \otimes \zeta_\nu]$ is a well-defined element of $\widehat{J}$. Let $E_{\chi,\nu}$ denote its representation space. Let $\sigma \in \mathfrak{a}_{\mathbb{C}}^*$. The isotropy subgroup of G at z_0 is JAN, and the representation $\zeta_{\chi,\nu,\sigma}(jan) = e^\sigma(a)\zeta_{\chi,\nu}(b)$ of JAN defines a G-homogeneous vector bundle. $\mathbb{E}_{\chi,\nu,\sigma} \to F$ with fiber $E_{\chi,\nu,\sigma}$ at z_0, where $E_{\chi,\nu,\sigma}$ is the representation space $E_{\chi,\nu}$ of $\zeta_{\chi,\nu,\sigma}$.

Note that $\mathbb{E}_{\chi,\nu,\sigma}|_{gS} \to gS$ is holomorphic, for every fiber gS of $F \to G/P$. Initially one is tempted to define the corresponding sheaf $\mathcal{O}_{\mathfrak{n}}(\mathbb{E}_{\chi,\nu,\sigma}) \to F$ of partially holomorphic sections as the sheaf of germs of C^∞ functions $h : G \to E_{\chi,\nu,\sigma}$ such that (i) $h(gjan) = \zeta_{\chi,\nu,\sigma}(jan)^{-1}(h(g))$ for $g \in G$ and $jan \in JAN$, and (ii) $h(g;\xi) + d\zeta_{\chi,\nu,\sigma}(\xi)h(g) = 0$ for $g \in G$ and $\xi \in (\mathfrak{j}+\mathfrak{a}+\mathfrak{n})_{\mathbb{C}}$. However, that causes a number of technical problems, and it is better to use hyperfunctions as in [Sch86, SW90] to ensure that the differentials in the cohomology of $\mathcal{O}_{\mathfrak{n}}(\mathbb{E}_{\chi,\nu,\sigma}) \to F$ have closed range. Thus, the appropriate definition is that $\mathcal{O}_{\mathfrak{n}}(\mathbb{E}_{\chi,\nu,\sigma}) \to F$ is the sheaf of germs of $E_{\chi,\nu,\sigma}$-valued hyperfunctions on G such that (i) $h(gjan) = \zeta_{\chi,\nu,\sigma}(jan)^{-1}(h(g))$ for $g \in G$ and $jan \in JAN$, and (ii) $h(g;\xi) + d\zeta_{\chi,\nu,\sigma}(\xi)h(g) = 0$ for $g \in G$ and $\xi \in (\mathfrak{j}+\mathfrak{a}+\mathfrak{n})_{\mathbb{C}}$.

Apply the Bott–Borel–Weil theorem to each $\mathbb{E}_{\chi,\nu,\sigma}|_{gS} \to gS$. By elliptic regularity, use of hyperfunction coefficients results in the same cohomology as use of smooth coefficients. The result is as follows.

Proposition 3.5 (Cf. [Wol74, Theorem 1.2.19]). *If $\nu \notin (\Lambda_{\mathfrak{j}}^+)'$ then $H^q(F;\mathcal{O}_{\mathfrak{n}}(\mathbb{E}_{\chi,\nu,\sigma})) = 0$ for every integer q. If $\nu \in (\Lambda_{\mathfrak{j}}^+)'$, then $H^q(F;\mathcal{O}_{\mathfrak{n}}(\mathbb{E}_{\chi,\nu,\sigma})) = 0$ for $q \neq q(\nu)$, and the natural action of G on $H^{q(\nu)}(F;\mathcal{O}_{\mathfrak{n}}(\mathbb{E}_{\chi,\nu,\sigma}))$ is infinitesimally equivalent (same underlying Harish-Chandra module) to the principal series representation $\pi_{\chi,\widetilde{\nu},\sigma}$.*

4. Principal series for direct limit groups

Consider a countable strict direct system $\{G_i, \phi_{k,i}\}_{i,k\in I}$ of reductive Lie groups. Thus, I is a countable partially ordered set. If $i,k \in I$ there exists $\gamma \in I$ with $i \leqslant \gamma$ and $k \leqslant \gamma$. Each G_i is a reductive Lie group. If $i \leqslant k$ then $\phi_{k,i} : G_i \to G_k$ is a continuous group homomorphism. Then we have the direct limit group $G = \varinjlim G_i$, with direct limit topology, and the $\phi_{k,i}$ specify continuous group homomorphisms $\phi_i : G_i \to G$. The strictness condition is that the homomorphisms ϕ_i are homeomorphisms onto their images. So we may in fact view the $\phi_{k,i}$ as inclusions and view G as the union of the G_i, and then the original topology on each G_i is the subspace topology. In particular, G_i sits in G as a closed (thus regularly embedded) submanifold.

Countability of I has two important consequences. First, it guarantees the existence of a C^ω (real analytic) Lie group structure on G. See [NRW91, NRW93, NRW01, Glo03]. Second, it guarantees that I either is finite or has a cofinal subset order-isomorphic to the positive integers. Whenever it is convenient we will replace I by that subset; this change in the defining direct system $\{G_i, \phi_{k,i}\}_{i,k\in I}$ has no effect on the direct limit group $G = \varinjlim G_i$.

We always assume that every G_i is a general reductive group in the sense of Definition 2.1.

We have the corresponding strict direct system $\{\mathfrak{g}_i, d\phi_{k,i}\}_{i,k\in I}$ of reductive Lie algebras, the direct limit algebra $\mathfrak{g} = \varinjlim \mathfrak{g}_i$ with the direct limit topology, and injective homomorphisms $d\phi_i : \mathfrak{g}_i \to \mathfrak{g}$ that are C^ω diffeomorphisms onto their images. We also have the exponential map $\exp : \mathfrak{g} \to G$, direct limit of the $\exp : \mathfrak{g}_i \to G_i$. The C^ω Lie group structure on the limit group G is specified by the condition that $\exp : \mathfrak{g} \to G$ is a C^ω diffeomorphism from a neighborhood on 0 in $\mathfrak{g}$ onto a neighborhood of 1 in G. Again see [NRW91, NRW93, NRW01, Glo03].

Consider a *compatible family of representations* $\{\pi_i, W_i, \psi_{k,i}\}_{i,k\in I}$ of $\{G_i, \phi_{k,i}\}$. Thus, W_i is a locally convex topological vector space (usually Hilbert or Fréchet), $\{W_i, \psi_{k,i}\}_{i,k\in I}$ is a strict direct system, π_i is a continuous representation of G_i on W_i and

$$\pi_k(\phi_{k,i}(g_i))(\psi_{k,i}(w_i)) = \psi_{k,i}(\pi_i(g_i)(w_i)) \quad \text{for } g_i \in G_i,\ w_i \in W_i \text{ and } i \leqslant k.$$

That, of course, results in continuous injective linear maps $\psi_i : W_i \to W$ with closed image, where $W = \varinjlim W_k$. We have the direct limit representation $\pi = \varinjlim \pi_i$ of G on W given by

$$\pi(g)w = \psi_i(\pi_i(g_i)(w_i)) \quad \text{for } g = \phi_i(g_i) \in G \text{ and } w = \psi_i(w_i) \in W.$$

We now examine the situation where the π_i are principal series representations of the G_i. For that we need direct limits of minimal parabolic subgroups.

As mentioned above we may assume $I = \{1, 2, 3, \dots\}$ with the usual order. Then we recursively construct Cartan involutions θ_i of $\mathfrak{g}_i$ such that if $i \leqslant k$ then $\theta_k|_{d\phi_{k,i}(\mathfrak{g}_i)}$ is θ_i, in other words $d\phi_{k,i}(\mathfrak{k}_i) = \mathfrak{k}_k \cap d\phi_{k,i}(\mathfrak{g}_i)$. We know that θ_i extends uniquely to G_i in such a way that its fixed point set K_i has Lie algebra $\mathfrak{k}_i$, contains the kernel of the adjoint representation of G_i and meets every component of G_i. Thus K_i is the G_i-normalizer of $K_i^0 = K_i \cap G_i^0$. Because of components, however, we must impose the following condition on our direct system.

Assumption 4.1. If $i \leqslant k$ then $\phi_{k,i}(K_i) \subset K_k$, so we have $K = \varinjlim K_i$.

While it is tempting to try to get around Assumption 4.1 by assuming that the G_i are connected, we would still meet the same problem with the groups M_i indicated below.

Now $d\phi_{k,i}$ maps the (-1)-eigenspace of θ_i into the (-1)-eigenspace of θ_k, so we can recursively construct maximal abelian subspaces $\mathfrak{a}_i \subset \{\xi \in \mathfrak{g}_i \mid \theta_i(\xi) = -\xi\}$ such that $d\phi_{k,i}(\mathfrak{a}_i) \subset \mathfrak{a}_k$ for $i \leqslant k$. Then the corresponding analytic subgroups satisfy $\phi_{k,i}(A_i) \subset A_k$ for $i \leqslant k$, so we have $A = \varinjlim A_i$.

Note $d\phi_{k,i}(\mathfrak{a}_i) = \mathfrak{a}_k \cap d\phi_{k,i}(\mathfrak{g}_i)$. This allows us to recursively construct a sequence of elements $\zeta_i \in \mathfrak{a}_i^*$ such that $\langle \zeta_i, \alpha_i \rangle \neq 0$ for all $\alpha_i \in \Sigma(\mathfrak{g}_i, \mathfrak{a}_i)$ and $d\phi_{k,i}(\zeta_k) = \zeta_i$ for $i \leqslant k$. Taking roots where that inner product is positive we have positive root systems $\Sigma(\mathfrak{g}_i, \mathfrak{a}_i)^+$ such that $d\phi_{k,i}$ maps every negative restricted root space $\mathfrak{g}_i^{-\alpha_i}$ into $\mathfrak{n}_k = \sum_{\beta_k \in \Sigma(\mathfrak{g}_k, \mathfrak{a}_k)^+} \mathfrak{g}_k^{-\beta_k}$. Again, the corresponding analytic subgroups satisfy $\phi_{k,i}(N_i) \subset N_k$ for $i \leqslant k$, so we have $N = \varinjlim N_i$.

Essentially as before, let M_i denote the centralizer $Z_{K_i}(A_i)$ of A_i in K_i. In general the behavior of the M_i (or even their identity components and Lie algebras) under the $\phi_{k,i}$ is unclear. Thus, we must impose one more condition on our direct system.

Assumption 4.2. If $i \leqslant k$ then $\phi_{k,i}(M_i) \subset M_k$, so we have $M = \varinjlim M_i$.

Now we put all this together.

LEMMA 4.3. *Suppose that Assumptions 4.1 and 4.2 hold. Then we have Iwasawa decompositions $G_i = K_i A_i N_i$ and minimal parabolics $P_i = M_i A_i N_i$ such that $\phi_{k,i}$ maps $K_i \to K_k$, $M_i \to M_k$, $A_i \to A_k$, and $N_i \to N_k$ for $i \leqslant k$. In particular, we have an Iwasawa decomposition $G = KAN$, and if $i \leqslant k$ then $\phi_{k,i}(P_i) \subset P_k$ so we also have the limit minimal parabolic $P = \varinjlim P_i = MAN$.*

Here $G = KAN$ and $P = MAN$ mean that (i) $(k, a, n) \mapsto kan$ is a C^ω diffeomorphism of $K \times A \times N$ onto G, and (ii) $(m, a, n) \mapsto man$ is a C^ω diffeomorphism of $M \times A \times N$ onto P.

Example 4.4. The diagonal embedding direct limit groups described in [NRW01, Section 5], and their extension to noncompact real forms, all satisfy (4.1) and (4.2), leading to the limits and decompositions $G = KAN$ and $P = \varinjlim P_i = MAN$ of Lemma 4.3. For example, let $\{r_n\}$ and $\{s_n\}$ be sequences of integers $\geqslant 0$ where $1 \leqslant n < \infty$ and $r_n + s_n \geqslant 1$. Fix $k_1 > 1$, and recursively define $k_{n+1} = r_n k_n + s_n$, define $G_n = SL(k_n; \mathbb{R})$ and $\phi_{n+1,n} : G_n \to G_{n+1}$ by $\phi_{n+1,n}(g) = \mathrm{diag}(g, \dots, g; 1, \dots, 1)$ with r_n of g's and s_n of 1's. Here K_n is the special orthogonal group $SO(k_n)$, A_n consists of the diagonal matrices of determinant 1 with positive diagonal entries in G_n, M_n consists of the diagonal matrices determinant 1 with diagonal entries ± 1 in G_n, and N_n consists of the lower triangular matrices in G_n with all diagonal entries equal to 1. The limit groups depend on the choice of sequences $\{r_n\}$ and $\{s_n\}$, and it is quite nontrivial to see when pairs of sequences lead to isomorphic limits.

In order to discuss representations of M we need direct systems of Cartan subalgebras and appropriate root orders. With $I = \{1, 2, \dots\}$ we recursively construct Cartan subalgebras $\mathfrak{t}_i$ in $\mathfrak{m}_i$

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such that $d\phi_{k,i}\mathfrak{t}_i \subset \mathfrak{t}_k$ for $i \leqslant k$, and positive systems $\Sigma(\mathfrak{m}_{i,\mathbb{C}},\, \mathfrak{t}_{i,\mathbb{C}})^+$ such that

$$d\phi_{k,i}\Bigg(\sum_{\alpha\in\Sigma(\mathfrak{m}_{i,\mathbb{C}},\, \mathfrak{t}_{i,\mathbb{C}})^+} \mathfrak{m}_i^\alpha\Bigg) \subset \sum_{\beta\in\Sigma(\mathfrak{m}_{k,\mathbb{C}},\, \mathfrak{t}_{k,\mathbb{C}})^+} \mathfrak{m}_k^\beta \quad \text{for } i \leqslant k. \tag{4.5}$$

Then $\mathfrak{t} = \varinjlim \mathfrak{t}_i$ is a Cartan subalgebra of $\mathfrak{m}$, the root system $\Sigma(\mathfrak{m}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) = \varprojlim \Sigma(\mathfrak{m}_{i,\mathbb{C}},\, \mathfrak{t}_{i,\mathbb{C}})$ (inverse limit), and the positive system $\Sigma(\mathfrak{m}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})^+ = \varprojlim \Sigma(\mathfrak{m}_{i,\mathbb{C}},\, \mathfrak{t}_{i,\mathbb{C}})^+$ is well defined. The Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{m}$ defines a Cartan subgroup $T = \{m \in M \mid \mathrm{Ad}(m)\xi = \xi \text{ for all } \xi \in \mathfrak{t}\}$, and $T^0 = T \cap M^0$ is the corresponding Cartan subgroup of M^0.

Each M_i/M_i^0 is discrete, so $\varinjlim M_i^0$ is connected, closed and open in M. Thus $\varinjlim M_i^0 = M^0$ and M/M^0 is discrete. The same considerations hold for G, K and T. Also, since each $M_i = T_iM_i^0$ we have $M = TM^0$. Here note $T^0 = T\cap M^0$. In the special case where each $\phi_{k,i} : Z_{M_i}(M_i^0) \to Z_{M_k}(M_k^0)$ we have $Z_M(M^0) = \varinjlim Z_{M_i}(M_i^0)$ and $M = Z_M(M^0)M^0$.

A linear functional $\nu \in \mathfrak{t}_{\mathbb{C}}^*$ is called *integral* (or $\mathfrak{m}$-integral) if e^ν is a well-defined homomorphism $T^0 \to \mathbb{C}^\times$, in other words if the pull-backs $\nu_i = \phi_i^*(\nu) \in \mathfrak{t}_{i,\mathbb{C}}^*$ are integral. Here note $\nu = \varprojlim \nu_i$. The functional ν is called *dominant* (or $\mathfrak{m}$-dominant) if $\langle \nu, a\rangle \geqslant 0$ for every $a \in \Sigma(\mathfrak{m}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})^+$, in other words if ν_i is $\mathfrak{m}_i$-dominant for each i. We use these notions for a small variation on the Mackey little-group method.

Proposition 4.6. *Let $\nu \in \mathfrak{t}_{\mathbb{C}}^*$ be a dominant integral linear functional. It determines an irreducible unitary representation η_ν of M^0 as follows. Let $\eta_{i,\nu}$ denote the irreducible unitary representation of M_i^0 with lowest weight $-\nu_i = \phi_i^*(-\nu)$. Choose a unit lowest weight vector $v_{i,\nu}$ in the representation space $V_{i,\nu}$ of $\eta_{i,\nu}$. For $i \leqslant k$ extend the Lie algebra monomorphism $d\phi_{k,i} : \mathfrak{g}_i \hookrightarrow \mathfrak{g}_k$ as usual to an enveloping algebra monomorphism $\mathcal{U}(\mathfrak{g}_i) \hookrightarrow \mathcal{U}(\mathfrak{g}_k)$, which we also denote $d\phi_{k,i}$, and define $\psi_{k,i} : V_{i,\nu} \to V_{k;\nu}$ by $\psi_{k,i}(d\eta_{i,\nu}(\Xi_i)(v_{i,-\nu})) = d\eta_{k,\nu}(d\phi_{k,i}(\Xi_i))(v_{k,-\nu})$ for $\Xi \in \mathcal{U}(\mathfrak{m}_i)$.*

1. *$\{\eta_{i,\nu}, V_{i,\nu}, \psi_{k,i}\}$ is a compatible system of irreducible representations of $\{M_i^0, \phi_{k,i}\}$, so $\eta_\nu = \varinjlim \eta_{i,\nu}$ is a well-defined irreducible unitary representation of M^0, with representation space $V_\nu = \varinjlim V_{i,\nu}$. Further, η_ν is a lowest weight representation with lowest weight $-\nu$, and $v_{-\nu} = \varinjlim v_{i,-\nu}$ is a lowest weight unit vector.*
2. *If $m \in M$ then $\eta_\nu \circ \mathrm{Ad}(m)^{-1}$ is unitarily equivalent to η_ν.*
3. *Denote $\widehat{M_\nu}$: equivalence classes of irreducible unitary representations η of M such that $\eta|_{M^0}$ weakly contains η_ν in the sense that the kernel of $d\eta$ on the enveloping algebra $\mathcal{U}(\mathfrak{m})$ is contained in the kernel of $d\eta_\nu$. Then $\widehat{M_\nu} = \{[\eta] \in \widehat{M} \mid [\eta|_{M^0}] \text{ is a multiple of } \eta_\nu\}$.*
4. *Let $[\eta_{\chi,\nu}] \in \widehat{M_\nu}$. Let $V_{\chi,\nu}$ denote its representation space. Choose a subspace $V_\nu' \subset V_{\chi,\nu}$ on which M^0 acts by η_ν, let $v \mapsto v'$ denote the intertwining map of V_ν onto V_ν' and let $v_{-\nu}'$ be the image of the lowest weight unit vector $v_{-\nu}$ of η_ν. Then the image of $V_{i,\nu}$ in $V_{\chi,\nu}$ is $d\eta_{i,\nu}(\mathcal{U}(\mathfrak{m}_i))(v_{-\nu}')$, and $V_\nu' = \varinjlim d\eta_{i,\nu}(\mathcal{U}(\mathfrak{m}_i))(v_{-\nu}') = d\eta_\nu(\mathcal{U}(\mathfrak{m}))(v_{-\nu}')$.*
5. *In the special case where $M = Z_M(M^0)M^0$, the set $\widehat{M_\nu}$ consists of all $[\chi\otimes\eta_\nu]$ such that $\chi \in (\widehat{Z_M(M^0)})_\xi$ where $\xi = e^{-\nu}|_{Z_{M^0}}$.*

Note. In general we write the elements of $\widehat{M_\nu}$ as $[\eta_{\chi,\nu}]$ where χ is just a parameter. In the case of Statement (5) the parameter χ is interpreted as an element of $(\widehat{Z_M(M^0)})_\xi$.

Proof. Statement (1) is satisfied by construction.

For statement (2) let $m \in M$ and $\eta_\nu' = \eta_\nu \circ \mathrm{Ad}(m)^{-1}$. We view M as the union of the M_i. Then m belongs to some M_δ, hence to M_i for $i \geqslant \delta$. Altering m by an element of M_δ^0 we may assume that $m \in Z_{M_\delta}(M_\delta^0)$, so $\mathrm{Ad}(m)^*(\nu_\delta) = \nu_\delta$, and thus $\eta_{\delta,\nu}(v_\delta)$ is some multiple $c_\delta v_\delta$ of v_δ. Apply the enveloping algebra now to see that $v \mapsto c_\delta v$ intertwines $\eta_{\delta,\nu}'$ with $\eta_{\delta,\nu}$. The point here is that

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we may replace c_δ by any other modular scalar, for example by 1. Now $v \mapsto v$ intertwines $\eta'_{i,\nu}$ with $\eta_{i,\nu}$ for every $i \geqslant \delta$, and thus intertwines η'_ν with η_ν.

For statement (3) let $\mathcal{K}$ denote the kernel of $d\eta$ on $\mathcal{U}(\mathfrak{m})$ and let $\mathcal{K}_\nu$ denote the kernel of $d\eta_\nu$. If $\eta|_{M^0}$ is a multiple of η_ν then $\mathcal{K} = \mathcal{K}_\nu$. Now let $\mathcal{K} \subset \mathcal{K}_\nu$. Then the associative algebra $\mathcal{U}(\mathfrak{m})/\mathcal{K}_\nu$ is a quotient of $\mathcal{U}(\mathfrak{m})/\mathcal{K}$. Remember that η_ν is irreducible. Since M^0 is connected and generated by $\exp(\mathfrak{m})$ now η_ν is equivalent to a quotient representation of $\eta|_{M^0}$. By unitarity now η_ν is equivalent to a subrepresentation of $\eta|_{M^0}$. Let w be a cyclic unit vector for that irreducible subrepresentation and let W be a set of representatives of M modulo M^0. Then the representation space of η is generated by the $\eta(M)(\eta(x)w), x \in W$. By statement (2), the action of M^0 on the closed span of $\eta(M)(\eta(x)w)$ is equivalent to η_ν. Thus $\eta|_{M^0}$ is a multiple of η_ν.

Statements (4) and (5) follow from statements (1) and (3). □

Fix $[\eta_{\chi,\nu}] \in \widehat{M_\nu}$ as in Proposition 4.6. In the notation of Proposition 4.6, identify V_ν with its image $V'_\nu = d\eta_\nu(\mathcal{U}(\mathfrak{m}))(v'_{-\nu})$ in $V_{\chi,\nu}$ and identify the lowest weight vector $v_{-\nu}$ of η_ν with its image $v'_{-\nu}$ in $V_{\chi,\nu}$. Let $V_{i,\chi,\nu}$ denote the closed span of $\eta_{\chi,\nu}(M_i)(v_{-\nu})$ and let $\eta_{i,\chi,\nu}$ denote the representation of M_i on $V_{i,\chi,\nu}$. Unwinding the definitions one sees that

$$\eta_{\chi,\nu} = \varinjlim \eta_{i,\chi,\nu}. \tag{4.7}$$

Now let

$$\sigma \in \mathfrak{a}^*_{\mathbb{C}} \quad \text{and} \quad \sigma_i = \phi^*_i(\sigma) \in (\mathfrak{a}_i)^*_{\mathbb{C}}. \tag{4.8}$$

As in Section 2 that is equivalent to the data $\eta_{\chi,\nu,\sigma} \in \widehat{P}$ and $\eta_{i,\chi,\nu,\sigma} \in \widehat{P}_i$ defined by

$$\begin{aligned} &\eta_{\chi,\nu,\sigma}(man) = e^{\sigma}(a)\eta_{\chi,\nu}(m) \quad \text{for } m \in M, a \in A \text{ and } n \in N, \quad \text{and} \\ &\eta_{i,\chi,\nu,\sigma}(man) = e^{\sigma_i}(a)\eta_{i,\chi,\nu}(m) \quad \text{for } m \in M_i, a \in A_i \text{ and } n \in N_i. \end{aligned} \tag{4.9}$$

Again as in Section 2 we write $V_{\chi,\nu,\sigma}$ for the representation space of $\eta_{\chi,\nu,\sigma}$; as a vector space it just $V_{\chi,\nu}$. Similarly we write $V_{i,\chi,\nu,\sigma}$ for the representation space of $\eta_{i,\chi,\nu,\sigma}$.

The *principal series* representation of G defined by $[\eta_{\chi,\nu}] \in \widehat{M_\nu}$ and $\sigma \in \mathfrak{a}^*_{\mathbb{C}}$ is

$$\pi_{\chi,\nu,\sigma} = \mathrm{Ind}_P^G(\eta_{\chi,\nu,\sigma}) \quad \text{(induced representation)}. \tag{4.10}$$

This representation is always given by the formula $\pi_{\chi,\nu,\sigma}(g)(f(g')) = f(g^{-1}g')$. Of course, we also have the principal series representations $\pi_{i,\chi,\nu,\sigma} = \mathrm{Ind}_{P_i}^{G_i}(\eta_{i,\chi,\nu,\sigma})$ of G_i.

The principal series representations $\pi_{\chi,\nu,\sigma}$ of (4.10) has representation space that consists of an appropriate class of functions $f : G \to V_{\chi,\nu,\sigma}$ such that $f(gman) = e^{-\sigma}(a)\eta_{\chi,\nu}(m)^{-1} \cdot f(g)$ for $g \in G$ and $man \in MAN = P$. Here recall that $V_{\chi,\nu,\sigma}$ is the representation space of $\eta_{\chi,\nu,\sigma}$. View the representation space $V_{i,\chi,\nu,\sigma}$ of $\eta_{i,\chi,\nu,\sigma}$ as the closed M_i-invariant subspace of $V_{\chi,\nu,\sigma}$ generated by $\eta_{\chi,\nu,\sigma}(M_i)(v_{-\nu})$. Then the representation space of $\pi_{i,\chi,\nu,\sigma}$ is the subspace of the representation space of $\pi_{\chi,\nu,\sigma}$, given by $f(G_i) \subset V_{i,\chi,\nu,\sigma}$. Since G is the union of the G_i and $V_{\chi,\nu,\sigma}$ is the union of the $V_{i,\chi,\nu,\sigma}$ we have proved the following.

Proposition 4.11. *The principal series representations of a countable strict direct limit are just the direct limits of principal series representations. Specifically,* $\pi_{\chi,\nu,\sigma} = \varinjlim \pi_{i,\chi,\nu,\sigma}$.

In dealing with principal series representations one must be very careful about the category in which they take the induced representation. Smoothness categories such as $C^k, 0 \leqslant k \leqslant \infty$, C^∞_c (test functions), $C^{-\infty}$ (distributions), C^ω (analytic) or $C^{-\omega}$ (hyperfunctions) are still available for principal series representations of G, but anything involving integration over G/P is excluded. We will get around this problem by constructing geometric realizations that provide L_p versions of the principal series for G.

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5. Groups and spaces for the limit principal series

The Iwasawa decompositions of Lemma 4.3, and the Cartan subalgebras $\mathfrak{t}_i \subset \mathfrak{m}_i$ and the recursively constructed positive root systems that led to (4.5) define Cartan subalgebras $\mathfrak{h}_i = \mathfrak{t}_i \oplus \mathfrak{a}_i$ in $\mathfrak{g}_i$ such that $d\phi_{k,i}$ maps $\mathfrak{h}_i \to \mathfrak{h}_k$ for $i \leqslant k$. That leads directly to positive root systems $\Sigma(\mathfrak{g}_{i,\mathbb{C}}, \mathfrak{h}_{i,\mathbb{C}})^+$ given by (2.3) such that

$$d\phi_{k,i}\biggl(\sum_{a\in\Sigma(\mathfrak{g}_{i,\mathbb{C}},\,\mathfrak{h}_{i,\mathbb{C}})^+} \mathfrak{g}_{i,a}\biggr) \subset \sum_{b\in\Sigma(\mathfrak{g}_{k,\mathbb{C}},\,\mathfrak{h}_{k,\mathbb{C}})^+} \mathfrak{g}_{k,b} \quad \text{for } i \leqslant k. \tag{5.1}$$

Then $\mathfrak{h} = \varinjlim \mathfrak{h}_i$ is a Cartan subalgebra of $\mathfrak{g}$, $\Sigma(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) = \varprojlim \Sigma(\mathfrak{g}_{i,\mathbb{C}},\, \mathfrak{h}_{i,\mathbb{C}})$ is its root system, and $\Sigma(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})^+ = \varprojlim \Sigma(\mathfrak{g}_{i,\mathbb{C}},\, \mathfrak{h}_{i,\mathbb{C}})^+$ is a positive subsystem. Further, we will need a system of parabolic subalgebras $\mathfrak{q}_i \subset \mathfrak{g}_{i,\mathbb{C}}$, where the $\mathfrak{q}_i$ are defined by sets Ψ_i of $\Sigma(\mathfrak{m}_{i,\mathbb{C}},\, \mathfrak{t}_{i,\mathbb{C}})^+$-simple roots as in Section 3, and $d\phi_{k,i}^{-1}(\mathfrak{q}_k) = \mathfrak{q}_i$ for $i \leqslant k$.

We decompose $\mathfrak{q}_i = \mathfrak{l}_{i,\mathbb{C}} + \mathfrak{u}_i$ where $\mathfrak{l}_{i,\mathbb{C}}$ is a reductive complement to the nilradical, such that $\mathfrak{h}_i \subset \mathfrak{l}_{i,\mathbb{C}}$ and $\mathfrak{l}_i = \mathfrak{g}_i \cap \mathfrak{l}_{i,\mathbb{C}}$ is a real form of $\mathfrak{l}_{i,\mathbb{C}}$. Then $\mathfrak{g}_i \cap \mathfrak{q}_i = \mathfrak{l}_i + \mathfrak{n}_i$ as in Section 3.

Let $G_{i,\mathbb{C}}$ denote the connected simply connected Lie group with Lie algebra $\mathfrak{g}_{i,\mathbb{C}}$. In general, G_i will not be a real form of $G_{i,\mathbb{C}}$ because, in general, $\mathfrak{g}_i \hookrightarrow \mathfrak{g}_{i,\mathbb{C}}$ will not integrate to a homomorphism $G_i \to G_{i,\mathbb{C}}$, but at least we have the connected complex simply connected group $G_{\mathbb{C}} = \varinjlim G_{i,\mathbb{C}}$ with Lie algebra $\mathfrak{g}_{\mathbb{C}} = \varinjlim \mathfrak{g}_{i,\mathbb{C}}$.

Let Q_i be the parabolic subgroup with Lie algebra $\mathfrak{q}_i$, and let Z_i denote the complex flag manifold $G_{i,\mathbb{C}}/Q_i$. Note that we would get the same Z_i if we did this construction starting with arbitrary complex Lie groups ${}'G_{i,\mathbb{C}}$ for which the $G_{i,\mathbb{C}}$ are the universal covering groups, in particular if we started with any connected complex Lie group ${}'G_{i,\mathbb{C}}$ for which G_i is a real form. For Z_i can be identified as the set of all $\mathrm{Int}(\mathfrak{g}_{i,\mathbb{C}})$-conjugates of $\mathfrak{q}_i$ in $\mathfrak{g}_{i,\mathbb{C}}$, with the action of G_i given by conjugation.

The reason for this indirection is that, in general, we cannot choose a family of complex Lie groups ${}'G_{i,\mathbb{C}}$, for which the $G_{i,\mathbb{C}}$ are the universal covering groups, such that the ${}'G_{i,\mathbb{C}}$ constitute a well-defined direct system of complex Lie groups and holomorphic homomorphisms ${}'\phi_{k,i}$ with $d\phi_{k,i} = d\,{}'\phi_{k,i}$.

We recall some structural information on limit groups and limit flags from [NRW01, Sections 1 and 2].

The parabolic $Q_i = L_{i,\mathbb{C}}U_i$, semidirect product, where $L_{i,\mathbb{C}}$ and U_i are the respective complex analytic subgroups of $G_{i,\mathbb{C}}$ for $\mathfrak{l}_{i,\mathbb{C}}$ and $\mathfrak{u}_i$. The direct systems $\{G_{i,\mathbb{C}}, \phi_{k,i}\}$ and $\{\mathfrak{q}_i, d\phi_{k,i}\}$ define direct systems $\{L_{i,\mathbb{C}}, \phi_{k,i}\}$ and $\{U_i, \phi_{k,i}\}$. Let $Q = \varinjlim Q_i$, $L_{\mathbb{C}} = \varinjlim L_{i,\mathbb{C}}$ and $U = \varinjlim U_i$. Then Q, $L_{\mathbb{C}}$ and U are closed complex analytic subgroups of G, and $Q = L_{\mathbb{C}}U$ semidirect product.

We define a direct system $\{Z_i, \phi'_{k,i}\}$ by $Z_i = \{\mathrm{Ad}(g_i)\mathfrak{q}_i \mid g_i \in G_i\}$ and $\phi'_{k,i}(z_i) = z_k$, where $z_i = \mathrm{Ad}(g_i)(\mathfrak{q}_i)$ gives $z_k = \mathrm{Ad}(\phi_{k,i}(g_i))(\mathfrak{q}_k)$. Then $\{Z_i, \phi'_{k,i}\}$ is a strict direct system of complex manifolds and holomorphic maps, so the limit $Z = \varinjlim Z_i$ is a complex manifold and the $\phi'_i : Z_i \to Z$ are holomorphic injections with closed image. The Z_i are complex homogeneous spaces $G_{i,\mathbb{C}}/Q_i$, and it follows that the limit flag manifold Z is a complex homogeneous space $G_{\mathbb{C}}(z_0) = G_{\mathbb{C}}/Q$ where z_0 is the base point in Z, i.e. $\phi'_i(z_{i,0}) = z_0$ for every i. Further, the action $G \times Z \to Z$ is holomorphic.

Let F_i denote the closed orbit $G_i(z_{i,0}) = K_i(z_{i,0})$ in Z_i. Let $S_i = M_i(z_{i,0})$, complex flag manifold $M_{i,\mathbb{C}}/R_i$ where R_i is the parabolic subgroup of $M_{i,\mathbb{C}}$ for the set Ψ_i of simple $(\mathfrak{m}_{i,\mathbb{C}}, \mathfrak{t}_{i,\mathbb{C}})$-roots whose extension to $\mathfrak{h}_{i,\mathbb{C}}$ defines $\mathfrak{q}_i$ as in Section 5. We have $\mathfrak{r}_i = \mathfrak{j}_{i,\mathbb{C}} + \mathfrak{n}_{i,\mathfrak{m}}$, reductive part and nilradical, and $\mathfrak{j}_{i,\mathbb{C}} = \mathfrak{r}_i \cap \mathfrak{j}_{i,\mathbb{C}}$ and $\mathfrak{n}_{\mathfrak{m}} = \mathfrak{r}_i \cap \mathfrak{n}_{i,\mathfrak{m}}$. Thus $R_i = J_{i,\mathbb{C}}N_{i,\mathfrak{m}}$. Up to finite covering, let L_i denote the real form $\phi_i^{-1}(L_{i,\mathbb{C}})$ of $L_{i,\mathbb{C}}$ and let J_i denote the real form $\phi_i^{-1}(J_{i,\mathbb{C}})$ of $J_{i,\mathbb{C}}$. Then $L_i = J_i \times A_i$.

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Now go to the limit: $F = \varinjlim F_i$, $L = \varinjlim L_i$, $S = \varinjlim S_i$, $R = \varinjlim R_i$ and $J_{\mathbb{C}} = \varinjlim J_{i,\mathbb{C}}$. Then $L = J \times A$, $G/LN = G/JAN \cong F \cong K/J$, and $S \cong M/J$. Further, two translates gS and $g'S$ either coincide or are disjoint, and $P = \{g \in G \mid gS = S\}$. Thus we have a fibration exactly as in Proposition 3.1.

PROPOSITION 5.2. *Define $k : F \to G/P = \{gS \mid g \in G\}$ by $k(gz_0) = gS$. Then $k : F \to G/P$ is a well-defined C^ω fiber bundle with structure group P, where the fiber over gS is the complex submanifold gS of Z that is contained in F.*

6. Bundles and sheaves for the limit principal series

Retain the notation of Section 5. In order to construct a coherent family of homogeneous vector bundles $\mathbb{E}_{i,\chi,\nu,\sigma} \to F_i$, we start with a coherent family of representations, as in Proposition 4.6. The proof of Proposition 6.1 below is the same as the proof of Proposition 4.6.

PROPOSITION 6.1. *Let $\nu \in \mathfrak{t}^*_{\mathbb{C}}$ be a $\mathfrak{j}_{\mathbb{C}}$-dominant integral linear functional. It determines an irreducible unitary representation ζ_ν of J^0 as follows. Let $\zeta_{i,\nu}$ denote the irreducible unitary representation of J^0_i with lowest weight $-\nu_i = \phi^*_i(-\nu)$. Choose a unit lowest weight vector $e_{i,-\nu}$ in the representation space $E_{i,\nu}$ of $\zeta_{i,\nu}$. For $i \leqslant k$ define $\psi_{k,i} : E_{i,\nu} \to E_{k,\nu}$ by $\psi_{k,i}(d\zeta_{i,\nu}(\Xi_i)(e_{i,-\nu})) = d\zeta_{k,\nu}(d\phi_{k,i}(\Xi_i))(e_{k,-\nu})$ for Ξ in the enveloping algebra $\mathcal{U}(\mathfrak{j}_i)$.*

(1) *$\{\zeta_{i,\nu}, E_{i,\nu}, \psi_{k,i}\}$ is a compatible system of irreducible representations of $\{J^0_i, \phi_{k,i}\}$, so $\zeta_\nu = \varinjlim \zeta_{i,\nu}$ is a well-defined irreducible unitary representation of J^0, with representation space $E_\nu = \varinjlim V_{i,\nu}$.*

(2) *If $j \in J$ then $\zeta_\nu \circ \mathrm{Ad}(j)^{-1}$ is unitarily equivalent to ζ_ν.*

(3) *Denote $\widehat{J_\nu}$: equivalence classes of irreducible unitary representations ζ of J such that $\zeta|_{J^0}$ weakly contains ζ_ν in the sense that the kernel of $d\zeta$ on the enveloping algebra $\mathcal{U}(\mathfrak{j})$ is contained in the kernel of $d\zeta_\nu$. Then $\widehat{J_\nu} = \{[\zeta] \in \widehat{J} \mid [\zeta|_{J^0}]$ is a multiple of $\zeta_\nu\}$.*

(4) *Let $[\zeta_{\chi,\nu}] \in \widehat{J_\nu}$. Let $E_{\chi,\nu}$ denote its representation space. Choose a subspace $E'_\nu \subset E_{\chi,\nu}$ on which J^0 acts by ζ_ν, let $e \mapsto e'$ denote the intertwining map of E_ν onto E'_ν and let $e'_{-\nu}$ be the image of the lowest weight unit vector $e_{-\nu}$ of ζ_ν. Then the image of $E_{i,\nu}$ in $E_{\chi,\nu}$ is $d\zeta_{i,\nu}(\mathcal{U}(\mathfrak{j}_i))(e'_{-\nu})$, and $E'_\nu = \varinjlim d\zeta_{i,\nu}(\mathcal{U}(\mathfrak{j}_i))(v'_{-\nu}) = d\zeta_\nu(\mathcal{U}(\mathfrak{j}))(e'_{-\nu})$.*

(5) *In the special case where $J = Z_J(J^0)J^0$, the set $\widehat{J_\nu}$ consists of all $[\chi \widehat{\otimes} \zeta_\nu]$ such that $\chi \in (\widehat{Z_J(J^0)})_\xi$ where $\xi = e^{-\nu}|_{Z_{J^0}}$.*

Note. In general, we write the elements of $\widehat{J_\nu}$ as $[\zeta_{\chi,\nu}]$ where χ is just a parameter. In the case of statement (5) the parameter χ is interpreted as an element of $(\widehat{Z_J(J^0)})_\xi$.

Now let $[\zeta_{\chi,\nu}] \in \widehat{J_\nu}$ as in Proposition 6.1, and let $\sigma \in \mathfrak{a}^*_{\mathbb{C}}$. As in Sections 2 and 4 that is equivalent to the datum $\zeta_{\chi,\nu,\sigma} \in \widehat{JAN}$ defined by

$$\zeta_{\chi,\nu,\sigma}(jan) = e^{\sigma}(a)\zeta_{\chi,\nu}(b) \quad \text{for } j \in J,\ a \in A \text{ and } n \in N. \tag{6.2}$$

As in Sections 2 and 4 we write $E_{\chi,\nu,\sigma}$ for the representation space of $\zeta_{\chi,\nu,\sigma}$. Now we have the G-homogeneous vector bundle $\mathbb{E}_{\chi,\nu,\sigma} \to F$ with fiber $E_{\chi,\nu,\sigma}$ at z_0 as in Section 3. If $g \in G$ then $\mathbb{E}_{\chi,\nu,\sigma}|_{gS} \to gS$ is a holomorphic vector bundle.

Note that the limit $E_{\chi,\nu,\sigma} = \varinjlim E_{i,\chi,\nu,\sigma}$ where $E_{i,\chi,\nu,\sigma}$ is the subspace of $E_{\chi,\nu,\sigma}$ generated by $\zeta_{\chi,\nu,\sigma}(J_iA_iN_i)(e'_{-\nu})$. Let $\mathbb{E}_{i,\chi,\nu,\sigma} \to F_i$ denote the associated G_i-homogeneous vector bundle. It is holomorphic over each g_iS_i. The maps

$$\phi_{k,i} \times \psi_{k,i} : G_i \times E_{i,\chi,\nu,\sigma} \to G_k \times E_{k,\chi,\nu,\sigma}$$

induce bundle maps $(\phi_{k,i}, \psi_{k,i}) : \mathbb{E}_{i,\chi,\nu,\sigma} \to \mathbb{E}_{k,\chi,\nu,\sigma}$. These bundle maps form a coherent system and give us

$$\mathbb{E}_{\chi,\nu,\sigma} = \varinjlim \mathbb{E}_{i,\chi,\nu,\sigma}. \tag{6.3}$$

Write $E^*_{\chi,\nu,\sigma}$ for the strong topological dual of $E_{\chi,\nu,\sigma}$ and write $\mathbb{E}^*_{\chi,\nu,\sigma} \to F$ for the associated homogeneous vector bundle. Again, the restricted bundles $\mathbb{E}^*_{\chi,\nu,\sigma}|_{gS} \to gS$ are holomorphic vector bundles, for every fiber gS of $F \to G/P$, by [NRW01, Lemma 2.2]. By elliptic regularity for hyperfunctions, Dolbeault cohomology is the same for C^∞ coefficients as for $C^{-\omega}$ coefficients. Thus, the corresponding sheaf $\mathcal{O}_\mathfrak{n}(E_{\chi,\nu,\sigma}) \to F$ is the sheaf of germs of $E_{\chi,\nu,\sigma}$-valued hyperfunctions h on G such that (i) $h(gjan) = \zeta_{\chi,\nu,\sigma}(jan)^{-1}(h(g))$ for $g \in G$ and $jan \in JAN$ and (ii) $h(g;\xi) + d\zeta_{\chi,\nu,\sigma}(\xi)h(g) = 0$ for $g \in G$ and $\xi \in (\mathfrak{j} + \mathfrak{a} + \mathfrak{n})_\mathbb{C}$ as in Section 3. Of course, we also have the sheaf $\mathcal{O}_\mathfrak{n}(\mathbb{E}^*_{\chi,\nu,\sigma}) \to F$ corresponding to the dual bundle. These are the sheaves of germs of $C^{-\omega}$ sections that are holomorphic over the fibers gS of $F \to G/P$.

For simplicity of notation, we write $\mathcal{O}_\mathfrak{n}(\mathbb{E}_{i,\chi,\nu,\sigma}) \to F_i$, instead of $\mathcal{O}_{\mathfrak{n}_i}(\mathbb{E}_{i,\chi,\nu,\sigma}) \to F_i$, for the sheaf over F_i analogous to $\mathcal{O}_\mathfrak{n}(E_{\chi,\nu,\sigma}) \to F$.

We recall the definition of the inverse limit sheaf $\varprojlim \mathcal{O}_\mathfrak{n}(\mathbb{E}^*_{i,\chi,\nu,\sigma})$. First, identify Z_i with $\phi_i(Z_i) \subset Z$, thus also identifying F_i with $\phi_i(F_i) \subset F$, and view $\mathcal{O}_\mathfrak{n}(\mathbb{E}^*_{i,\chi,\nu,\sigma})$ as a sheaf over F with stalk $\{0\}$ over every point $z \notin F_i$. The open subsets of F_i are the sets $U_i = U \cap F_i$ where U is open in F. Let $\Gamma_i(U)$ denote the abelian group of sections of $\mathcal{O}_\mathfrak{n}(\mathbb{E}^*_{i,\chi,\nu,\sigma})|_{U_i}$. The $\Gamma_i(U)$ form a complete presheaf, corresponding to $\mathbb{E}^*_{i,\chi,\nu,\sigma}$. Also, the abelian group $\Gamma(U)$ of sections of $\mathcal{O}_\mathfrak{n}(\mathbb{E}^*_{\chi,\nu,\sigma})|_U$ is the inverse limit, $\Gamma(U) = \varprojlim \Gamma_i(U)$ corresponding to the inverse system given by restriction of sections and then extension by zero. Also, the $\Gamma(U)$ form a complete presheaf corresponding to $\mathcal{O}_\mathfrak{n}(\mathbb{E}^*_{\chi,\nu,\sigma})$. Thus, by definition,

$$\mathcal{O}_\mathfrak{n}(\mathbb{E}^*_{\chi,\nu,\sigma}) = \varprojlim \mathcal{O}_\mathfrak{n}(\mathbb{E}^*_{i,\chi,\nu,\sigma}). \tag{6.4}$$

Proposition 6.5 (Cf. [NRW01, Proposition 2.4]). *Let $q \geqslant 0$. Then there is a natural G-equivariant isomorphism from the cohomology $H^q(F; \mathcal{O}_\mathfrak{n}(\mathbb{E}^*_{\chi,\nu,\sigma}))$ of the inverse limit onto the inverse limit $\varprojlim H^q(F_i; \mathcal{O}_\mathfrak{n}(\mathbb{E}^*_{i,\chi,\nu,\sigma}))$ of the cohomologies.*

Proof. Apply [Har76, Ch. I, Theorem 4.5] with the global section functor Γ in place of T to see that our sheaf cohomologies are the derived functors of Γ. Our neighborhood bases on F and the F_i are properly aligned, as described in the above description of the definition of the inverse limit sheaf, so that we have a base $\mathcal{B}$ for the topology of F such that each $\mathcal{B}_i = \{U_i = U \cap F_i \mid U \in \mathcal{B}\}$ forms a base for the topology of F_i. We can refine $\mathcal{B}$ so that the neighborhoods $U \in \mathcal{B}$ have the following property. If $U \in \mathcal{B}$ and $g_i \in G_i$ such that $U \cap g_iS_i \neq \emptyset$ then each $U \cap g_iS_i$ is Stein, and U_i is the product of $(U \cap g_iS_i)$ with a cell. Then, for every $U \in \mathcal{B}$:

(a) the inverse system $\{\Gamma_i(U)\}$ is surjective, in other words if $i \leqslant k$ and $s_i \in \Gamma_i(U)$ then there exists $s_k \in \Gamma_k(U)$ such that $s_i = s_k|_{U_i}$; and

(b) if $q > 0$ then $H^q(U_i, \mathcal{O}_\mathfrak{n}(\mathbb{E}^*_{i,\chi,\nu,\sigma})|_{U_i}) = 0$ for all i.

The properties just noted are conditions (a) and (b) of [Har76, Ch. I, Theorem 4.5]. Thus, we have G-equivariant exact sequences

$$0 \to \varprojlim{}^{(1)} H^{q-1}(F_i; \mathcal{O}_\mathfrak{n}(\mathbb{E}^*_{i,\chi,\nu,\sigma})) \to H^q(F; \mathcal{O}_\mathfrak{n}(\mathbb{E}^*_{\chi,\nu,\sigma})) \to \varprojlim H^q(F_i; \mathcal{O}_\mathfrak{n}(\mathbb{E}^*_{i,\chi,\nu,\sigma})) \to 0 \tag{6.6}$$

where $\varprojlim^{(1)}$ denotes the first right derived functor of the $\varprojlim$ functor. The proof now is reduced to the proof that $\varprojlim^{(1)} H^{q-1}(F_i; \mathcal{O}_\mathfrak{n}(\mathbb{E}^*_{i,\chi,\nu,\sigma})) = 0$. Following [Har76, Ch. I, Theorem 4.3] it suffices to check the Mittag–Leffler condition: for each i the filtration of $H^{q-1}(F_i; \mathcal{O}_\mathfrak{n}(\mathbb{E}^*_{i,\chi,\nu,\sigma}))$ by the $H^{q-1}(F_k; \mathcal{O}_\mathfrak{n}(\mathbb{E}^*_{k,\chi,\nu,\sigma}))$ is eventually constant.

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Let $\eta_{i,\chi,\nu}$ denote the representation of M_i on $V_i = H^{q-1}(S_i; \mathcal{O}_{\mathfrak{n}}(\mathbb{E}^*_{i,\chi,\nu,\sigma})|_{S_i})$. If ν_i is $\mathfrak{m}_i$-singular then $V_i = 0$, so $H^{q-1}(F_i; \mathcal{O}_{\mathfrak{n}}(\mathbb{E}^*_{i,\chi,\nu,\sigma})) = 0$ and the Mittag–Leffler condition is trivially satisfied. Now assume that ν_i is $\mathfrak{m}_i$–nonsingular. From Proposition 3.5 (or see [Wol74, Theorem 1.2.19]) the action of G_i on $H^{q-1}(F_i; \mathcal{O}_{\mathfrak{n}}(\mathbb{E}^*_{i,\chi,\nu,\sigma}))$ is a certain principal series representation. Those representations have finite composition series: see [Wol74, Theorem 4.4.4] for the unitary case, and note that its proof suffices for the general case. The point there is that the infinitesimal character and the K_i-restriction are fixed, and that forces finiteness for the composition series. Since each subspace in the filtration $\{H^{q-1}(F_k; \mathcal{O}_{\mathfrak{n}}(\mathbb{E}^*_{k,\chi,\nu,\sigma})) \mid k \geqslant i\}$ of $H^{q-1}(F_i; \mathcal{O}_{\mathfrak{n}}(\mathbb{E}^*_{i,\chi,\nu,\sigma}))$ is an M_i-submodule, there are only finitely many possible composition factors, and the Mittag–Leffler condition is immediate. That completes the proof of Proposition 6.5. □

7. Geometric realization of the limit principal series

In this section we establish the geometric realization of principal series representations of direct limit groups, and look at some of the consequences. In effect we combine Propositions 4.11, 5.2, 6.1 and 6.5, and use ideas of Bott–Borel–Weil theory from [NRW01].

We first look at a limit construction for principal series representations in the geometric style of the limit Borel–Weil theorem, where there is no problem of cohomology degree.

Theorem 7.1. *Let $\nu \in \mathfrak{t}^*_{\mathbb{C}}$ be an $\mathfrak{j}_{\mathbb{C}}$-dominant integral linear functional. Let $\zeta_{\chi,\nu} \in \widehat{J_\nu}$ as in Proposition 6.1. Let $\zeta_{\chi^*,\nu^*} = \zeta^*_{\chi,\nu}$, the dual (contragredient) of $\zeta_{\chi,\nu}$. For each i suppose that $\nu_i = \phi_i^*(\nu)$ is $\mathfrak{m}_{i,\mathbb{C}}$-dominant. Let $\sigma \in \mathfrak{a}^*_{\mathbb{C}}$. Let σ^* denote its complex conjugate, using conjugation of $\mathfrak{a}^*_{\mathbb{C}}$ over $\mathfrak{a}^*$. Then the natural action of G on $H^0(F; \mathcal{O}_{\mathfrak{n}}(\mathbb{E}^*_{\chi,\nu,\sigma}))$ is infinitesimally equivalent to the principal series representation $\pi_{\chi^*,\nu^*,\sigma^*} = \varprojlim \pi_{i,\chi^*,\nu^*,\sigma^*}$ of G, and its dual is infinitesimally equivalent to the principal series representation $\pi_{\chi,\nu,\sigma} = \varinjlim \pi_{i,\chi,\nu,\sigma}$ of G.*

Proof. Apply Proposition 3.5 to each $\mathbb{E}_{i,\chi,\nu,\sigma} \to F_i$. Since ν is dominant, $\nu_i \in (\Lambda^+_{\mathfrak{j}_i})'$, $q(\nu_i) = 0$, and $\widetilde{\nu_i} = \nu_i$. Thus, the natural action of G_i on $H^0(F_i; \mathcal{O}_{\mathfrak{n}}(\mathbb{E}_{i,\chi,\nu,\sigma}))$ is the principal series representation $\pi_{i,\chi,\nu,\sigma}$.

Note $\zeta^*_{\chi,\nu} = \zeta_{\chi^*,\nu^*}$ for some index χ^*, and χ^* is in fact the dual of χ when we are in the situation $J = Z_J(J^0)$ of Proposition 6.1. Also, e^{σ^*} is the dual of e^σ. Thus, the bundles $\mathbb{E}_{\chi,\nu,\sigma}$ and $\mathbb{E}_{\chi^*,\nu^*,\sigma^*}$ are dual, at least at the K-finite level. Now $\pi_{\chi^*,\nu^*,\sigma^*}$ and $\pi_{\chi,\nu,\sigma}$ are dual, so the natural action of G on $H^0(F; \mathcal{O}_{\mathfrak{n}}(\mathbb{E}^*_{\chi,\nu,\sigma}))$ is $\pi_{\chi^*,\nu^*,\sigma^*}$, and the natural action of G on $H^0(F; \mathcal{O}_{\mathfrak{n}}(\mathbb{E}_{\chi,\nu,\sigma}))$ is $\pi_{\chi,\nu,\sigma}$.

Similarly $\pi_{i,\chi,\nu,\sigma}$ and $\pi_{i,\chi^*,\nu^*,\sigma^*}$ are dual, so the natural action of G_i on $H^0(F_i; \mathcal{O}_{\mathfrak{n}}(\mathbb{E}^*_{i,\chi,\nu,\sigma}))$ is $\pi_{i,\chi^*,\nu^*,\sigma^*}$. Now Proposition 6.5 says that $\pi_{\chi^*,\nu^*,\sigma^*} = \varprojlim \pi_{i,\chi^*,\nu^*,\sigma^*}$, and thus also $\pi_{\chi,\nu,\sigma} = \varinjlim \pi_{i,\chi,\nu,\sigma}$. □

In order to extend Theorem 7.1 to higher cohomology we face the same problem as in [NRW01]. We have to find conditions under which the cohomology degrees

$$q_i = q(\nu_i) = |\{\gamma_i \in \Sigma(\mathfrak{m}_{i,\mathbb{C}}, \mathfrak{t}_{i,\mathbb{C}})^+ \mid \langle \nu_i + \rho_{i,\mathfrak{m},\mathfrak{t}}, \gamma_i \rangle < 0\}|$$

remain constant as i increases indefinitely. So we recall some definitions from [NRW01, Section 4].

Suppose that $\nu_i + \rho_{i,\mathfrak{m},\mathfrak{t}}$ is nonsingular. Then there is a unique element w_i in the Weyl group $W(\mathfrak{m}_i, \mathfrak{t}_i)$ that carries $\nu_i + \rho_{i,\mathfrak{m},\mathfrak{t}}$ to a dominant weight, and $q_i = q_i(\nu_i)$ is the length $\ell(w_i)$.

The *Weyl group* $W = W(\mathfrak{m}, \mathfrak{t})$ is defined to be the group of all $w|_{\mathfrak{t}}$ where w is an automorphism of $\mathfrak{m}$ such that (i) $w(\mathfrak{t}) = \mathfrak{t}$ and (ii) for some index i_0 if $i \geqslant i_0$ then $w(d\phi_i(\mathfrak{m}_i)) = d\phi_i(\mathfrak{m}_i)$ and $w|_{d\phi_i(\mathfrak{m}_i)}$ is an inner automorphism of $\mathfrak{m}_i$.

The construction leading to (4.5) amounts to a choice of Borel subalgebra $\mathfrak{b} = \varinjlim \mathfrak{b}_i$ of $\mathfrak{m}$ such that $\mathfrak{t}_i \subset \mathfrak{b}_i \subset \mathfrak{r}_i$ and $d\phi_{k,i}(\mathfrak{b}_i) \subset \mathfrak{b}_k$, where $\mathfrak{r}_i = d\phi_i^{-1}(\mathfrak{r})$. This choice determines the *finite Weyl* *group* $W_F = W_F(\mathfrak{m}, \mathfrak{b}, \mathfrak{t})$ consisting of all $w \in W$ such that $w(\mathfrak{b}) \cap \mathfrak{b}$ has finite codimension in $\mathfrak{b}$. We define this codimension to be the *length* $\ell(w)$.

Let $w \in W(\mathfrak{m}, \mathfrak{t})$. Then we have the classically defined lengths $\ell(w_i)$ relative to the positive root systems $\Sigma(\mathfrak{m}_{i,\mathbb{C}}, \mathfrak{t}_{i,\mathbb{C}})^+$. If $w \in W_F(\mathfrak{m}, \mathfrak{b}, \mathfrak{t})$ then there is an index i_0, which in general depends on w, such that $\ell(w_i) = \ell(w_k)$ for $k \geqslant i \geqslant i_0$, and this common length is $\ell(w)$.

A linear functional $\nu \in \mathfrak{t}_{\mathbb{C}}^*$ is *classically cohomologically finite* if there exist $w \in W_F(\mathfrak{m}, \mathfrak{b}, \mathfrak{t})$ and i_0 as above, and an integral linear functional $\widetilde{\nu} \in \mathfrak{t}_{\mathbb{C}}^*$, with the following property. If $i \geqslant i_0$ then $d\phi_i^*(\widetilde{\nu})$ is dominant relative to the positive root system $\Sigma(\mathfrak{m}_{i,\mathbb{C}}, \mathfrak{t}_{i,\mathbb{C}})^+$, and $d\phi_i^*(\widetilde{\nu}) = w_i(\nu_i + \rho_{i,\mathfrak{m},\mathfrak{t}}) - \rho_{i,\mathfrak{m},\mathfrak{t}}$. A linear functional $\nu \in \mathfrak{t}_{\mathbb{C}}^*$ is *cohomologically finite* of *degree* q_ν if, whenever i is sufficiently large, say $i \geqslant i_0$: (i) $\nu_i + \rho_{i,\mathfrak{m},\mathfrak{t}}$ is nonsingular and (ii) $q_i = q_\nu$ constant in i. If ν is classically cohomologically finite by means of $w \in W_F$ then it is cohomologically finite of degree $\ell(w)$. By contrast, there are cases where ν is cohomologically finite of degree $q > 0$ while $W_F = \{1\}$, so ν is not classically cohomologically finite.

Drawing on [NRW01, Theorem 4.6] we now have a limit construction for principal series representations in the geometric style of the Bott–Borel–Weil theorem, as follows.

Theorem 7.2. *Let $\nu \in \mathfrak{t}_{\mathbb{C}}^*$ be a $\mathfrak{j}_{\mathbb{C}}$-dominant integral linear functional. Let $\sigma \in \mathfrak{a}_{\mathbb{C}}^*$.*

(1) *If ν is not cohomologically finite then every $H^q(F; \mathcal{O}_{\mathfrak{n}}(\mathbb{E}^*_{\chi,\nu,\sigma})) = 0$.*
(2) *Assume that ν is cohomologically finite of degree q_ν. Then:*
   (a) *$H^q(F; \mathcal{O}_{\mathfrak{n}}(\mathbb{E}^*_{\chi,\nu,\sigma})) = 0$ for $q \neq q_\nu$; and*
   (b) *the natural action of G on $H^{q_\nu}(F; \mathcal{O}_{\mathfrak{n}}(\mathbb{E}^*_{\chi,\nu,\sigma}))$ is infinitesimally equivalent to a principal series representation of the form $\pi_{\chi^*,\mu^*,\sigma^*} = \varprojlim \pi_{i,\chi^*,\mu^*,\sigma^*}$, and its dual is infinitesimally equivalent to a principal series representation of the form $\pi_{\chi,\mu,\sigma} = \varinjlim \pi_{i,\chi,\mu,\sigma}$.*
(3) *If further ν is classically cohomologically finite, say by means of $w \in W_F$, then $q_\nu = \ell(w)$ and in assertion (2) we may take $\mu = \widetilde{\nu}$, defined by $\mu_i = w_i(\nu_i + \rho_{i,\mathfrak{m},\mathfrak{t}}) - \rho_{i,\mathfrak{m},\mathfrak{t}}$ for i sufficiently large.*

Proof. Suppose that ν is not cohomologically finite. Fix an integer $p \geqslant 0$. If $\nu_i + \rho_{i,\mathfrak{m},\mathfrak{t}}$ is singular then $H^p(S_i; \mathcal{O}_{\mathfrak{n}}(\mathbb{E}^*_{\chi,\nu,\sigma}|_{S_i})) = 0$. If $\nu_i + \rho_{i,\mathfrak{m},\mathfrak{t}}$ is nonsingular, then $H^p(S_i; \mathcal{O}_{\mathfrak{n}}(\mathbb{E}^*_{\chi,\nu,\sigma}|_{S_i})) = 0$ unless $q(\nu_i) = p$. The $q(\nu_i)$ are increasing in i. Since ν is not cohomologically finite, the $q(\nu_i)$ are unbounded. Thus, $H^p(S_i; \mathcal{O}_{\mathfrak{n}}(\mathbb{E}^*_{\chi,\nu,\sigma}|_{S_i}))$ becomes 0 and stays 0 as i increases. Let η_i^* denote the representation of M_i on $H^p(S_i; \mathcal{O}_{\mathfrak{n}}(\mathbb{E}^*_{\chi,\nu,\sigma}|_{S_i}))$, and let η^* denote the representation of M on $H^p(S; \mathcal{O}_{\mathfrak{n}}(\mathbb{E}^*_{\chi,\nu,\sigma}|_S))$. We have just seen that $H^p(S; \mathcal{O}_{\mathfrak{n}}(\mathbb{E}^*_{\chi,\nu,\sigma}|_S)) = \varprojlim H^p(S_i; \mathcal{O}_{\mathfrak{n}}(\mathbb{E}^*_{\chi,\nu,\sigma}|_{S_i})) = 0$, so the representation space of η^* is 0, and thus the representation space $H^p(F; \mathcal{O}_{\mathfrak{n}}(\mathbb{E}^*_{\chi,\nu,\sigma}))$ of $\operatorname{Ind}_{MAN}^G(\eta \otimes e^{\sigma^*})$ is zero. That proves assertion (1).

Assertion (2a) follows by an argument used for (1), and (2b) and (3) follow by the argument of Theorem 7.1. □

Theorem 7.2 leaves us with two tasks:

(1) find conditions on ν for cohomological finiteness; and
(2) investigate boundedness and unitarity for the limit principal series representations.

The first is studied extensively in [NRW01], and we now turn to the second.

8. Unitarity, L_p boundedness, and related questions

According to Proposition 2.8, the L_p condition for $\pi_{i,\chi,\nu,\sigma}$ is $\sigma_i \in \mathbf{i}\mathfrak{a}_i^* + (2/p)\rho_{i,\mathfrak{a}}$. So the L_∞ condition is transparent: $\sigma_i \in \mathbf{i}\mathfrak{a}_i^*$ for all i if and only if $\sigma \in \mathbf{i}\mathfrak{a}^*$. Now we set that case aside and suppose $1 \leqslant p < \infty$.

LEMMA 8.1. *Let $1 \leqslant p < \infty$. If $k \geqslant i$ view $d\phi_{k,i} : \mathfrak{g}_i \to \mathfrak{g}_k$ as an inclusion $\mathfrak{g}_i \hookrightarrow \mathfrak{g}_k$. Then the $\pi_{i,\chi,\nu,\sigma}$ satisfy the L_p condition for all $i \geqslant i_0$ if and only if (i) $\sigma_{i_0} \in \mathbf{i}\mathfrak{a}_{i_0}^* + \frac{2}{p}\rho_{i_0,\mathfrak{a}}$ and (ii) if $k \geqslant i \geqslant i_0$ then $\rho_{k,\mathfrak{a}}|_{\mathfrak{a}_i} = \rho_{i,\mathfrak{a}}$. In that case $\rho_{\mathfrak{a}} = \varprojlim \rho_{i,\mathfrak{a}} \in \mathfrak{a}^*$ is well defined.*

Proof. The $\pi_{i,\chi,\nu,\sigma}$ satisfy the L_p condition for all $i \geqslant i_0$ if and only if $\sigma_i \in \mathbf{i}\mathfrak{a}_i^* + (2/p)\rho_{i,\mathfrak{a}}$ for $i \geqslant i_0$, in other words $Re\, \sigma_i = \frac{2}{p}\rho_{i,\mathfrak{a}}$. If conditions (i) and (ii) hold, it is obvious that the $\pi_{i,\chi,\nu,\sigma}$ satisfy the L_p condition for all i sufficiently large, say $i \geqslant i_0$. Conversely, suppose that $k \geqslant i \geqslant i_0$ and that $\pi_{\ell,\chi,\nu,\sigma}$ satisfies the L_p condition for $\ell = k, i, i_0$. Then $(2/p)\rho_{i,\mathfrak{a}} = (2/p)\rho_{k,\mathfrak{a}}|_{\mathfrak{a}_i}$, in other words $\rho_{i,\mathfrak{a}} = \rho_{k,\mathfrak{a}}|_{\mathfrak{a}_i}$, in addition to $\sigma_{i_0} \in \mathbf{i}\mathfrak{a}_{i_0}^* + (2/p)\rho_{i_0,\mathfrak{a}}$. The last assertion follows. □

Recall the structure theory for real parabolic subalgebras. Let Ψ_i denote the set of simple roots of $\Sigma(\mathfrak{g}_i, \mathfrak{a}_i)^+$. The G_i^0-conjugacy classes of (real) parabolic subalgebras of $\mathfrak{g}_i$ are in one-to-one correspondence $\Phi_i \leftrightarrow \mathfrak{p}_{i,\Phi}$ with the subsets $\Phi_i \subset \Psi_i$ by

$$\mathfrak{p}_{i,\Phi} = \mathfrak{m}_{i,\Phi} + \mathfrak{a}_{i,\Phi} + \mathfrak{n}_{i,\Phi} \tag{8.2}$$

where

$$\begin{aligned} &\mathfrak{a}_{i,\Phi} = \{\xi \in \mathfrak{a}_i \mid \psi_i(\xi) = 0 \text{ for all } \psi_i \in \Phi_i\}, \\ &\mathfrak{m}_{i,\Phi} = \theta(\mathfrak{m}_{i,\Phi}) \text{ and } \mathfrak{m}_{i,\Phi} \oplus \mathfrak{a}_{i,\Phi} \text{ is the centralizer of } \mathfrak{a}_{i,\Phi} \text{ in } \mathfrak{g}_i, \\ &\mathfrak{n}_{i,\Phi} \text{ is the sum of the negative } \mathfrak{a}_i\text{-root spaces not in } \mathfrak{m}_{i,\Phi}. \end{aligned} \tag{8.3}$$

Here $\mathfrak{n}_{i,\Phi}$ is the nilradical, $\mathfrak{m}_{i,\Phi} \oplus \mathfrak{a}_{i,\Phi}$ is the Levi component (reductive part) and $\mathfrak{p}_{i,\Phi}$ is the normalizer of $\mathfrak{n}_{i,\Phi}$ in $\mathfrak{g}_i$. The subset Φ_i is the simple root system for $\mathfrak{m}_{i,\Phi} \oplus \mathfrak{a}_{i,\Phi}$. The minimal parabolic is the case $\Phi_i = \emptyset$. The derived algebra $\mathfrak{m}'_{i,\Phi} = [\mathfrak{m}_{i,\Phi}, \mathfrak{m}_{i,\Phi}]$ is a maximal semisimple subalgebra of $\mathfrak{p}_{i,\Phi}$, and we refer to it as the *semisimple component* of $\mathfrak{p}_{i,\Phi}$.

If $\gamma \in \Sigma(\mathfrak{g}_i, \mathfrak{a}_i)$ we write $\mathrm{mult}(\gamma)$ for the multiplicity of γ as an $\mathfrak{a}_i$-root, in other words for the dimension $\dim \mathfrak{g}_i^\gamma$ of the root space. Thus $\rho_{i,\mathfrak{g},\mathfrak{a}} = \sum_{\gamma \in \Sigma(\mathfrak{g}_i,\mathfrak{a}_i)^+} \mathrm{mult}(\gamma)\gamma$.

The following lemma is standard in the context on non-restricted roots, but we have not been able to find it in the literature, so we give a proof for the convenience of the reader.

LEMMA 8.4. *If $\psi \in \Psi_i$ then $2\langle \rho_{i,\mathfrak{g},\mathfrak{a}}, \psi\rangle / \langle \psi, \psi\rangle = \mathrm{mult}(\psi) + 2\,\mathrm{mult}(2\psi)$.*

Proof. Let w_ψ denote the Weyl group reflection for the simple restricted root ψ. Then $w_\psi \Sigma(\mathfrak{g}_i, \mathfrak{a}_i)^+ = \Sigma(\mathfrak{g}_i, \mathfrak{a}_i)^+ \setminus S(\psi)$ where $S(\psi)$ is $\{\psi\}$ if 2ψ is not a restricted root, $\{\psi, 2\psi\}$ if 2ψ is a restricted root. Now compute

$$\begin{aligned} 2(\rho_{i,\mathfrak{g},\mathfrak{a}} - \mathrm{mult}(\psi)\psi - \mathrm{mult}(2\psi)2\psi) &= 2w_\psi(\rho_{i,\mathfrak{g},\mathfrak{a}}) \\ &= \sum_{\gamma \in \Sigma(\mathfrak{g}_i,\mathfrak{a}_i)^+} w_\psi(\gamma) \\ &= \sum_{\gamma \in \Sigma(\mathfrak{g}_i,\mathfrak{a}_i)^+} \left(\gamma - \frac{2\langle \gamma, \psi\rangle}{\langle \psi, \psi\rangle}\psi\right) \\ &= 2\rho_{i,\mathfrak{g},\mathfrak{a}} - \frac{2\langle 2\rho_{i,\mathfrak{g},\mathfrak{a}}, \psi\rangle}{\langle \psi, \psi\rangle}. \end{aligned}$$

Thus $\mathrm{mult}(\psi) + 2\,\mathrm{mult}(2\psi) = 2\langle \rho_{i,\mathfrak{g},\mathfrak{a}}, \psi\rangle / \langle \psi, \psi\rangle$, as asserted. □

Now we are ready to look at condition (ii) of Lemma 8.1.

PROPOSITION 8.5. *Let $\mathfrak{g}_i \subset \mathfrak{g}_k$, real semisimple Lie algebras. Choose a Cartan involution θ of $\mathfrak{g}_k$ that preserves $\mathfrak{g}_i$, let $\mathfrak{a}_i$ be a maximal abelian subspace of $\{\xi \in \mathfrak{g}_i \mid \theta(\xi) = -\xi\}$, and enlarge $\mathfrak{a}_i$ to a maximal abelian subspace $\mathfrak{a}_k$ of $\{\xi \in \mathfrak{g}_k \mid \theta(\xi) = -\xi\}$. Suppose that $\mathfrak{a}_k = \mathfrak{a}_i \oplus \mathfrak{a}_{k,i}$ where $\mathfrak{a}_{k,i}$ centralizes $\mathfrak{g}_i$, in other words that $\mathfrak{g}_i \oplus \mathfrak{a}_{k,i}$ is a subalgebra of $\mathfrak{g}_k$. Then following conditions are equivalent.*

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(1) *The restriction* $\rho_{k,\mathfrak{g},\mathfrak{a}}|_{\mathfrak{a}_i} = \rho_{i,\mathfrak{g},\mathfrak{a}}$.

(2) $(\mathfrak{g}_i + \mathfrak{m}_k) \oplus \mathfrak{a}_{k,i}$ *is the centralizer of* $\mathfrak{a}_{k,i}$ *in* $\mathfrak{g}_k$.

(3) *Modulo* $\mathfrak{m}_k$, *the algebra* $\mathfrak{g}_i$ *is the semisimple component of a real parabolic subalgebra of* $\mathfrak{g}_k$ *that contains* $\mathfrak{a}_k$.

Proof. Assume condition (3). Then there is a subset $\Phi \subset \Psi_k$ such that, modulo $\mathfrak{m}_k$, $\mathfrak{g}_i$ is the semisimple component $\mathfrak{s}$ of $\mathfrak{p}_{k,\Phi}$. In particular Φ is the simple root system for $\Sigma(\mathfrak{g}_i \oplus \mathfrak{a}_{k,i}, \mathfrak{a}_k)^+$, so $\Sigma(\mathfrak{g}_i \oplus \mathfrak{a}_{k,i}, \mathfrak{a}_k) = \Sigma(\mathfrak{s} \oplus \mathfrak{a}_{k,i}, \mathfrak{a}_k)$, and the multiplicities $\mathrm{mult}_{\mathfrak{g}_i}(\gamma) = \mathrm{mult}_{\mathfrak{s}}(\gamma)$ for every root $\gamma \in \Sigma(\mathfrak{s} \oplus \mathfrak{a}_{k,i}, \mathfrak{a}_k)$. Thus $\rho_{i,\mathfrak{g},\mathfrak{a}} = \rho_{\mathfrak{s},\mathfrak{a}_i}$. However, Lemma 8.4 shows that $\langle \rho_{k,\mathfrak{g},\mathfrak{a}}, \varphi \rangle = \langle \rho_{\mathfrak{s}\oplus\mathfrak{a}_{k,i},\mathfrak{a}}, \varphi \rangle$ for every $\varphi \in \Phi$, so $\rho_{k,\mathfrak{g},\mathfrak{a}}|_{\mathfrak{a}_i} = \rho_{\mathfrak{s},\mathfrak{a}_i}$. That proves condition (1).

Assume condition (1). Denote $\mathfrak{r} = (\mathfrak{g}_i + \mathfrak{m}_k) \oplus \mathfrak{a}_{k,i}$. We have not yet proved that $\mathfrak{r}$ is an algebra, but we do have $\rho_{\mathfrak{r},\mathfrak{a}_k} = \frac{1}{2}\sum_{\gamma\in\Sigma(\mathfrak{r},\mathfrak{a}_k)} \dim(\mathfrak{r} \cap \mathfrak{g}_k^{\gamma})\gamma$, and $\rho_{\mathfrak{r},\mathfrak{a}_k} = \rho_{\mathfrak{g}_i\oplus\mathfrak{a}_{k,i},\mathfrak{a}_k}$ by definition of $\mathfrak{r}$.

Let $\mathfrak{z}$ denote the centralizer of $\mathfrak{a}_{k,i}$ in $\mathfrak{g}_k$. Then $\rho_{k,\mathfrak{g},\mathfrak{a}}|_{\mathfrak{a}_i} = \rho_{\mathfrak{z},\mathfrak{a}_k}|_{\mathfrak{a}_i}$. Using assumption (1) now $\rho_{\mathfrak{r},\mathfrak{a}_k} = \rho_{\mathfrak{z},\mathfrak{a}_k}$. By construction of $\mathfrak{r}$ and of $\mathfrak{z}$, if $\gamma \in \Sigma(\mathfrak{r}, \mathfrak{a}_k)^+$ then $\gamma \in \Sigma(\mathfrak{z}, \mathfrak{a}_k)^+$ and its multiplicities satisfy $\mathrm{mult}_{\mathfrak{r}}(\gamma) \leqslant \mathrm{mult}_{\mathfrak{z}}(\gamma)$. As $\rho_{\mathfrak{r},\mathfrak{a}_k} = \rho_{\mathfrak{z},\mathfrak{a}_k}$ now $\sum_{\gamma\in\Sigma(\mathfrak{z},\mathfrak{a}_k)^+}[\mathrm{mult}_{\mathfrak{z}}(\gamma) - \mathrm{mult}_{\mathfrak{r}}(\gamma)]\gamma = 0$. Take inner product with $\rho_{\mathfrak{z},\mathfrak{a}_k}|_{\mathfrak{a}_i}$. Since each $\langle \rho_{\mathfrak{z},\mathfrak{a}_k}|_{\mathfrak{a}_i}, \gamma \rangle > 0$ and each $\mathrm{mult}_{\mathfrak{z}}(\gamma) \geqslant \mathrm{mult}_{\mathfrak{r}}(\gamma)$ it follows that $\mathrm{mult}_{\mathfrak{z}}(\gamma) = \mathrm{mult}_{\mathfrak{r}}(\gamma)$. That proves $\mathfrak{r} = \mathfrak{z}$, which is the assertion of condition (2).

Assume condition (2). Then $\mathfrak{z} = (\mathfrak{g}_i + \mathfrak{m}_k) \oplus \mathfrak{a}_{k,i}$ is the reductive component of a parabolic subalgebra of $\mathfrak{g}_k$ and the corresponding semisimple component is $[\mathfrak{z}, \mathfrak{z}] = [\mathfrak{g}_i + \mathfrak{m}_k, \mathfrak{g}_i + \mathfrak{m}_k]$. If $\gamma \in \Sigma(\mathfrak{g}_i \oplus \mathfrak{a}_{k,i}, \mathfrak{a}_k)$ then $(\mathfrak{g}_i \oplus \mathfrak{a}_{k,i})^{\gamma} = \mathfrak{g}_k^{\gamma}$, so $[\mathfrak{m}_k, \mathfrak{g}_i] \subset \mathfrak{g}_i$. Now $[\mathfrak{z}, \mathfrak{z}] = \mathfrak{g}_i + [\mathfrak{m}_k, \mathfrak{m}_k]$. That proves condition (3), completing the proof of the Proposition. ☐

Corollary 8.6. *Let* $\mathfrak{g}_i$ *be the semisimple component of a real parabolic subalgebra of* $\mathfrak{g}_k$ *that contains* $\mathfrak{a}_k$. *Then the restriction* $\rho_{k,\mathfrak{g},\mathfrak{a}}|_{\mathfrak{a}_i} = \rho_{i,\mathfrak{g},\mathfrak{a}}$.

Definition 8.7. The strict direct system $\{G_i, \phi_{k,i}\}$ of reductive Lie groups is *weakly parabolic* if for every pair $k \geqslant i$: (1) $d\phi_{k,i}$ sends the center of $\mathfrak{g}_i$ into the center of $\mathfrak{g}_k$; and (2) the subalgebra $d\phi_{k,i}(\mathfrak{g}_i') \hookrightarrow \mathfrak{g}_k'$ satisfies the conditions of Proposition 8.5 where $\mathfrak{g}_\gamma'$ denotes the derived algebra $[\mathfrak{g}_\gamma', \mathfrak{g}_\gamma']$. It is *parabolic* if for every pair $k \geqslant i$ the subalgebra $d\phi_{k,i}(\mathfrak{g}_i')$ is the semisimple component of a real parabolic subalgebra of $\mathfrak{g}_k$. Note that condition (1) is vacuous if the G_i are semisimple.

Remark 8.8. The condition that $\{G_i, \phi_{k,i}\}$ be weakly parabolic is slightly less restrictive than the corresponding condition (7.1) of 'coherent root orderings' in [NRW01]. The context and applications are different, but the core idea is similar.

Now we come to the main result of this section.

Theorem 8.9. *Suppose that the direct system* $\{G_i, \phi_{k,i}\}$ *is weakly parabolic. Let* $\nu \in \mathfrak{t}_{\mathbb{C}}^*$ *be a* $\mathfrak{j}_{\mathbb{C}}$*-dominant integral linear functional that is cohomologically finite of degree* q_ν. *Let* $\sigma \in \mathfrak{a}_{\mathbb{C}}^*$. *Recall the principal series representation* $\pi_{\chi^*,\nu^*,\sigma^*} = \varprojlim \pi_{i,\chi^*,\nu^*,\sigma^*}$ *of* G *on* $H^{q_\nu}(F; \mathcal{O}_{\mathfrak{n}}(\mathbb{E}^*_{\chi,\nu,\sigma}))$ *and its dual* $\pi_{\chi,\nu,\sigma} = \varinjlim \pi_{i,\chi,\nu,\sigma}$. *Let* $1 \leqslant p \leqslant \infty$. *Suppose that* $\sigma \in \mathbf{i}\mathfrak{a}^* + (2/p)\rho$, *or equivalently that there is an index* i_0 *such that* $\sigma_i \in \mathbf{i}\mathfrak{a}_i^* + (2/p)\rho_{i,\mathfrak{a}}$ *for all* $i \geqslant i_0$. *Then* $\pi_{\chi,\mu,\sigma}$ *is infinitesimally equivalent to a Banach space representation on* $\varinjlim L_p(G_i, P_i : V_{i,\chi,\mu,\sigma})$. *In particular, if* $\sigma_i \in \mathbf{i}\mathfrak{a}_i^* + \rho_{i,\mathfrak{a}}$ *for* $i \geqslant i_0$ *then* $\pi_{\chi,\mu,\sigma}$ *is infinitesimally equivalent to a unitary representation on* $\varinjlim L_2(G_i, P_i : V_{i,\chi,\mu,\sigma})$.

Proof. Combine Theorem 7.2 with Lemma 8.1, Proposition 8.5 and Definition 8.7. ☐

In Theorem 8.9 it would be better to derive the L_p norm directly from the limit bundle $\mathbb{E}_{\chi,\mu,\sigma} \to F$. We do this by using a partially holomorphic cohomology space, as in [Wol74]. The fibers gS_i of $F_i \to G_i/P_i$ are compact, so any cohomology class $c_{gS_i} \in H^q(gS_i, \mathcal{O}_{\mathfrak{n}}(\mathbb{E}_{\chi,\mu,\sigma}|_{gS_i}))$ is represented by a harmonic $\mathbb{E}_{\chi,\mu,\sigma}|_{gS_i}$-valued $(0,q)$-form ω_{gS_i}, and ω_{gS_i} has a well-defined L_p

norm $\|\omega_{gS_i}\|_p = (\int_{M_i} \|\omega_{gS_i}(m)\|^p\, dm)^{1/p}$. Thus we have the Banach space $\mathcal{B}^q_{i,p}(F_i; \mathbb{E}_{i,\chi,\mu,\sigma})$ consisting of all measurable $\mathbb{E}_{i,\chi,\mu,\sigma}$-valued $(0,q)$-forms ω on F_i such that $\omega|_{gS_i}$ is harmonic in the sense of Hodge and Kodaira, $\|\omega|_{gS_i}\|_p$ is a measurable function of $gS_i \in G_i/P_i = K_i/M_i$, and $\int_{K_i/M_i}(\|\omega_{gS_i}\|_p)^p\, dk < \infty$ where

$$\|\omega\|_p = \left(\int_{K_i/M_i} (\|\omega_{kS_i}\|_p)^p\, dk\right)^{1/p} = \left(\int_{K_i/J_i} (\|\omega(kJ_i)\|)^p\, dk\right)^{1/p}.$$

For $p = 2$ the norm is given by the inner product

$$\langle \omega, \omega' \rangle = \int_{K_i/M_i} \left(\int_{M_i/J_i} \omega(kmz_0) \mathbin{\bar{\wedge}} \#\omega(kmz_0)\, dm\right) dk. \tag{8.10}$$

There $\#\omega$ is the $\mathbb{E}^*_{i,\chi,\mu,\sigma}$-valued $(s, s-q)$-form, $s = \dim S_i$, which along kS_i is the Hodge–Kodaira orthogonal of ω, and $\bar{\wedge}$ is exterior product followed by the pairing of $E_{i,\chi,\mu,\sigma}$ with $E^*_{i,\chi,\mu,\sigma}$. That gives us a Hilbert space

$$\mathcal{H}^q_{i,2}(F_i; \mathbb{E}_{i,\chi,\mu,\sigma}) = (\mathcal{B}^q_{i,2}(F_i; \mathbb{E}_{i,\chi,\mu,\sigma}), \langle\cdot,\cdot\rangle). \tag{8.11}$$

Now, if we stay with a cofinal weakly parabolic subsystem of $\{G_i, \phi_{k,i}\}$ as in Theorem 8.9, we have Banach space representations $\pi_{\chi^*,\mu^*,\sigma^*}$ on $\mathcal{B}^q_p(F; \mathbb{E}_{\chi^*,\mu^*,\sigma^*}) = \varprojlim \mathcal{B}^q_{i,p}(F_i; \mathbb{E}_{i,\chi^*,\mu^*,\sigma^*})$ and $\pi_{\chi,\nu,\sigma}$ on $\mathcal{B}^q_{p'}(F; \mathbb{E}_{\chi,\mu,\sigma}) = \varinjlim \mathcal{B}^q_{i,p'}(F_i; \mathbb{E}_{i,\chi,\mu,\sigma})$ where $1/p + 1/p' = 1$ for $1 < p < \infty$. In the case $p = 2$ we have unitary representations $\pi_{\chi^*,\mu^*,\sigma^*}$ on $\mathcal{H}^q_2(F; \mathbb{E}_{\chi^*,\mu^*,\sigma^*}) = \varprojlim \mathcal{H}^q_{i,2}(F_i; \mathbb{E}_{i,\chi^*,\mu^*,\sigma^*})$ and $\pi_{\chi,\nu,\sigma}$ on $\mathcal{H}^q_2(F; \mathbb{E}_{\chi,\mu,\sigma}) = \varinjlim \mathcal{H}^q_{i,2}(F_i; \mathbb{E}_{i,\chi,\mu,\sigma})$. Here note that $\varinjlim$ and $\varprojlim$ are the same in the Hilbert space category.

9. Diagonal embedding direct limits

In this section we study an important class of direct limit groups that includes those obtained from weakly parabolic direct systems. These diagonal embedding direct limits were introduced on the complex Lie algebra level (see, for example, [BS95a, BS95b, Bar98, BZ99, YZ96, Zhd96]). This topic now plays a central role in the theory of locally finite Lie algebras. The idea was somewhat extended and applied on both the compact and the complex group level in [NRW01, Section 5], and that is our starting point.

Linear groups

We consider limits of real, complex and quaternionic special linear groups. Fix sequences $\mathbf{r} = \{r_n\}_{n\geqslant 1}$, $\mathbf{s} = \{s_n\}_{n\geqslant 1}$ and $\mathbf{t} = \{t_n\}_{n\geqslant 1}$ of nonnegative integers with all $r_n + s_n > 0$. Start with $d_0 > 0$ and recursively define $d_{n+1} = d_n(r_{n+1} + s_{n+1}) + t_{n+1}$. Let $\mathbb{F}$ be one of $\mathbb{R}$ (real), $\mathbb{C}$ (complex) or $\mathbb{H}$ (quaternions) and define $G_n = SL(d_n; \mathbb{F})$. Let δ denote the outer automorphism of G_n given by

$$\delta(g) = J(g^t)^{-1}J^{-1} \quad \text{where } J = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}. \tag{9.1}$$

(The point of J here is that δ, as defined, preserves the standard positive root system.) Then we have strict direct systems $\{G_m, \phi_{n,m}\}_{n\geqslant m\geqslant 0}$ given by

$$\phi_{n+1,n} : G_n \to G_{n+1} \text{ by } \phi_{n+1,n}(g) = \mathrm{diag}\{g, \dots, g; \delta(g), \dots, \delta(g); 1, \dots, 1\} \tag{9.2}$$

with r_{n+1} blocks g, with s_{n+1} blocks $\delta(g)$, and with t_{n+1} entries 1. The given $\mathbf{r}, \mathbf{s}$ and $\mathbf{t}$ define

$$G = SL_{\mathbf{r},\mathbf{s},\mathbf{t}}(\infty; \mathbb{F}) = \varinjlim\{G_m, \phi_{n,m}\}. \tag{9.3}$$

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Thus we have $SL_{\mathbf{r},\mathbf{s},\mathbf{t}}(\infty;\mathbb{R})$, $SL_{\mathbf{r},\mathbf{s},\mathbf{t}}(\infty;\mathbb{C})$ and $SL_{\mathbf{r},\mathbf{s},\mathbf{t}}(\infty;\mathbb{H})$. Of course, the situation is exactly the same to construct infinite general linear groups $GL_{\mathbf{r},\mathbf{s},\mathbf{t}}(\infty;\mathbb{R})$ and $GL_{\mathbf{r},\mathbf{s},\mathbf{t}}(\infty;\mathbb{C})$.

Unitary groups

We consider limits of real, complex and quaternionic unitary groups. Here $SU(p,q;\mathbb{R})$ denotes the special orthogonal group $SO(p,q)$ for a nondegenerate bilinear form of signature (p,q), $SU(p,q;\mathbb{C})$ denotes the usual complex special unitary $SU(p,q)$ for a nondegenerate hermitian form of signature (p,q) and $SU(p,q;\mathbb{H})$ is the quaternionic special unitary group for a nondegenerate hermitian form signature (p,q). In each case we write the form as $b(z,w) = \sum_{1\leqslant 1\leqslant p} \overline{w_i} z_i - \sum_{1\leqslant 1\leqslant q} \overline{w_{p+i}} z_{p+i}$, reflecting the fact that we view $\mathbb{F}^{p+q}$ as a right vector space over $\mathbb{F}$ so that linear transformations act on the left.

Fix sequences $\mathbf{r} = \{r_n\}_{n\geqslant 1}$, $\mathbf{s} = \{s_n\}_{n\geqslant 1}$, plus two new sequences $\mathbf{t}' = \{t'_n\}_{n\geqslant 1}$ and $\mathbf{t}'' = \{t''_n\}_{n\geqslant 1}$, all of nonnegative integers with each $r_n + s_n > 0$, and $d''_{n+1} = d''_n(r_{n+1}+s_{n+1}) + t''_{n+1}$ and denote $d_{n+1} = d'_{n+1} + d''_{n+1}$. Let G_n be the real special unitary group $SU(d'_n, d''_n;\mathbb{F})$ over $\mathbb{F}$. If $\mathbb{F} = \mathbb{H}$, or if $\mathbb{F} = \mathbb{R}$ and d_n is odd, then $G_{n,\mathbb{C}}$ has no outer automorphism, and we denote $\delta = 1 \in \mathrm{Aut}(G_n)$. Otherwise (except when $\mathbb{F} = \mathbb{R}$ and $d_n = 8$) $G_{n,\mathbb{C}}$ has outer automorphism group generated modulo inner automorphisms by $\delta_0 = \mathrm{Ad}\left(\begin{smallmatrix} -1 & 0 \\ 0 & +I \end{smallmatrix}\right)$ if $\mathbb{F} = \mathbb{R}$, by $\delta_0 : g \mapsto {}^t g^{-1}$ if $\mathbb{F} = \mathbb{C}$, and we choose $\delta \in \delta_0 \,\mathrm{Int}(G_n)$ that preserves the standard positive root system. Then we have $\phi_{n+1,n} : SU(d'_n, d''_n;\mathbb{F}) \to SU(d'_{n+1}, d''_{n+1};\mathbb{F})$ given by

$$\phi_{n+1,n}(g) = \mathrm{diag}\{1,\dots,1; g,\dots,g; \delta(g),\dots,\delta(g); 1,\dots,1\} \tag{9.4}$$

with t'_{n+1} entries 1, r_{n+1} blocks g, s_{n+1} blocks $\delta(g)$, and finally t''_{n+1} entries 1, where all $s_n = 0$ in the case $\mathbb{F} = \mathbb{H}$. Now (9.4) defines a strict direct system $\{G_m, \phi_{n,m}\}$. Let $d' = \lim d'_n$ and $d'' = \lim d''_n$; both usually are ∞ but of course it can happen that one is finite, even zero. In any case the given $\mathbf{t}', \mathbf{r}, \mathbf{s}$ and $\mathbf{t}''$ define

$$G = SU_{\mathbf{t}',\mathbf{r},\mathbf{s},\mathbf{t}''}(d', d'';\mathbb{F}) = \varinjlim\{G_m, \phi_{n,m}\}. \tag{9.5}$$

Thus, we have the groups $SO_{\mathbf{t}',\mathbf{r},\mathbf{s},\mathbf{t}''}(d', d'')$, $SU_{\mathbf{t}',\mathbf{r},\mathbf{s},\mathbf{t}''}(d', d'')$ and $Sp_{\mathbf{t}',\mathbf{r},\mathbf{s},\mathbf{t}''}(d', d'')$.

The same process gives us real limit orthogonal groups $O_{\mathbf{t}',\mathbf{r},\mathbf{s},\mathbf{t}''}(d', d'')$ and the complex limit unitary groups $U_{\mathbf{t}',\mathbf{r},\mathbf{s},\mathbf{t}''}(d', d'')$. In the $O_{\mathbf{t}',\mathbf{r},\mathbf{s},\mathbf{t}''}(d', d'')$ case, Definition 2.1 requires that each d_n should be odd.

Symplectic groups

We consider limits of real and complex symplectic groups. Fix sequences $\mathbf{r} = \{r_n\}_{n\geqslant 1}$ and $\mathbf{t} = \{t_n\}_{n\geqslant 1}$ with all $r_n > 0$. Start with $d_0 > 0$ and recursively define $d_{n+1} = d_n r_{n+1} + t_{n+1}$. Our convention is that $Sp(n;\mathbb{F})$ is the automorphism group of $\mathbb{F}^{2n}$ with a nondegenerate antisymmetric bilinear form; that forces $\mathbb{F}$ to be $\mathbb{R}$ or $\mathbb{C}$. Let $G_n = Sp(d_n;\mathbb{F})$, either the real symplectic group $Sp(d_n;\mathbb{R})$ or the complex symplectic group $Sp(n;\mathbb{C})$. Then we have strict direct systems $\{G_m, \phi_{n,m}\}_{m\geqslant n\geqslant 0}$ with

$$\phi_{n+1,n} : G_n \to G_{n+1} \text{ by } \phi_{n+1,n}(g) = \mathrm{diag}\{g,\dots,g; 1,\dots,1\} \tag{9.6}$$

with r_{n+1} blocks g and with $2t_{n+1}$ entries 1. Thus the given $\mathbf{r}$ and $\mathbf{t}$ define

$$G = Sp_{\mathbf{r},2\mathbf{t}}(\infty;\mathbb{F}) = \varinjlim\{G_m, \phi_{n,m}\}. \tag{9.7}$$

Complex orthogonal groups

Now consider the complex special orthogonal groups $G_n = SO(d_n;\mathbb{C})$. The formula (9.3) defines maps $\phi_{n+1,n} : SO(d_n;\mathbb{C}) \to SO(d_{n+1};\mathbb{C})$ so it defines a strict direct system $\{G_m, \phi_{n,m}\}$. Now, for the given $\mathbf{r}, \mathbf{s},$ and $\mathbf{t}$, we have

$$G = SO_{\mathbf{r},\mathbf{s},\mathbf{t}}(\infty;\mathbb{C}) = \varinjlim\{G_m, \phi_{n,m}\}. \tag{9.8}$$

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The same process gives us complex limit orthogonal groups $O_{\mathbf{r},\mathbf{s},\mathbf{t}}(\infty;\mathbb{C})$; as before, here Definition 2.1 requires that each d_n be odd.

The remaining classical series

There is one other series of real classical groups, the groups $SO^*(2n)$, real form of $SO(2n;\mathbb{C})$ with maximal compact subgroup $U(n)$. The usual definition is

$$SO^*(2n) = \left\{ g \in U(n,n) \;\middle|\; g \text{ preserves } (x,y) = \sum_{1}^{n}(x_a y_{n+a} + x_{n+a} y_a) \text{ on } \mathbb{C}^{2n} \right\}.$$

It will be more convenient for us to use the alternate formulation of [Wol78, Section 8], which is

$$SO^*(2n) = \{g \in SL(n;\mathbb{H}) \mid b(gx,gy) = b(x,y) \text{ for all } x,y \in \mathbb{H}^n\} \tag{9.9}$$

where b is the skew-hermitian form on $\mathbb{H}^n$ given by $b(x,y) = \sum_{a=1}^{n} \overline{x}_a i y_a$. For then (9.1) defines an outer automorphism δ of each $SO^*(2n)$. Now fix sequences $\mathbf{r} = \{r_n\}_{n\geqslant 1}$, $\mathbf{s} = \{s_n\}_{n\geqslant 1}$ and $\mathbf{t} = \{t_n\}_{n\geqslant 1}$ of nonnegative integers with $r_0 > 0$ and all $r_n + s_n > 0$. Start with $d_0 > 0$ and recursively define $d_{n+1} = d_n(r_{n+1} + s_{n+1}) + t_{n+1}$. Define $G_n = SO^*(2d_n)$. Then we have strict direct systems $\{G_m, \phi_{n,m}\}_{n\geqslant m\geqslant 0}$ given by

$$\phi_{n+1,n} : G_n \to G_{n+1} \text{ by } \phi_{n+1,n}(g) = \mathrm{diag}\{g,\ldots,g;\delta(g),\ldots,\delta(g);1,\ldots,1\} \tag{9.10}$$

with r_{n+1} blocks g, with s_{n+1} blocks $\delta(g)$, and with t_{n+1} entries 1. Fr the given $\mathbf{r},\mathbf{s}$ and $\mathbf{t}$ we have

$$G = SO^*_{\mathbf{r},\mathbf{s},\mathbf{t}}(\infty) = \varinjlim\{G_m, \phi_{n,m}\}. \tag{9.11}$$

We refer to the direct limit groups (9.3), (9.5), (9.7), (9.8) and (9.11) as *diagonal embedding direct limit groups* and to the associated direct systems as *diagonal embedding direct systems*. Note that the groups G_n in the corresponding direct systems all are semisimple. In the unitary symplectic case of (9.5) we made the convention that we have the sequence $\mathbf{s}$ but each $s_n = 0$. We say that a diagonal embedding direct limit group and the associated diagonal embedding direct system are of *classical type* if $r_n + s_n = 1$ for all n sufficiently large.

Now we collect some basic properties of diagonal embedding direct limit groups.

Proposition 9.12. *Let $G = \varinjlim\{G_m, \phi_{n,m}\}_{n\geqslant m\geqslant 0}$ be a diagonal embedding direct limit group. Then the conditions of Definition 2.1 hold, and if the G_n are not (special) unitary groups over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ then the hypothesis of Lemma 4.3 holds, so G has principal series representations. If the G_n are (special) unitary groups over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, then the hypothesis of Lemma 4.3 holds for a cofinal subsystem (which, of course, yields the same limit group G) of $\{G_m, \phi_{n,m}\}$. In any case, if $\{G_m, \phi_{n,m}\}$ is weakly parabolic then it is of classical type.*

Proof. The conditions of Definition 2.1 are clear because the G_n are semisimple Lie groups, connected except possibly for the case of orthogonal groups where the second is obvious and we have explicitly ensured the first. Now we look at the hypothesis of Lemma 4.3.

We first consider the special linear groups $G_n = SL(d_n;\mathbb{F})$. Fix a basis $\mathcal{B}$ of $\mathbb{F}^{d_n}$. Relative to $\mathcal{B}$, A_n will consist of the diagonal real matrices in G_n that have all entries > 0, M_n will consist of the diagonal matrices in G_n that have all entries of absolute value 1, and N_n will consist of all lower triangular matrices in G_n that have all diagonal entries $= 1$. It is immediate that $\phi_{n+1,n}$ maps A_n into A_{n+1}, maps M_n into M_{n+1} and maps N_n into N_{n+1}. That is the hypothesis of Lemma 4.3.

Now consider the symplectic groups $G_n = Sp(d_n;\mathbb{F})$. The standard basis $\{e_i\}$ of $\mathbb{F}^{2d_n}$, in which the antisymmetric bilinear form b_n that defines G_n has matrix $\left(\begin{smallmatrix} O & I \\ -I & 0 \end{smallmatrix}\right)$, specifies a new basis $\mathcal{B} = \{v_1,\ldots v_{d_n}; v'_1,\ldots,v'_{d_n}\}$ by $v_i = \frac{1}{\sqrt{2}}(e_i + e_{d_n+i})$ and $v'_i = \frac{1}{\sqrt{2}}(e_i - e_{d_n+i})$. Relative to $\mathcal{B}$, the group A_n will consist of the diagonal real matrices in G_n with all entries > 0, in other words

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all $\mathrm{diag}\{a_1, \ldots, a_{d_n}, a_1^{-1}, \ldots, a_{d_n}^{-1}\}$ with each $a_i > 0$. Then, as above, M_n will consist of the diagonal matrices in G_n that have all entries of absolute value 1, and N_n will consist of all lower triangular matrices in G_n that have all diagonal entries $= 1$, so $\phi_{n+1,n}$ maps A_n into A_{n+1}, maps M_n into M_{n+1} and maps N_n into N_{n+1}. That is the hypothesis of Lemma 4.3.

Next consider the complex special orthogonal groups $G_n = SO(d_n; \mathbb{C})$. Let $m_n = [d_n/2]$, let $\{e_i\}$ be a basis of $\mathbb{C}^{d_n}$ in which the symmetric bilinear form b_n that defines G_n has matrix I. Define $v_i = \frac{1}{\sqrt{2}}(e_i + e_{m_n+i}\mathbf{i})$ and $v_i' = \frac{1}{\sqrt{2}}(e_i - e_{m_n+i}\mathbf{i})$ for $1 \leqslant i \leqslant m_n$. Consider the basis $\mathcal{B}$ of $\mathbb{C}^{d_n}$ given by $\{v_1, \ldots v_{m_n}; v_1', \ldots, v_{m_n}'\}$ if d_n is even (hence $= 2m_n$), by $\{v_1, \ldots v_{m_n}; v_1', \ldots, v_{m_n}'; e_{d_n}\}$ if d_n is odd (hence $= 2m_n + 1$). Relative to $\mathcal{B}$, if d_n is even then A_n will consist of all matrices $\mathrm{diag}\{a_1, \ldots, a_{m_n}, a_1^{-1}, \ldots, a_{m_n}^{-1}\}$ with each $a_i > 0$, and if d_n is odd it will consist of all $\mathrm{diag}\{a_1, \ldots, a_{m_n}, a_1^{-1}, \ldots, a_{m_n}^{-1}, 1\}$. Then M_n will consist of the diagonal matrices in G_n that have all entries of absolute value 1, and N_n will consist of all lower triangular matrices in G_n that have all diagonal entries $= 1$, so $\phi_{n+1,n}$ maps A_n into A_{n+1}, maps M_n into M_{n+1} and maps N_n into N_{n+1}. That is the hypothesis of Lemma 4.3.

We now consider the groups $G_n = SO^*(2d_n)$, essentially as above. Let U_n be the underlying right vector space over $\mathbb{H}$ on which G_n acts. Let $\{e_i\}$ be a basis of $U_n = \mathbb{H}^{d_n}$ in which the skew-hermitian form b_n that defines G_n is given by $b_n(z, w) = \sum_{1 \leqslant i \leqslant d_n} \overline{w_i}\mathbf{i}z_i$. Define $v_i = \frac{1}{\sqrt{2}}(e_{2i-1} + e_{2i}\mathbf{j})$ and $v_i' = \frac{1}{\sqrt{2}}(e_{2i-1}\mathbf{i} + e_{2i}\mathbf{k})$ for $1 \leqslant i \leqslant m_n$ where $m_n = [d_n/2]$. Let $\mathcal{B} = \{v_1, \ldots, v_{m_n}; v_1', \ldots v_{m_n}'\}$ if d_n is even, i.e. $d_n = 2m_n$, and let $\mathcal{B} = \{v_1, \ldots, v_{m_n}; v_1', \ldots v_{m_n}'; e_{d_n}\}$ if d_n is odd, i.e. $d_n = 2m_n + 1$. Then $V_n = \mathrm{Span}\{v_i\}$ and $V_n' = \mathrm{Span}\{v_i'\}$ are maximal totally b_n-isotropic subspaces of U_n, paired by $b_n(v_i, v_j') = \delta_{ij}$. If d_n is even then $U_n = V_n + V_n'$, and if d_n is odd then $U_n = V_n + V_n' + W_n$ where $W_n = \mathrm{Span}\{e_{d_n}\} = (V_n + V_n')^\perp$ relative to b_n. In the basis $\mathcal{B}$ the groups A_n, M_n and N_n are given as in the case of the complex special orthogonal groups, so $\phi_{n+1,n}$ maps A_n into A_{n+1}, maps M_n into M_{n+1} and maps N_n into N_{n+1}. That gives us the hypotheses of Lemma 4.3.

Finally we come to the case $G_n = SU(d_n', d_n''; \mathbb{F})$ of the real orthogonal, complex unitary and unitary symplectic (quaternion unitary) groups. Let U_n be the underlying right vector space, over $\mathbb{F}$ on which G_n acts. Then G_n is essentially the group of automorphisms of (U_n, b_n) where b_n is the nondegenerate $\mathbb{F}$-hermitian form on U_n that defines G_n. Let $\{e_i\}$ be a basis of U_n in which b_n has matrix $\left(\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right)$. Let $m_n = \min(d_n', d_n'')$, the real rank $\dim \mathfrak{a}_n$ of G_n. Define $v_i = \frac{1}{\sqrt{2}}(e_i + e_{m_n+i})$ and $v_i' = \frac{1}{\sqrt{2}}(e_i - e_{m_n+i})$ for $1 \leqslant i \leqslant m_n$. Let $r_n = d_n - 2m_n$, and let $\{w_1, \ldots, w_{r_n}\}$ denote the ordered set of those e_i not involved in the v_j. Then we have the basis $\mathcal{B} = \{v_1', \ldots, v_{m_n}'; w_1, \ldots w_{r_n}; v_{m_n}, \ldots, v_1\}$ of U_n. On the subspace level, $V_n = \mathrm{Span}\{v_i\}$ and $V_n' = \mathrm{Span}\{v_i'\}$ are maximal totally b_n-isotropic subspaces of U_n, paired by $b_n(v_i, v_j') = \delta_{ij}$, and $U_n = V_n' + W_n + V_n$ where $W_n = \mathrm{Span}\{w_i\} = (V_n + V_n')^\perp$ relative to b_n. Note that W_n is zero if $d_n' = d_n''$, positive definite if $d_n' > d_n''$, negative definite if $d_n' < d_n''$.

We choose A_n to consist of all linear transformations of U_n with matrix, relative to $\mathcal{B}$, of the form $\mathrm{diag}\{a_1, \ldots, a_{m_n}; 1, \ldots 1; a_{m_n}^{-1}, \ldots, a_1^{-1}\}$ with a_i all real and positive. Then M_n consists of all linear transformations $m \in G_n$ such that

$$m(v_i') = m_i v_i', \quad m(W_n) = W_n \quad \text{and} \quad m(v_i) = m_i v_i \quad \text{where the } m_i \in \mathbb{F} \text{ with } |m_i| = 1.$$

Thus $\phi_{n+1,n}(A_n) \subset A_{n+1}$.

The description of N_n is a little more complicated. Let $\mathcal{V}_n = \{V_{n,1}, \ldots, V_{n,m_n}\}$ be the maximal isotropic flag in V_n given by $V_{n,j} = \mathrm{Span}\{v_1, \ldots, v_j\}$. In almost every case we may take the minimal parabolic subgroup P_n of G_n to be the G_n-stabilizer of $\mathcal{V}_n$. That done, let $P_{n,j}$ denote the maximal real parabolic subgroup of G_n that is the stabilizer of $V_{n,j}$. Then the nilradicals of these parabolics satisfy $\mathfrak{n}_n = \sum_{1 \leqslant j \leqslant m_n} \mathfrak{n}_{n,j}$. The point of this is that we know the $\mathfrak{n}_{n,j}$ in a convenient form. Let $X_{n,j} = \mathrm{Span}\{v_{j+1}, \ldots, v_{m_n}\}$ so that $V_n = V_{n,j} \oplus X_{n,j}$. Define $V_{n,j}' = \mathrm{Span}\{v_1', \ldots, v_j'\}$

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and $X'_{n,j} = \mathrm{Span}\{v'_{j+1}, \ldots, v'_{m_n}\}$ so that $V'_n = V'_{n,j} \oplus X'_{n,j}$. Denote $W_{n,j} = X_{n,j} \oplus W_n \oplus X'_{n,j}$ so that $U_n = V_{n,j} + W_{n,j} + V'_{n,j}$. According to [Wol76, Lemma 3.4], the nilradical $\mathfrak{n}_{n,j}$ of $\mathfrak{p}_{n,j}$ is the sum of its two subspaces

$$\mathfrak{p}^2_{n,j} = \{\xi \in \mathfrak{g}_n \mid \xi(V'_{n,j}) \subset V_{n,j}, \xi(W_{n,j}) = 0, \xi(V_{n,j}) = 0\} \quad \text{and}$$
$$\mathfrak{p}^1_{n,j} = \{\xi \in \mathfrak{g}_n \mid \xi(V'_{n,j}) \subset W_{n,j}, \xi(W_{n,j}) \subset V_{n,j}, \xi(V_{n,j}) = 0\}$$

while the reductive component consists of those ξ in $\mathfrak{g}_n$ that stabilize each of $V'_{n,j}$, $W_{n,j}$ and $V_{n,j}$. Thus, relative to the basis $\mathcal{B}$, the elements of $\mathfrak{n}_{n,j}$ have block form $\left(\begin{smallmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ * & * & 0 \end{smallmatrix}\right)$ along $U_n = V'_{n,j} + W_{n,j} + V_{n,j}$. Summing over j, the elements of $\mathfrak{n}_n$ are precisely those elements of $\mathfrak{p}_n$ whose matrix relative to $\mathcal{B}$ has block form $\left(\begin{smallmatrix} \ell & 0 & 0 \\ * & 0 & 0 \\ * & * & \ell' \end{smallmatrix}\right)$ along $U_n = V'_n + W_n + V_n$, where ℓ and ℓ' are lower triangular with zeroes on their diagonals. In the case $G_n = SO(d'_n, d''_n)$ one must be a bit more careful and take some orientation into account, as in [WZ00], but the result is the same. Thus $\phi_{n+1,n}(N_n) \subset N_{n+1}$.

It certainly cannot be automatic that $\phi_{n+1,n}(M_n) \subset M_{n+1}$. Define a difference $\mu_n = \max(d'_n, d''_n) - \min(d'_n, d''_n)$. Then $\mu_n = \dim W_n$, so $M^0_n \cong SU(\mu_n; \mathbb{F})$ and $M^0_{n+1} \cong SU(\mu_{n+1}; \mathbb{F})$, and consequently $\phi_{n+1,n}(M_n) \not\subset M_{n+1}$ whenever $\mu_n > \mu_{n+1}$.

Now we pin this down. The map $\phi_{n+1,n} : G_n \to G_{n+1}$ is implemented by a unitary injection $k_n : (U_n, b_n) \hookrightarrow (U_{n+1}, \pm b_{n+1})$. We have set things up so that, possibly after interchanging the v_i and the v'_i in U_{n+1}, $k_n(V_{n,j}) \subset V_{n+1,j}$ and $k_n(V'_{n,j}) \subset V'_{n+1,j}$ for $1 \leqslant j \leqslant m_n$. We used that to prove that $\phi_{n+1,n}$ maps A_n into A_{n+1} and N_n into N_{n+1}. However, $\phi_{n+1,n}(M_n) \subset M_{n+1}$ if and only if we can make the choices of V_{n+1} and V'_{n+1} so that $k_n(W_n) \hookrightarrow W_{n+1}$. That is possible if and only if $\mu_n \leqslant \mu_{n+1}$.

If $\mu_n \leqslant \mu_{n+1}$ for infinitely many indices n, then we have a cofinal subsystem of $\{G_m, \phi_{n,m}\}$ in which $\mu_n \leqslant \mu_{n+1}$ for all n, and thus $\phi_{n+1,n}(M_n) \subset M_{n+1}$. If $\mu_n \leqslant \mu_{n+1}$ for only finitely many indices n, then we have an index n_0 such that $\mu_n > \mu_{n+1} \geqslant 0$ for all $n \geqslant n_0$. That is impossible. Thus the hypothesis of Lemma 4.3 is always valid for a cofinal subsystem of $\{G_m, \phi_{n,m}\}$.

Suppose that $\{G_m, \phi_{n,m}\}$ is weakly parabolic. View the $\phi_{n+1,n}$ as inclusions $G_n \hookrightarrow G_{n+1}$. Then $\mathfrak{a}_{n+1} = \mathfrak{a}_n \oplus \mathfrak{a}_{n+1,n}$ as in Proposition 8.5, and $(\mathfrak{a}_n + \mathfrak{m}_n) \oplus \mathfrak{a}_{n+1,n}$ is the centralizer of $\mathfrak{a}_{n+1,n}$ in $\mathfrak{g}_{n+1}$. In particular, $\Sigma(\mathfrak{g}_n \oplus \mathfrak{a}_{n+1,n}, \mathfrak{a}_{n+1}) \subset \Sigma(\mathfrak{g}_{n+1}, \mathfrak{a}_{n+1})$. Thus, if $\gamma_n \in \Sigma(\mathfrak{g}_n, \mathfrak{a}_n)$ there is a unique $\gamma_{n+1} \in \Sigma(\mathfrak{g}_{n+1}, \mathfrak{a}_{n+1})$ such that $\gamma_{n+1}|_{\mathfrak{a}_n} = \gamma_n$. However, if $r_{n+1} + s_{n+1} \geqslant 2$ then at least two distinct elements of $\Sigma(\mathfrak{g}_{n+1}, \mathfrak{a}_{n+1})$ restrict to γ_n. Thus $r_{n+1} + s_{n+1} = 1$. $\square$

PROPOSITION 9.13. *Let $G = \varinjlim\{G_m, \phi_{n,m}\}_{n \geqslant m \geqslant 0}$ be a diagonal embedding direct limit group. Then the following conditions are equivalent.*

(1) *$G = \varinjlim\{G_m, \phi_{n,m}\}$ is of classical type, in other words $r_n + s_n = 1$ for $n \geqslant n_0$.*
(2) *The root system $\Sigma(\mathfrak{g}, \mathfrak{a}) = \varprojlim \Sigma(\mathfrak{g}_n, \mathfrak{a}_n)$ is countable.*
(3) *$\Sigma(\mathfrak{g}, \mathfrak{a}) = \bigcup_{n \geqslant 0} \Sigma(\mathfrak{g}_n, \mathfrak{a}_n)$.*
(4) *$\mathfrak{g}$ is restricted-root-reductive in the sense that $\mathfrak{g} = (\mathfrak{m} + \mathfrak{a}) + \sum_{\gamma \in \Sigma(\mathfrak{g}, \mathfrak{a})} \mathfrak{g}^\gamma$.*

Proof. Let G be of classical type. View the ϕ_n as inclusions $G_n \hookrightarrow G$ as inclusions. Let Ψ_n denote the simple system of $\Sigma(\mathfrak{g}_n, \mathfrak{a}_n)^+$. If $n \geqslant m \geqslant n_0$ then Proposition 9.12 shows how $\Psi_m \subset \Psi_n$ when we extend the elements of Ψ_m by zero on $\mathfrak{a}_{n,m}$. Thus $\Sigma(\mathfrak{g}, \mathfrak{a}) = \bigcup_{n \geqslant 0} \Sigma(\mathfrak{g}_n, \mathfrak{a}_n)$, which is countable, and if $\gamma \in \Sigma(\mathfrak{g}, \mathfrak{a})$ then $\phi_n(\mathfrak{g}_n^{\phi^*_n(\gamma)})$ is a well-defined subspace of the root space for γ. We have just seen that condition (1) implies conditions (2), (3) and (4). On the other hand, condition (4) implies condition (3), and condition (3) implies condition (2), at a glance. Thus we need only prove that condition (2) implies condition (1).

Suppose that $G = \varinjlim\{G_m, \phi_{n,m}\}$ is not of classical type. Then we can pass to a cofinal subsystem in which every $r_n + s_n \geqslant 2$. That done, every root $\gamma \in \Sigma(\mathfrak{g}_n, \mathfrak{a}_n)^+$ is the restriction of at least 2 roots

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in $\Sigma(\mathfrak{g}_{n+1}, \mathfrak{a}_{n+1})^+$, thus is the restriction of at least $2^{\aleph_0}$ roots in $\Sigma(\mathfrak{g}, \mathfrak{a})$. In particular, $\Sigma(\mathfrak{g}, \mathfrak{a})$ is not countable. Thus condition (2) implies condition (1). $\square$

Recall the notion of Satake diagram. We use the Cartan subalgebra $\mathfrak{h}_n = \mathfrak{t}_n + \mathfrak{a}_n$ of $\mathfrak{g}_n$ and the positive root system $\Sigma(\mathfrak{g}_{n,\mathbb{C}}, \mathfrak{h}_{n,\mathbb{C}})^+$ as in (2.3). Write $\Psi(\mathfrak{g}_{n,\mathbb{C}}, \mathfrak{h}_{n,\mathbb{C}})$ for the corresponding simple $\mathfrak{h}_{n,\mathbb{C}}$-root system, and write $\Psi(\mathfrak{g}_n, \mathfrak{a}_n)$ for the simple $\mathfrak{a}_n$-root system corresponding to $\Sigma(\mathfrak{g}_n, \mathfrak{a}_n)^+$. Every $\psi \in \Psi(\mathfrak{g}_n, \mathfrak{a}_n)$ is of the form $\widetilde{\psi}|_{\mathfrak{a}_n}$ for some $\widetilde{\psi} \in \Psi(\mathfrak{g}_{n,\mathbb{C}}, \mathfrak{h}_{n,\mathbb{C}})$. More or less conversely, if $\widetilde{\psi} \in \Psi(\mathfrak{g}_{n,\mathbb{C}}, \mathfrak{h}_{n,\mathbb{C}})$ then either $\widetilde{\psi}|_{\mathfrak{a}_n} = 0$ or $\widetilde{\psi}|_{\mathfrak{a}_n} \in \Psi(\mathfrak{g}_n, \mathfrak{a}_n)$. The Satake diagram describes the restriction process. Start with the Dynkin diagram $\mathcal{D}_n$ of $\mathfrak{g}_{n,\mathbb{C}}$ whose vertices are the elements of $\Psi(\mathfrak{g}_{n,\mathbb{C}}, \mathfrak{h}_{n,\mathbb{C}})$. If there are two root lengths this is indicated by arrows rather than darkening the vertices for short roots. Now darken those $\widetilde{\psi} \in \Psi(\mathfrak{g}_{n,\mathbb{C}}, \mathfrak{h}_{n,\mathbb{C}})$ such that $\widetilde{\psi}|_{\mathfrak{a}_n} = 0$. It can happen that two (but never more than two) distinct elements $\widetilde{\psi}, \widetilde{\psi}' \in \Psi(\mathfrak{g}_{n,\mathbb{C}}, \mathfrak{h}_{n,\mathbb{C}})$ have the same $\mathfrak{a}_n$-restriction. In that case, join them by an arrow. The result is the Satake diagram of $\mathfrak{g}_n$. The white vertices and vertex pairs corresponding to simple $\mathfrak{a}_n$-roots of $\mathfrak{g}_n$. The black vertices correspond to simple $\mathfrak{t}_{n\mathbb{C}}$-roots of $\mathfrak{m}_{n,\mathbb{C}}$. See [Wol80, pp. 90–93] or [Ara62, pp. 32–33], for Araki's list of Satake diagrams.

We use the Satake diagrams to see just which $G = \varinjlim\{G_m, \phi_{n,m}\}$ of classical type are weakly parabolic. The description (8.2) and (8.3) of real parabolic subalgebras gives us the following.

Lemma 9.14. *The semisimple components of real parabolic subalgebras of $\mathfrak{g}_{n+1}$ are characterized up to $\mathrm{Int}(G_{n+1})$-conjugacy by their Satake diagrams, and those Satake diagrams are obtained from the Satake diagram of $\mathfrak{g}_{n+1}$ by deleting (i) an arbitrary set of white vertices and then (ii) all white vertices joined by arrows (meaning the same restriction to $\mathfrak{a}_{n+1}$) to vertices deleted in (i).*

The black vertices (restriction 0 to $\mathfrak{a}_{n+1}$) remain because they represent the simple roots of $\mathfrak{m}_{n+1}$, which is contained in every real parabolic subalgebra that contains $\mathfrak{a}_{n+1}$.

Let $G = \varinjlim\{G_m, \phi_{n,m}\}$ be a diagonal embedding direct limit group of classical type. From Araki's list of Satake diagrams one sees that the possible inclusions $\phi_{n+1,n} : \mathfrak{g}'_n \to \mathfrak{g}_{n+1}$ of weakly parabolic type are given, modulo $\mathfrak{m}_{n+1}$, by

$$SL(d_n;\mathbb{F}) \hookrightarrow SL(d_{n+1},\mathbb{F}) \quad \text{by } \phi_{n+1,n}(g) = \mathrm{diag}\{g \text{ or } \delta(g), 1, \dots 1\}, d_{n+1} > d_n, \tag{9.15a}$$
$$SO(d'_n, d''_n) \hookrightarrow SO(d'_n + u_n, d''_n + u_n) \quad \text{by } \phi_{n+1,n}(g) = \mathrm{diag}\{g \text{ or } \delta(g), 1, \dots 1\}, u_n > 0, \tag{9.15b}$$
$$SO(d_n;\mathbb{C}) \hookrightarrow SO(d_n + 2u_n;\mathbb{C}) \quad \text{by } \phi_{n+1,n}(g) = \mathrm{diag}\{g \text{ or } \delta(g), 1, \dots 1\}, u_n > 0, \tag{9.15c}$$
$$SU(d'_n, d''_n) \hookrightarrow SU(d'_n + u_n, d''_n + u_n) \quad \text{by } \phi_{n+1,n}(g) = \mathrm{diag}\{g \text{ or } \delta(g), 1, \dots 1\}, u_n > 0, \tag{9.15d}$$
$$Sp(d'_n, d''_n) \hookrightarrow Sp(d'_n + u_n, d''_n + u_n) \quad \text{by } \phi_{n+1,n}(g) = \mathrm{diag}\{g, 1, \dots 1\}, u_n > 0, \tag{9.15e}$$
$$Sp(d_n;\mathbb{F}) \hookrightarrow Sp(d_{n+1};\mathbb{F}) \quad \text{by } \phi_{n+1,n}(g) = \mathrm{diag}\{g, 1, \dots 1\}, d_{n+1} > d_n \text{ and } \mathbb{F} = \mathbb{R} \text{ or } \mathbb{C}, \tag{9.15f}$$
$$SO^*(2d_n) \hookrightarrow SO^*(2d_n + 4u_n) \quad \text{by } \phi_{n+1,n}(g) = \mathrm{diag}\{g \text{ or } \delta(g), 1, \dots 1\}, u_n > 0. \tag{9.15g}$$

In order to pin things down we make use of the fact that $G = \varinjlim\{G_m, \phi_{n,m}\}$ is determined by any cofinal subsequence of indices. Denote

$$\mathbf{0} = \{0, 0, 0, \dots\}, \quad \mathbf{1} = \{1, 1, 1, \dots\} \quad \text{and} \quad \mathbf{2} = \{2, 2, 2, \dots\}.$$

Consider, for example, the case of (9.15a). Suppose first that there are only finitely many indices n for which $\phi_{n+1,n}(g) = \mathrm{diag}\{\delta(g), 1, \dots, 1\}$. Pass to the subsequence starting just after the last $\phi_{n+1,n}$ that involves δ. That done, we interpolate and arrive at the same limit with each $\phi_{n+1,n}(g) = \left(\begin{smallmatrix} g & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Now suppose that there are infinitely many indices n for which $\phi_{n+1,n}(g) = \mathrm{diag}\{\delta(g), 1, \dots, 1\}$. Pass to the cofinal subsequence obtained by deleting the G_n for which $\phi_{n+1,n}(g) = \mathrm{diag}\{g, 1, \dots, 1\}$, so now every $\phi_{n+1,n}(g)$ is of the form $g \mapsto \mathrm{diag}\{\delta(g), 1, \dots, 1\}$. If $t_{n+1} > 1$ for an infinite number of t_{n+1} then, recursively, we take the smallest index n for which $t_{n+1} > 1$, insert $t_{n+1} - 1$ steps $g \mapsto \left(\begin{smallmatrix} \delta(g) & 0 \\ 0 & 1 \end{smallmatrix}\right)$ between G_n and G_{n+1}, and proceed to insert steps $g \mapsto \left(\begin{smallmatrix} \delta(g) & 0 \\ 0 & 1 \end{smallmatrix}\right)$ at the next $t_{n+1} - 1$

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possible places. Then we arrive at the same limit with each $\phi_{n+1,n}(g) = \left(\begin{smallmatrix} \delta(g) & 0 \\ 0 & 1 \end{smallmatrix}\right)$. If $t_{n+1} > 1$ for only finitely many n we just pass to the subsequence starting just after the last $\phi_{n+1,n}$ involving a t_{n+1} that is > 1. Thus $G = SL_{\mathbf{1},\mathbf{0},\mathbf{1}}(\infty, \mathbb{F}) = \varinjlim SL(n+1; \mathbb{F})$ in the first case, with $\phi_{n+1,n}(g) = \left(\begin{smallmatrix} g & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and $G = SL_{\mathbf{0},\mathbf{1},\mathbf{1}}(\infty, \mathbb{F}) = \varinjlim SL(n+1; \mathbb{F})$ in the second case, with $\phi_{n+1,n}(g) = \left(\begin{smallmatrix} \delta(g) & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

Similar considerations hold in the other six cases (9.15b)–(9.15g). The final result is as follows.

Proposition 9.16. *The weakly parabolic diagonal embedding direct limit groups $G = \{G_m, \phi_{n,m}\}$ of classical type, with G_m noncompact and simple for m large, are given, up to isomorphism, by one of the following:*

$$SL_{\mathbf{1},\mathbf{0},\mathbf{1}}(\infty; \mathbb{F}) \text{ with } g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad SL_{\mathbf{0},\mathbf{1},\mathbf{1}}(\infty, \mathbb{F}) \text{ with } g \mapsto \begin{pmatrix} \delta(g) & 0 \\ 0 & 1 \end{pmatrix}; \tag{9.17a}$$

here we may take G_m to be $SL(m+1; \mathbb{F})$;

$$SO_{\mathbf{1},\mathbf{1},\mathbf{0},\mathbf{1}}(\infty, \infty) \text{ with } g \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad SO_{\mathbf{1},\mathbf{0},\mathbf{1},\mathbf{1}}(\infty, \infty) \text{ with } g \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & \delta(g) & 0 \\ 0 & 0 & 1 \end{pmatrix}; \tag{9.17b}$$

here we may take G_m to be an $SO(d_1' + m, d_1'' + m)$ where $d_1', d_1'' \geqslant 1$;

$$SO_{\mathbf{1},\mathbf{0},\mathbf{2}}(\infty; \mathbb{C}) \text{ with } g \mapsto \begin{pmatrix} g & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad SO_{\mathbf{0},\mathbf{1},\mathbf{2}}(\infty; \mathbb{C}) \text{ with } g \mapsto \begin{pmatrix} \delta(g) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \tag{9.17c}$$

here we may take G_m to be $SO(2m+1; \mathbb{C})$ (type B) or $SO(2m; \mathbb{C})$ (type D);

$$SU_{\mathbf{1},\mathbf{1},\mathbf{0},\mathbf{1}}(\infty; \infty) \text{ with } g \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad SU_{\mathbf{1},\mathbf{0},\mathbf{1},\mathbf{1}}(\infty; \infty) \text{ with } g \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & \delta(g) & 0 \\ 0 & 0 & 1 \end{pmatrix}; \tag{9.17d}$$

here we may take G_m to be an $SU(d_1' + m, d_1'' + m)$ where $d_1', d_1'' \geqslant 1$;

$$Sp_{\mathbf{1},\mathbf{1},\mathbf{0},\mathbf{1}}(\infty; \infty) \text{ with } g \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & 1 \end{pmatrix}; \tag{9.17e}$$

here we may take G_m to be an $Sp(d_1' + m, d_1'' + m)$ where $d_1', d_1'' \geqslant 1$;

$$Sp_{\mathbf{1},\mathbf{2}}(\infty; \mathbb{F}) \text{ with } g \mapsto \begin{pmatrix} g & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \mathbb{F} = \mathbb{R} \text{ or } \mathbb{C}; \tag{9.17f}$$

here we may take G_m to be $Sp(m; \mathbb{F})$;

$$SO^*_{\mathbf{1},\mathbf{0},\mathbf{1}}(\infty) \text{ with } g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad SO^*_{\mathbf{0},\mathbf{1},\mathbf{1}}(\infty) \text{ with } g \mapsto \begin{pmatrix} \delta(g) & 0 \\ 0 & 1 \end{pmatrix} \quad \text{(quaternionic matrices)}; \tag{9.17g}$$

here we may take G_m to be $SO^*(2m)$.

10. The other tempered series

The finite-dimensional real reductive Lie groups G that satisfy the conditions of Definition 2.1 have a series of unitary representations for each conjugacy class of Cartan subgroups. Those are the 'tempered' representations, those that occur in the decomposition of $L_2(G)$ under the left translation action of G. The principal series is the tempered series corresponding to the conjugacy class of a maximally noncompact Cartan subgroup, but in general there are others. If G has a Cartan subgroup with compact image under the adjoint representation, then the corresponding series is the discrete series. In general, these series are constructed by combining the ideas of the discrete series and the principal series. See [HCh66, HCh75, HCh76a, HCh76b] for the case where G is Harish-Chandra class and [Wol74, HW86a, HW86b] for the general case. We now recall a few relevant facts from these papers in order to indicate the corresponding extension of our principal series results.

Fix a Cartan involution θ of G and let K denote its fixed point set. As usual, $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$, decomposition into (± 1)-eigenspaces of θ. Every G^0-conjugacy class of Cartan subgroups contains a θ-stable Cartan. Fix a θ-stable Cartan subgroup H of G. Then $\mathfrak{h} = \mathfrak{t} + \mathfrak{a}$ where $\mathfrak{t} = \mathfrak{h} \cap \mathfrak{k}$ and $\mathfrak{a} = \mathfrak{h} \cap \mathfrak{s}$. Here $H = T \times A$ where $T = H \cap K$ and $A = \exp(\mathfrak{a})$. Earlier we had only considered the case where $\mathfrak{a}$ is maximal abelian in $\mathfrak{s}$; here the situation is more general. The centralizer of A in G has form $M \times A$ where $\theta(M) = M$. In our earlier discussions M was compact modulo $Z_G(G^0)$ (*relatively compact*), but here the situation is more general. In any case, M satisfies the conditions of Definition 2.1, and T is relatively compact, so M has relative discrete series representations. In the principal series setting these will be all the irreducible representations of M and will necessarily be finite dimensional, but here the situation is more general.

We have the $\mathfrak{a}$-root system $\Sigma(\mathfrak{g}, \mathfrak{a}) = \{\alpha|_{\mathfrak{a}} \mid \alpha \in \Sigma(\mathfrak{g}, \mathfrak{h}) \text{ and } \alpha|_{\mathfrak{a}} \neq 0\}$. Fix a positive subsystem $\Sigma(\mathfrak{g}, \mathfrak{a})^+$ and define $\mathfrak{n} = \sum_{\beta \in \Sigma(\mathfrak{g}, \mathfrak{a})^+} \mathfrak{g}^{-\beta}$. Then $\mathfrak{p} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n}$ is a particular kind of (real) parabolic subalgebra of $\mathfrak{g}$, distinguished by the fact that $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{m}$. Those are the *cuspidal parabolic* subalgebras of $\mathfrak{g}$. Let $N = \exp(\mathfrak{n})$. It is the analytic subgroup of G with Lie algebra $\mathfrak{n}$, and $P = MAN$ is the parabolic subgroup of G with Lie algebra $\mathfrak{p}$. Those are the *cuspidal parabolic* subgroups of G.

We use the cuspidal parabolic subgroup $P = MAN$ to describe the H-series representations of G. The analog of Proposition 2.2 is the parameterization of the relative discrete series of M. Fix a positive $\mathfrak{t}_{\mathbb{C}}$-root system $\Sigma(\mathfrak{m}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})^+$ on $\mathfrak{m}_{\mathbb{C}}$. Let $\nu \in i\mathfrak{t}^*$ such that $e^{\nu - \rho_{\mathfrak{m},\mathfrak{t}}}$ is well defined on T^0 and $\langle \nu, \alpha \rangle \neq 0$ for all $\alpha \in \Sigma(\mathfrak{m}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. Then M^0 has a unique unitary equivalence class of relative discrete series representations, $[\eta^0_\nu]$, with Harish-Chandra parameter ν. Here ν is the infinitesimal character of $[\eta^0_\nu]$; if η^0_ν has a highest weight, that weight is $\nu - \rho_{\mathfrak{m},\mathfrak{t}}$. Set $M^\dagger = Z_M(M^0)M^0$. Then the relative discrete series classes of $M^\dagger$ are the $[\eta^\dagger_{\chi,\nu}] = [\chi \otimes \eta_\nu]$ where $[\chi] \in \widehat{Z_M(M^0)}_\xi$ with $\xi = e^{\nu - \rho_{\mathfrak{m},\mathfrak{t}}}|_{Z_{M^0}}$. The relative discrete series classes of M are the $[\eta_{\chi,\nu}]$ where $\eta_{\chi,\nu} = \mathrm{Ind}_{M^\dagger}^M(\eta^\dagger_{\chi,\nu})$. Let $\sigma \in \mathfrak{a}^*_{\mathbb{C}}$. Then we have $[\eta_{\chi,\nu,\sigma}] \in \widehat{P}$ defined by $\eta_{\chi,\nu,\sigma}(man) = e^\sigma(a)\eta_{\chi,\nu}(m)$ for $m \in M, a \in A$ and $n \in N$. The corresponding H-series representation of G is $\pi_{\chi,\nu,\sigma} = \mathrm{Ind}_P^G(\eta_{\chi,\nu,\sigma})$. Its equivalence class does not depend on the choice of $\Sigma(\mathfrak{g}, \mathfrak{a})^+$. The H-series of G consists of all such representations or, depending on context, the unitary ones. The principal series of G is the case where $\mathfrak{a}$ is maximal, equivalently where M is relatively compact. The relative discrete series of G is the case $\mathfrak{a} = 0$; it exists if and only if G has a relatively compact Cartan subgroup.

Now we return to our strict direct system $\{G_i, \phi_{k,i}\}$ of reductive Lie groups. Fix a Cartan subgroup $H = \varinjlim H_i$ of $G = \varinjlim G_i$. We consider limit representations $\pi = \varprojlim \pi_i$ of G, where, for each i, π_i is an H_i-series representation of G_i.

Here there are several problems. First, we need the discrete series analog of [NRW01] in order to construct the M-component of any $\varprojlim \pi_i$. That falls into two parts. The first is to realize the discrete series representations of the M_i on some appropriate cohomology spaces, such as spaces of L_2 harmonic forms. This is done, for example, in [Sch71, Sch76, Wol74]. The second is to make sure that these representations all appear on cohomologies of the same degree, and to line them up properly so that one can take limits. This was done in [Nat94] for holomorphic discrete series; there the cohomology degree is 0, the alignment is done using the universal enveloping algebra description of highest weight representations, and the result is analyzed by use of [EHW83]. It was done in [Hab01] for other discrete series of certain diagonal embedding direct limit groups $Sp(p, \infty)$ and $SO(2p, \infty)$ of classical type using Zuckerman derived functor modules $A_q(\lambda)$ for the cohomologies. We address these matters in some generality in [Wol05a].

Second, we need an analog of the considerations of Section 8. This is not so difficult, but one has to be careful. We address this matter in [Wol05b].

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Joseph A. Wolf jawolf@math.berkeley.edu
Department of Mathematics, University of California, Berkeley, CA 94720–3840, USA

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