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Mixed Perverse Sheaves on Flag Varieties for Coxeter Groups

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Abstract. In this paper we construct an abelian category of mixed perverse sheaves attached to any realization of a Coxeter group, in terms of the associated Elias–Williamson diagrammatic category. This construction extends previous work of the first two authors, where we worked with parity complexes instead of diagrams, and we extend most of the properties known in this case to the general setting. As an application we prove that the split Grothendieck group of the Elias–Williamson diagrammatic category is isomorphic to the corresponding Hecke algebra, for any choice of realization.

1 Introduction

1.1 Categorifications of Hecke algebras

Let (W, S) be a Coxeter system and let $\mathcal{H}_W$ be the associated Hecke algebra. When W is crystallographic, *i.e.*, is the Weyl group of a Kac–Moody group G, a fact of fundamental importance, going back to 1980, is the existence of a remarkable geometric categorification of $\mathcal{H}_W$: it can be realized as the split Grothendieck group of the additive monoidal category of B-equivariant semisimple complexes (with complex coefficients) on the flag variety G/B of G, where $B \subset G$ is the Borel subgroup [KL2, Sp].[1] The main point of this categorification is that it realizes the Kazhdan–Lusztig basis of $\mathcal{H}_W$ as the classes of simple perverse sheaves.

In the 2000s, this categorification was generalized into two different directions:

• Soergel [So3] showed that semisimple complexes on flag varieties can be replaced by Soergel bimodules, thereby providing a categorification of $\mathcal{H}_W$ for any Coxeter group.

• Juteau–Mautner–Williamson [JMW] introduced parity complexes, which provide the appropriate replacement for semisimple complexes when we take coefficients in an arbitrary field (this leads naturally to the notion of p-canonical bases).

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[1] The papers [KL2] and [Sp] only mention finite or affine Kac–Moody groups. However, thanks to the subsequent development of the general theory of Kac–Moody groups [Ku, Ti], their methods now apply in this generality.

These two generalizations were recently united by the introduction of the Elias–Williamson diagrammatic category [EW2]: a certain monoidal category attached to any Coxeter group equipped with a *realization*. For certain realizations (coming from reflection faithful representations), this category is equivalent to the category of Soergel bimodules. On the other hand, for realizations constructed from Kac–Moody root data, one recovers the corresponding category of parity complexes on the associated flag variety. (This result was suggested in [EW2, JMW], and formally proved in [RW, Part 3].) Let us note that the reflection faithful requirement is rather restrictive, justifying the interest of a construction avoiding this condition.

1.2 Triangulated Categories

The categorifications considered above take us away from the very comfortable world of perverse sheaves. The main goal of the present paper is to explain how perverse sheaves can be reintroduced into the picture. This paper draws inspiration from [AR1], which, in the setting of parity complexes on flag varieties, introduced the notions of *mixed derived category* and *mixed perverse sheaves*. These notions have since found important applications in modular representation theory [AR1, ARd2, MaR, AR2, AMRW2].

The first step is to embed the diagrammatic category in a suitable triangulated category. This was done by Makisumi, Williamson, and the first two authors [AMRW1]. That paper defines the *biequivariant* derived category $\mathsf{BE}(\mathfrak{h}, W)$ attached to a Coxeter group W and a realization $\mathfrak{h}$ as the bounded homotopy category of the Elias–Williamson category. The same paper also defines the *right-equivariant* derived category $\mathsf{RE}(\mathfrak{h}, W)$, which plays the role of the *B-constructible* derived category of G/B in the usual picture.

1.3 Perverse Sheaves

In the present paper, we build on this approach and construct the perverse t-structure on $\mathsf{BE}(\mathfrak{h}, W)$ and $\mathsf{RE}(\mathfrak{h}, W)$. One would like to follow the model of [AR1], but that paper exploits the fact that parity complexes are already defined in terms of sheaves on some topological space, where it makes sense to restrict to or push forward from a locally closed subspace. Thus, one key step in the present paper is to understand the correct analogue of *locally closed subspace* in the diagrammatic setting. The solution, explained in Section 4 and inspired by [ARd1], is to work with certain naive subquotients of the diagrammatic category.

The proof that these subquotients have the appropriate behaviour relies on some properties of the *double leaves basis* for morphism spaces, introduced by Libedinsky [Li] for Soergel bimodules, and studied in the context of the diagrammatic category in [EW2, §6, 7]. From our point of view, this study provides another illustration of the power of these methods.

Once we have made the correct definitions, we will construct a *recollement* formalism for these categories (following essentially the same ideas as in [AR1]), and use it to define the desired t-structure.

1.4 Standard and Costandard Objects

An important property of B-equivariant perverse sheaves on G/B is that the *standard* and *costandard* objects (the $*$- and !-extensions of the constant perverse sheaves on Bruhat strata) are perverse sheaves. In the usual topological context, this property follows from the fact that the embeddings of these strata are affine morphisms. A different argument was needed for the mixed derived categories of [AR1]. The proof given there is based on the study of pushforward to and pullback from partial flag varieties.

In the context of the present paper, we have no analogue of sheaves on partial flag varieties,[2] so some new ideas are needed, which, once again, rely to some extent on the properties of the light-leaves basis. The proof that standard and costandard objects are perverse in the diagrammatic setting appears in Section 7.

1.5 Some Other Properties

Let $\mathsf{P}^{\mathsf{BE}}(\mathfrak{h}, W)$ and $\mathsf{P}^{\mathsf{RE}}(\mathfrak{h}, W)$ denote the hearts of the perverse t-structures on

$$\mathsf{BE}(\mathfrak{h}, W) \quad \text{and} \quad \mathsf{RE}(\mathfrak{h}, W),$$

respectively. These categories share many properties with their traditional counterparts. In particular, we prove the following.

- In the case of field coefficients, the simple objects in the abelian categories $\mathsf{P}^{\mathsf{BE}}(\mathfrak{h}, W)$ and $\mathsf{P}^{\mathsf{RE}}(\mathfrak{h}, W)$ can be described in terms of $!*$-extensions (Sections 8.1 and 9.5).
- The forgetful functor $\mathsf{P}^{\mathsf{BE}}(\mathfrak{h}, W) \to \mathsf{P}^{\mathsf{RE}}(\mathfrak{h}, W)$ is fully faithful (Proposition 9.4).
- If $\Bbbk$ is a field, the category $\mathsf{P}^{\mathsf{RE}}(\mathfrak{h}, W)$ has a natural structure of highest weight category (Theorem 9.6).
- If $\Bbbk$ is a field and W is finite, one can construct a Ringel duality, exchanging projective and tilting objects in $\mathsf{P}^{\mathsf{RE}}(\mathfrak{h}, W)$; moreover, the indecomposable tilting object associated with the longest element in W is both projective and injective (Section 10).

1.6 Applications

One classical motivation for studying mixed perverse sheaves on flag varieties (with complex coefficients) is that they provide a mixed version of (a regular block of) the Bernstein–Gelfand–Gelfand category $\mathcal{O}$ associated with a semisimple complex Lie algebra [BGS, So1]. In this spirit, $\mathsf{P}^{\mathsf{RE}}(\mathfrak{h}, W)$ may be thought of as a generalized mixed category $\mathcal{O}$ attached to W and $\mathfrak{h}$.

As a more concrete application of our results, we prove that for any realization of W, the split Grothendieck group of the Elias–Williamson diagrammatic category is isomorphic to the Hecke algebra $\mathcal{H}_W$. (Note that in [EW2] this result was proved only in the case that the base ring $\Bbbk$ is a field or a complete local ring.) This application

[2] A definition of such a category would require a diagrammatic version of the singular Soergel bimodules of [W]; no definition of such objects is available at the moment.

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illustrates the interest in our formalism for the study of the diagrammatic category, beyond the simple game of defining perverse objects.

One of the main results of [AMRW2] is that when the realization $\mathfrak{h}$ comes from a Kac–Moody root datum, there is an equivalence of triangulated categories between $\mathsf{RE}(\mathfrak{h}, W)$ and $\mathsf{RE}(\mathfrak{h}^*, W)$, known as the *Koszul duality*. The main reason for the restriction to the Kac–Moody setting is that some of the arguments make use of the perverse t-structure from [AR1]. We expect that the methods developed in the present paper will allow one to drop this restriction.

1.7 Relation With Previous Work

As mentioned already, the idea of using the recollement formalism in this kind of setting comes from [AR1]. Makisumi [Mak] showed how to adapt the constructions of [AR1] to the setting of sheaves on moment graphs. In general, this notion, which arose as a kind of combinatorial model for torus-equivariant geometry, takes us away from the world of Coxeter groups, but it overlaps with the results of the present paper in the following special situation: for a Coxeter group equipped with a reflection faithful representation, the category of Soergel bimodules (and hence the Elias–Williamson diagrammatic category) is equivalent to the category of sheaves on the Bruhat moment graph. In this setting, Makisumi's constructions and ours are equivalent. Under the further assumption that Soergel's conjecture holds for the representation under consideration, this t-structure can also be defined purely in terms of Soergel bimodules [Mak, Remark 5.7]. Because moment graphs are closer to geometry (both in spirit and because of the existence of moment graphs modelling partial flag varieties), the arguments in [Mak] avoid some of the difficulties mentioned in Sections 1.3, 1.4.

Separately, a different approach to defining a category $\mathcal{O}$ for a general Coxeter group was proposed by Fiebig [Fi] in terms of sheaves on the Bruhat moment graph, and studied further by Abe [Ab]. Compared to their point of view, ours is Koszul dual; in their picture the indecomposable Soergel bimodules correspond to projective objects, whereas for us they correspond to parity objects, *i.e.*, semisimple complexes when $\Bbbk$ is a field of characteristic 0.

1.8 Contents

Section 2 contains notation and conventions related to graded modules, and Section 3 contains background on the Elias–Williamson diagrammatic category and on the categories $\mathsf{BE}(\mathfrak{h}, W)$ and $\mathsf{RE}(\mathfrak{h}, W)$. In Section 4, we study the diagrammatic analogues of parity complexes on locally closed subsets of the flag variety. This is needed in order to formulate the recollement theorem, which is proved in Section 5. Next, Section 6 is devoted to the study of standard and costandard objects. This section also contains the proof of the categorification result mentioned in §1.6.

The definition and some basic properties of the perverse t-structure on $\mathsf{BE}(\mathfrak{h}, W)$ appear in Section 7. In Section 8, we specialize to the case of field coefficients. Much of the work in this section is aimed at understanding the composition factors of standard and costandard perverse objects.

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In Section 9, we turn our attention to $\mathsf{RE}(\mathfrak{h}, W)$. Many statements carry over from $\mathsf{BE}(\mathfrak{h}, W)$, but there are two new results here: one about the full faithfulness of the forgetful functor, and another about the highest weight structure on $\mathsf{P}^{\mathsf{RE}}(\mathfrak{h}, W)$ for field coefficients. One may then ask what the Ringel dual of this highest weight category is. We conclude the paper in Section 10 with a proof that $\mathsf{P}^{\mathsf{RE}}(\mathfrak{h}, W)$ is self-Ringel-dual.

2 Preliminaries

2.1 Graded Categories

Let $\Bbbk$ be a commutative ring, and let $\mathscr{A}$ be a small $\Bbbk$-linear category that is enriched over $\mathbb{Z}$-graded $\Bbbk$-modules. Recall that this means that for any X, Y in $\mathscr{A}$, the set of morphisms from X to Y in $\mathscr{A}$ is a graded $\Bbbk$-module

$$\mathrm{Hom}^\bullet_{\mathscr{A}}(X, Y) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}^n_{\mathscr{A}}(X, Y),$$

and that composition is defined by morphisms of graded $\Bbbk$-modules, which implies that identity morphisms have degree 0. To such a category one can attach a category $\mathscr{A}^\circ$ whose objects are symbols $X(n)$, where X is an object of $\mathscr{A}$ and $n \in \mathbb{Z}$, and whose morphisms are defined by $\mathrm{Hom}_{\mathscr{A}^\circ}(X(n), Y(m)) := \mathrm{Hom}^{m-n}_{\mathscr{A}}(X, Y)$ (with the composition defined in the obvious way). This category admits a natural autoequivalence (1) sending the object $X(n)$ to $X(n+1)$; we will denote its j-th power by (j) for $j \in \mathbb{Z}$. Moreover, each orbit of the group $\{(j) : j \in \mathbb{Z}\}$ on the set of objects of $\mathscr{A}^\circ$ admits a distinguished representative $X(0)$.

On the other hand, let $\mathscr{B}$ be a small $\Bbbk$-linear category endowed with an autoequivalence (1), whose j-th power will be denoted (j), and a set of representatives of the orbits of $\{(j) : j \in \mathbb{Z}\}$ on the set of objects of $\mathscr{B}$. Then one can define a category $\mathscr{B}^{\mathbb{Z}}$ enriched over graded $\Bbbk$-modules as follows. The objects of $\mathscr{B}^{\mathbb{Z}}$ are the representatives considered above, and the morphisms are defined by

$$\mathrm{Hom}^\bullet_{\mathscr{B}^{\mathbb{Z}}}(X, Y) := \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathscr{B}}(X, Y(n)).$$

It is not difficult to check that the assignments $\mathscr{A} \mapsto \mathscr{A}^\circ$ and $\mathscr{B} \mapsto \mathscr{B}^{\mathbb{Z}}$ are inverse to each other, in the sense that there exist canonical equivalences $(\mathscr{A}^\circ)^{\mathbb{Z}} \cong \mathscr{A}$ and $(\mathscr{B}^{\mathbb{Z}})^\circ \cong \mathscr{B}$ of categories enriched over graded $\Bbbk$-modules and of $\Bbbk$-linear categories, respectively. For this reason, in the body of the paper we will sometimes not be very careful about the distinction between these points of view, and write *e.g.*, $\mathrm{Hom}^\bullet_{\mathscr{B}}(M, N)$ for $\bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathscr{B}}(M, N(n))$.

2.2 Tensor Products With R-modules

Let $\Bbbk$ and $\mathscr{B}$ be as in Section 2.1. We also let R be a commutative $\mathbb{Z}$-graded $\Bbbk$-algebra, and assume that R acts on the objects of the category $\mathscr{B}^{\mathbb{Z}}$ in the sense that for any M in $\mathscr{B}$ and $r \in R^n$, we have morphisms $\rho^M_r : M \to M(n)$ and $\lambda^M_r : M \to M(n)$ such that

$$\rho^{M(n)}_{r'} \circ \rho^M_r = \rho^M_{rr'} \quad \text{and} \quad \lambda^{M(n)}_{r'} \circ \lambda^M_r = \lambda^M_{r'r},$$

for $r \in R^n$ and $r' \in R^m$, and that satisfy $\rho_r^{M(1)} = \rho_r^M(1)$ and $\lambda_r^{M(1)} = \lambda_r^M(1)$, for any M in $\mathscr{B}$ and $r \in R^n$, $\rho_r^N \circ f = (f(n)) \circ \rho_r^M$ and $\lambda_r^N \circ f = (f(n)) \circ \lambda_r^M$ for any M, N in $\mathscr{B}$, $r \in R^n$, and $f \in \mathrm{Hom}_{\mathscr{B}}(M, N)$, and finally that if r is in the image of $\Bbbk$ in R^0 and M is in $\mathscr{B}$, then $\rho_r^M = \lambda_r^M$ is the action given by the $\Bbbk$-linear structure on $\mathscr{B}$.

If X is in $\mathscr{B}$ and if M is a $\mathbb{Z}$-graded left R-module that is free of finite rank, then we define $X \underline{\otimes}_R M$ as the object representing the functor

$$Y \mapsto (\mathrm{Hom}^\bullet_{\mathscr{B}}(Y, X) \otimes_R M)^0,$$

where the superscript 0 means the degree-0 part, and where the right action of R on $\mathrm{Hom}^\bullet_{\mathscr{B}}(Y, X)$ is defined by $f \cdot r = (f(n)) \circ \lambda_r^Y = \lambda_r^{X(m)} \circ f$ for $f \in \mathrm{Hom}^m_{\mathscr{B}}(Y, X)$ and $r \in R^n$. Then we have a natural isomorphism

$$\mathrm{Hom}^\bullet_{\mathscr{B}}(Y, X \underline{\otimes}_R M) \cong \mathrm{Hom}^\bullet_{\mathscr{B}}(Y, X) \otimes_R M. \tag{2.1}$$

In practice, any choice of a graded basis $(e_i)_{i \in I}$ of M as a left R-module defines an identification $X \underline{\otimes}_R M \cong \bigoplus_{i \in I} X(-\deg(e_i))$. Moreover, if $(f_j)_{j \in J}$ is another graded basis of M, then there exist unique homogeneous coefficients $a_{i,j} \in R$ such that $e_i = \sum_j a_{i,j} \cdot f_j$ for any i, j, and the matrix $(\lambda_{a_{ij}}^{X(-\deg f_j)})_{j \in J, i \in I}$ gives an isomorphism

$$\bigoplus_{i \in I} X(-\deg(e_i)) \xrightarrow{\sim} \bigoplus_{j \in J} X(-\deg(f_j)).$$

The morphisms $\lambda_r^{X \underline{\otimes}_R M}$ and $\rho_r^{X \underline{\otimes}_R M}$ are induced in the natural way by λ_r^X and ρ_r^X, respectively.

Now let X, Y be in $\mathscr{B}$. We consider the left R-action on $\mathrm{Hom}^\bullet_{\mathscr{B}}(X, Y)$ given by $r \cdot f = \lambda_r^{Y(m)} \circ f$ for $f \in \mathrm{Hom}^m_{\mathscr{B}}(X, Y)$ and $r \in R^n$. (In other words we consider the same action as before, but now considered as a *left* action.) We assume that this action makes $\mathrm{Hom}^\bullet_{\mathscr{B}}(X, Y)$ a graded free left R-module. We claim that in this situation there exists a canonical morphism

$$X \underline{\otimes}_R \mathrm{Hom}^\bullet_{\mathscr{B}}(X, Y) \to Y. \tag{2.2}$$

In fact, if $(\varphi_i)_{i \in I}$ is a graded basis of the left R-module $\mathrm{Hom}^\bullet_{\mathscr{B}}(X, Y)$, then this choice identifies the left-hand side with $\bigoplus_{i \in I} X(-\deg(\varphi_i))$, and (2.2) can be defined as

$$\bigoplus_{i \in I} \varphi_i(-\deg(\varphi_i)).$$

It can be easily checked that this morphism does not depend on the choice of basis, and hence is indeed canonical. For any Z in $\mathscr{B}$, the induced morphism

$$\mathrm{Hom}^\bullet_{\mathscr{B}}\big(Z, X \underline{\otimes}_R \mathrm{Hom}^\bullet_{\mathscr{B}}(X, Y)\big) \longrightarrow \mathrm{Hom}^\bullet_{\mathscr{B}}(Z, Y)$$

identifies, via (2.1), with the morphism induced by composition in $\mathscr{B}$.

2.3 Derived Category and Free Modules

For some results in this paper we will impose the following assumptions on our (commutative) base ring $\Bbbk$.

(1) $\Bbbk$ is an integral domain;
(2) $\Bbbk$ is Noetherian and of finite global dimension;
(3) every projective finitely generated $\Bbbk$-module is free.

Here Assumption (1) is needed in order to apply the results of [EW2].[3] Assumption (2) is standard, and ensures that the bounded derived category of finitely generated $\Bbbk$-modules has favorable behavior (and similarly for graded modules, and for rings of polynomials with coefficients in $\Bbbk$). Finally, Assumption (3) allows us to describe an appropriate derived category in terms of free modules; see Lemma 2.1 below. Of course, these assumptions are satisfied if $\Bbbk$ is a field or the ring of integers in a finite extension of $\mathbb{Q}_p$ or a finite localization of $\mathbb{Z}$. (These are the typical examples the reader can keep in mind.) Assumption (3) is also known to hold when $\Bbbk$ is local [Ma, Theorem 2.5]. (Note that here we only need the trivial special case of Kaplansky's theorem when the module is of finite type.)

So, in this subsection we assume that $\Bbbk$ satisfies properties (2)–(3) above. We let V be a left graded $\Bbbk$-module that is free of finite rank and concentrated in positive degrees. Then we denote by R the symmetric algebra of V, which we consider as a graded $\Bbbk$-algebra. We will denote by $\mathrm{Mod}^{\mathrm{fg},\mathbb{Z}}(R)$ the abelian category of finitely generated graded left R-modules, and by $\mathrm{Free}^{\mathrm{fg},\mathbb{Z}}(R)$ the full subcategory whose objects are the free finitely generated graded left R-modules.

Lemma 2.1 *The natural functor $K^{\mathrm{b}}\mathrm{Free}^{\mathrm{fg},\mathbb{Z}}(R) \to D^{\mathrm{b}}\mathrm{Mod}^{\mathrm{fg},\mathbb{Z}}(R)$ is an equivalence of triangulated categories.*

Proof Since $\Bbbk$ has finite global dimension, the same property holds for R. Hence any bounded complex of graded R-modules is quasi-isomorphic to a bounded complex of projective graded R-modules, and to conclude, it suffices to prove that any finitely generated projective graded R-module is, in fact, graded free. However, if M is a finitely generated projective graded R-module, then $\Bbbk \otimes_R M$ is a finitely generated projective graded $\Bbbk$-module (where $\Bbbk$ is concentrated in degree 0, and R acts on $\Bbbk$ via the quotient $R/V \cdot R = \Bbbk$), and hence is graded free by Assumption (3). Then we deduce that M is graded free by the graded Nakayama lemma. ∎

2.4 Terminology

In the body of the paper we will use the following terminology. If $(X, \le)$ is a poset, we will say that a subset $Y \subset X$ is *closed* if for any $x, x' \in X$ with $x' \in Y$ and $x \le x'$ we have $x \in Y$. A subset $Z \subset X$ will be called *open* if $X \smallsetminus Z$ is closed. Finally we will say that $Y \subset X$ is *locally closed* if it is the intersection of an open and a closed subset or, equivalently, if Y is open in $\overline{Y} = \{x \in X \mid \exists y \in Y \text{ such that } x \le y\}$ or, equivalently, if Y is closed in $X \smallsetminus (\overline{Y} \smallsetminus Y)$. A basic observation that we will use repeatedly is that if $x \in X$, then x is minimal for $\le$ if and only if $\{x\}$ is a closed subset of X, and x is maximal for $\le$ if and only if $\{x\}$ is an open subset of X.

In this context, if $x \in X$, we will write $\{\le x\}$, resp. $\{< x\}$, for $\{z \in X \mid z \le x\}$, resp. $\{z \in X \mid z < x\}$; these subsets are closed in X.

[3] This assumption is not explicit in [EW2] but, as noted in particular in [AMRW1, Footnote on p. 10], it is in fact needed.

3 The Elias–Williamson Diagrammatic Category

Henceforth, we let $\Bbbk$ be an integral domain.

3.1 Notation and Terminology Regarding Coxeter Systems

For the rest of this paper we fix a Coxeter system (W, S) with S finite. Then W is equipped with the Bruhat order $\leq$ and the length function ℓ.

A word $\underline{w}$ in S will be called an *expression*. The length $\ell(\underline{w})$ of an expression $\underline{w}$ is the number of letters in this word. We will by denote $\pi(\underline{w})$ the corresponding element in W (obtained as the product in W of the letters of $\underline{w}$); then we will say that $\underline{w}$ is *an expression for* (or that $\underline{w}$ *expresses*) $\pi(\underline{w}) \in W$. Recall also that an expression $\underline{w}$ is said to be *reduced* if $\ell(\underline{w}) = \ell(\pi(\underline{w}))$.

If $\underline{w} = (s_1, \dots, s_n)$ is an expression, a *subexpression* of $\underline{w}$ is defined to be a sequence $\mathbf{e} = (e_1, \dots, e_n)$ of 0's and 1's. Such a datum determines an expression $\underline{v} = (s_{i_1}, \dots, s_{i_m})$, where $1 \leq i_1 < \cdots < i_m \leq n$ are the indices such that $\{i_1, \dots, i_m\} = \{i \in \{1, \dots, n\} \mid e_i = 1\}$. In a minor abuse of language, we will also say that $\mathbf{e}$ expresses $\pi(\underline{v})$. If $x \in W$, we will denote by $M(\underline{w}, x)$ the set of subexpressions of $\underline{w}$ expressing x.

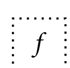
With this terminology, the Bruhat order on W can be described as follows: if $\underline{w}$ is a reduced expression for $w \in W$, then $v \leq w$ if and only if v is expressed by some subexpression of $\underline{w}$.

3.2 The Elias–Williamson Category

Let $\mathfrak{h} = (V, \{\alpha_s^\vee : s \in S\}, \{\alpha_s : s \in S\})$ be a balanced realization of (W, S) over $\Bbbk$ that satisfies Demazure surjectivity in the sense of [EW2] (see also [AMRW1, §2.1]). In particular, V is a free $\Bbbk$-module of finite rank, $\{\alpha_s^\vee : s \in S\}$ is a subset of V, and $\{\alpha_s : s \in S\}$ is a subset of $V^* := \mathrm{Hom}_\Bbbk(V, \Bbbk)$.

We let R be the symmetric algebra of V^*, considered as a graded ring with V^* in degree 2. Following Elias–Williamson [EW2] (see also [AMRW1, §2.2]), we associate a $\Bbbk$-linear monoidal category with (W, S) and $\mathfrak{h}$ as follows. First, one defines a $\Bbbk$-linear monoidal category $\widetilde{\mathscr{D}}_{\mathrm{BS}}(\mathfrak{h}, W)$ enriched over graded $\Bbbk$-modules as follows:

- objects are the symbols $B_{\underline{w}}$ for $\underline{w}$ an expression, with the monoidal product defined by $B_{\underline{v}} \star B_{\underline{w}} = B_{\underline{vw}}$;
- morphisms are generated (under composition, monoidal product, and $\Bbbk$-linear combinations) by the following elementary morphisms:
  - for any homogeneous $f \in R$, a morphism



    from $B_\varnothing$ to itself, of degree $\deg(f)$;
  - for any $s \in S$, dot morphisms

and

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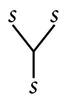
from B_s to $B_\varnothing$ and from $B_\varnothing$ to B_s, respectively, of degree 1;
- for any $s \in S$, trivalent morphisms



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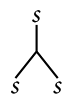
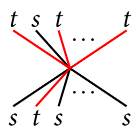
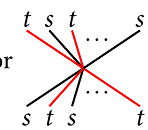


from B_s to $B_{(s,s)}$ and from $B_{(s,s)}$ to B_s, respectively, of degree -1;
- for any pair (s,t) of distinct simple reflections such that st has finite order m_{st} in W, a morphism



if m_{st} is odd or


if m_{st} is even

from $B_{(s,t,\cdots)}$ to $B_{(t,s,\cdots)}$ (where each expression has length m_{st}, and letters alternate), of degree 0,

subject to a number of relations [EW2] [AMRW1, §2.2].

Then we set $\mathscr{D}_{\mathrm{BS}}(\mathfrak{h}, W) := \big(\widetilde{\mathscr{D}}_{\mathrm{BS}}(\mathfrak{h}, W)\big)^{\circ}$ (where we use the notation from §2.1). We will also denote by $\mathscr{D}^{\oplus}_{\mathrm{BS}}(\mathfrak{h}, W)$ the additive hull of $\mathscr{D}_{\mathrm{BS}}(\mathfrak{h}, W)$.

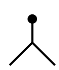
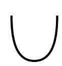
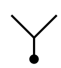
Typically, a morphism in $\mathscr{D}_{\mathrm{BS}}(\mathfrak{h}, W)$ or in $\widetilde{\mathscr{D}}_{\mathrm{BS}}(\mathfrak{h}, W)$ will be written as a linear combination of (equivalence classes of) diagrams, where horizontal concatenation corresponds to the monoidal product and vertical concatenation corresponds to composition. Such diagrams are to be read from bottom to top. We will sometimes omit the labels "s" or "t" in the diagrams for morphisms when they do not play any role.

Note that for X, Y in $\mathscr{D}^{\oplus}_{\mathrm{BS}}(\mathfrak{h}, W)$, the graded $\Bbbk$-module

$$\mathrm{Hom}^{\bullet}_{\mathscr{D}^{\oplus}_{\mathrm{BS}}(\mathfrak{h},W)}(X, Y) := \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{\mathscr{D}^{\oplus}_{\mathrm{BS}}(\mathfrak{h},W)}(X, Y(n)) \tag{3.1}$$

has a natural structure of a graded R-bimodule, where the left, resp., right, action of $f \in R^n$ is induced by adding a box labelled by f to the left, resp., right, of a diagram.

We set

 :=  ,  :=  .

These morphisms induce morphisms of functors

$$\big(B_s \star (-)\big) \circ \big(B_s \star (-)\big) \longrightarrow \mathrm{id} \quad \text{and} \quad \mathrm{id} \longrightarrow \big(B_s \star (-)\big) \circ \big(B_s \star (-)\big)$$

which make $(B_s \star (-), B_s \star (-))$ an adjoint pair. Similarly, $((-) \star B_s, (-) \star B_s)$ is an adjoint pair in a natural way. Let us recall also, from [EW2, (5.14)], the isomorphism

$$B_s \star B_s \cong B_s(1) \oplus B_s(-1). \tag{3.2}$$

This category has another symmetry that turns out to be very useful. We denote by $\mathbb{D} \colon \mathscr{D}^{\oplus}_{\mathrm{BS}}(\mathfrak{h}, W) \to \mathscr{D}^{\oplus}_{\mathrm{BS}}(\mathfrak{h}, W)^{\mathrm{op}}$ the anti-involution that fixes each $B_{\underline{w}}$ and flips diagrams upside-down [EW2, Definition 6.22]. Notice that $\mathbb{D} \circ (n) \simeq (-n) \circ \mathbb{D}$.

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3.3 The Double Leaves Basis

One of the main results of [EW2] states that for any two expressions $\underline{v}, \underline{w}$, the graded R-bimodule $\mathrm{Hom}^{\bullet}_{\mathscr{D}_{\mathrm{BS}}(\mathfrak{h},W)}(B_{\underline{v}}, B_{\underline{w}})$ is graded free of finite rank as a left R-module and as a right R-module [EW2, Corollary 6.14]. In fact, following an idea of Libedinsky, Elias and Williamson provided a way to produce a set $\mathbb{LL}_{\underline{v},\underline{w}}$ of homogeneous morphisms, called *double leaves morphisms*, which constitutes a graded basis of

$$\mathrm{Hom}^{\bullet}_{\mathscr{D}_{\mathrm{BS}}(\mathfrak{h},W)}(B_{\underline{v}}, B_{\underline{w}}),$$

both as a left R-module and as a right R-module. This construction is algorithmic in nature, and depends on many choices. We will not repeat the construction here, but we will recall certain properties that we will need below.

The set $\mathbb{LL}_{\underline{v},\underline{w}}$ is in natural bijection with the set $\bigcup_{x \in W} M(\underline{w}, x) \times M(\underline{v}, x)$. In fact, if $\mathbf{e}$ and $\mathbf{f}$ are subexpressions of $\underline{v}$ and $\underline{w}$, respectively, expressing the same element $x \in W$, then the procedure of [EW2, §6.1] produces homogeneous elements $\mathrm{LL}_{\underline{v},\mathbf{e}} \in \mathrm{Hom}^{\bullet}_{\mathscr{D}_{\mathrm{BS}}(\mathfrak{h},W)}(B_{\underline{v}}, B_{\underline{x}})$ and $\mathrm{LL}_{\underline{w},\mathbf{f}} \in \mathrm{Hom}^{\bullet}_{\mathscr{D}_{\mathrm{BS}}(\mathfrak{h},W)}(B_{\underline{w}}, B_{\underline{x}})$ for a certain reduced expression $\underline{x}$ for x (which can be chosen arbitrarily), and then one defines

$$\mathbb{LL}^{\underline{v},\underline{w}}_{x,\mathbf{f},\mathbf{e}} := \mathbb{D}(\mathrm{LL}_{\underline{w},\mathbf{f}}) \circ \mathrm{LL}_{\underline{v},\mathbf{e}}$$

and sets $\mathbb{LL}_{\underline{v},\underline{w}} = \left\{ \mathbb{LL}^{\underline{v},\underline{w}}_{x,\mathbf{f},\mathbf{e}} : (\mathbf{f},\mathbf{e}) \in \bigcup_{x \in W} M(\underline{w}, x) \times M(\underline{v}, x) \right\}$.

Note in particular that if $\underline{v}$ and $\underline{w}$ are reduced expressions, then the element x above must satisfy $x \leq \pi(\underline{v})$ and $x \leq \pi(\underline{w})$.

Example 3.1. Let $s \in S$. The (left or right) R-modules $\mathrm{Hom}^{\bullet}_{\mathscr{D}_{\mathrm{BS}}(\mathfrak{h},W)}(B_s, B_{\varnothing})$ and $\mathrm{Hom}^{\bullet}_{\mathscr{D}_{\mathrm{BS}}(\mathfrak{h},W)}(B_{\varnothing}, B_s)$ are of rank 1, with generators

and


,

respectively.

3.4 The Biequivariant and the Right Equivariant Categories

In [AMRW1], Makisumi, Williamson and the first two authors of the present paper studied various triangulated categories constructed out of $\mathscr{D}_{\mathrm{BS}}(\mathfrak{h}, W)$. The two cases that we will mainly consider in this paper are:

- the biequivariant[4] category $\mathsf{BE}(\mathfrak{h}, W)$, which can be defined as
$$\mathsf{BE}(\mathfrak{h}, W) := K^{\mathrm{b}} \mathscr{D}^{\oplus}_{\mathrm{BS}}(\mathfrak{h}, W);$$
- the right-equivariant category $\mathsf{RE}(\mathfrak{h}, W)$, which can be defined as
$$\mathsf{RE}(\mathfrak{h}, W) := K^{\mathrm{b}} \overline{\mathscr{D}}^{\oplus}_{\mathrm{BS}}(\mathfrak{h}, W).$$

Here, $\overline{\mathscr{D}}^{\oplus}_{\mathrm{BS}}(\mathfrak{h}, W)$ is the additive hull of the category $\overline{\mathscr{D}}_{\mathrm{BS}}(\mathfrak{h}, W)$ obtained by the procedure $(-)^{\circ}$ of Section 2.1 out of the category obtained from $\widetilde{\mathscr{D}}_{\mathrm{BS}}(\mathfrak{h}, W)$ by applying

[4]The motivation for our terminology comes from geometry; see [AMRW1] for details.

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$\Bbbk \otimes_R (-)$ to morphism spaces (where again $\Bbbk$ is in degree 0, and R acts via the quotient $R/V \cdot R = \Bbbk$). For an expression $\underline{w}$, we will denote by $\overline{B}_{\underline{w}}$ the image of $B_{\underline{w}}$ in $\overline{\mathscr{D}}^{\oplus}_{\mathrm{BS}}(\mathfrak{h}, W)$.

The category $\mathsf{BE}(\mathfrak{h}, W)$ has a natural monoidal structure, which extends the product $\star$ on $\mathscr{D}^{\oplus}_{\mathrm{BS}}(\mathfrak{h}, W)$, and whose product will be denoted $\underline{\star}$; see [AMRW1, §4.2] for details. (This construction involves some rather delicate sign conventions, which will not be recalled in detail here.) As in Section 3.2, the pairs of functors

$$(B_s \underline{\star} (-), B_s \underline{\star} (-)) \quad \text{and} \quad ((-) \underline{\star} B_s, (-) \underline{\star} B_s)$$

form adjoint pairs in a natural way. The unit for this product is $B_{\varnothing}$. The category $\mathsf{RE}(\mathfrak{h}, W)$ is in a natural way a right-module category over $\mathsf{BE}(\mathfrak{h}, W)$; this operation is also denoted $\underline{\star}$. There also exists a natural forgetful functor $\mathsf{For}^{\mathsf{BE}}_{\mathsf{RE}} \colon \mathsf{BE}(\mathfrak{h}, W) \to \mathsf{RE}(\mathfrak{h}, W)$ induced by tensoring morphism spaces with $\Bbbk$ (over R); this functor satisfies

$$\mathsf{For}^{\mathsf{BE}}_{\mathsf{RE}}(\mathscr{F} \underline{\star} \mathscr{G}) = \mathsf{For}^{\mathsf{BE}}_{\mathsf{RE}}(\mathscr{F}) \underline{\star} \mathscr{G} \tag{3.3}$$

for $\mathscr{F}, \mathscr{G}$ in $\mathsf{BE}(\mathfrak{h}, W)$.

The cohomological shift functors on the triangulated categories $\mathsf{BE}(\mathfrak{h}, W)$ and $\mathsf{RE}(\mathfrak{h}, W)$ will be denoted $[1]$. These categories possess two other shift autoequivalences denoted $\langle 1 \rangle$ and (1). Here (1) extends the operation on $\mathscr{D}^{\oplus}_{\mathrm{BS}}(\mathfrak{h}, W)$ denoted similarly in the following way: it sends a complex $(\mathscr{F}^n, d^n)_{n \in \mathbb{Z}}$ to the complex

$$(\mathscr{F}^n(1), -d^n)_{n \in \mathbb{Z}},$$

and we have $(1) = \langle -1 \rangle [1]$. The m-th power of $[1]$, resp., $\langle 1 \rangle$, resp., (1), is denoted $[m]$, resp., $\langle m \rangle$, resp., (m).

4 Diagrammatic Categories Associated With Locally Closed Subsets of W

We continue with the setting of Section 3. In particular, $\Bbbk$ is only required to be an integral domain.

4.1 The Diagrammatic Category Attached to a Closed Subset

Let $I \subset W$ be a closed subset. We define the category $\mathscr{D}_{\mathrm{BS},I}(\mathfrak{h}, W)$ as the full subcategory of $\mathscr{D}_{\mathrm{BS}}(\mathfrak{h}, W)$ whose objects are of the form $B_{\underline{w}}(n)$ for $n \in \mathbb{Z}$ and $\underline{w}$ a reduced expression for an element in I. We will also denote by $\mathscr{D}^{\oplus}_{\mathrm{BS},I}(\mathfrak{h}, W)$ the additive hull of $\mathscr{D}_{\mathrm{BS},I}(\mathfrak{h}, W)$; this category identifies in a natural way with the full subcategory of $\mathscr{D}^{\oplus}_{\mathrm{BS}}(\mathfrak{h}, W)$ whose objects are the direct sums of objects of the form $B_{\underline{w}}(n)$ with $\underline{w}$ a reduced expression for an element in I.

Remark 4.1. We warn the reader that in the case $I = W$, it is not clear (and most probably false) that the category $\mathscr{D}^{\oplus}_{\mathrm{BS},W}(\mathfrak{h}, W)$ is equivalent to $\mathscr{D}^{\oplus}_{\mathrm{BS}}(\mathfrak{h}, W)$, since the latter contains objects $B_{\underline{w}}$, where $\underline{w}$ is not a reduced expression. Nevertheless, we will see later on that their homotopy categories are equivalent (Remark 6.3 (1)).

Note that the anti-involution $\mathbb{D}$ stabilizes the subcategory $\mathscr{D}^{\oplus}_{\mathrm{BS},I}(\mathfrak{h}, W)$; its restriction will be denoted $\mathbb{D}_I$. As in $\mathscr{D}^{\oplus}_{\mathrm{BS}}(\mathfrak{h}, W)$, for B, B' in $\mathscr{D}^{\oplus}_{\mathrm{BS},I}(\mathfrak{h}, W)$ we set

$$\mathrm{Hom}^{\bullet}_{\mathscr{D}^{\oplus}_{\mathrm{BS},I}(\mathfrak{h},W)}(B, B') = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathscr{D}^{\oplus}_{\mathrm{BS},I}(\mathfrak{h},W)}(B, B'(n)).$$

If B, B' are objects of $\mathscr{D}^{\oplus}_{\mathrm{BS}}(\mathfrak{h}, W)$, we will denote by

$$\mathfrak{F}_I(B, B') \subset \mathrm{Hom}^{\bullet}_{\mathscr{D}^{\oplus}_{\mathrm{BS}}(\mathfrak{h},W)}(B, B')$$

the submodule of morphisms that factor through $\mathscr{D}^{\oplus}_{\mathrm{BS},I}(\mathfrak{h}, W)$.

Lemma 4.2 *If $\underline{v}$ and $\underline{w}$ are expressions, then $\mathfrak{F}_I(B_{\underline{v}}, B_{\underline{w}})$ is the R-span (under either the left or right action) of the double leaves morphisms $\mathbb{LL}^{\underline{v},\underline{w}}_{x,\mathbf{f},\mathbf{e}}$ with $x \in I$.*

Proof To fix notation, we consider the left action of R. It is clear from the definition that if $x \in I$, then $\mathbb{LL}^{\underline{v},\underline{w}}_{x,\mathbf{f},\mathbf{e}} \in \mathfrak{F}_I(B_{\underline{v}}, B_{\underline{y}})$. In particular, the R-span under consideration is contained in $\mathfrak{F}_I(B_{\underline{v}}, B_{\underline{w}})$.

For the opposite containment, we will prove that for any reduced expression $\underline{y}$ for an element of I, any morphism that factors through a shift of $B_{\underline{y}}$ belongs to the R-span of the light-leaves morphisms $\mathbb{LL}^{\underline{v},\underline{w}}_{x,\mathbf{f},\mathbf{e}}$ with $x \in I$. Let $\underline{y}$ be as above, and let $f \in \mathrm{Hom}^{\bullet}_{\mathscr{D}^{\oplus}_{\mathrm{BS}}(\mathfrak{h},W)}(B_{\underline{v}}, B_{\underline{w}})$ be a morphism that factors through a shift of $B_{\underline{y}}$. Since the light-leaves morphisms form an R-basis of $\mathrm{Hom}^{\bullet}_{\mathscr{D}^{\oplus}_{\mathrm{BS}}(\mathfrak{h},W)}(B_{\underline{v}}, B_{\underline{y}})$, we can assume that $f = g\mathbb{LL}^{\underline{v},\underline{y}}_{x,\mathbf{f},\mathbf{e}}$ for some subexpressions $\mathbf{e}, \mathbf{f}$ of $\underline{v}$ and $\underline{y}$, respectively, expressing some element x. Here $x \leq \pi(\underline{y})$, and hence $x \in I$. By [EW2, Claim 6.21], f is then an R-linear combination of light-leaves morphisms $\mathbb{LL}^{\underline{v},\underline{w}}_{x,\mathbf{f}',\mathbf{e}}$ for some subexpressions $\mathbf{f}'$ of $\underline{w}$ expressing x, and some light-leaves morphisms corresponding to subexpressions expressing certain elements $x' < x$. Here again $x' \in I$, so the result follows. ∎

4.2 The Diagrammatic Category Attached to a Locally Closed Subset

Let $I_0 \subset W$ be a closed subset, and let $I_1 \subset I_0$ be closed. Then I_1 is also closed in W, so that we can consider the categories $\mathscr{D}^{\oplus}_{\mathrm{BS},I_0}(\mathfrak{h}, W)$ and $\mathscr{D}^{\oplus}_{\mathrm{BS},I_1}(\mathfrak{h}, W)$. We set

$$\mathscr{D}^{\oplus}_{\mathrm{BS},I_0,I_1}(\mathfrak{h}, W) := \mathscr{D}^{\oplus}_{\mathrm{BS},I_0}(\mathfrak{h}, W) /\!\!/ \mathscr{D}^{\oplus}_{\mathrm{BS},I_1}(\mathfrak{h}, W),$$

where the naive quotient on the right-hand side is defined as follows: its objects are the same as those of $\mathscr{D}^{\oplus}_{\mathrm{BS},I_0}(\mathfrak{h}, W)$, and its morphisms are defined by

$$\mathrm{Hom}_{\mathscr{D}^{\oplus}_{\mathrm{BS},I_0,I_1}(\mathfrak{h},W)}(B, B') = \left(\mathrm{Hom}^{\bullet}_{\mathscr{D}^{\oplus}_{\mathrm{BS},I_0}(\mathfrak{h},W)}(B, B')/\mathfrak{F}_{I_1}(B, B')\right)^0$$

for B, B' in $\mathscr{D}^{\oplus}_{\mathrm{BS},I_0}(\mathfrak{h}, W)$, where the superscript "0" means the degree-0 part. Note that the objects $B_{\underline{w}}$ with $\underline{w}$ a reduced expression for an element in I_1 have trivial images in $\mathscr{D}^{\oplus}_{\mathrm{BS},I_0,I_1}(\mathfrak{h}, W)$. In particular, every object of $\mathscr{D}^{\oplus}_{\mathrm{BS},I_0,I_1}(\mathfrak{h}, W)$ is a direct sum of (images of) objects of the form $B_{\underline{w}}(m)$, where $m \in \mathbb{Z}$ and $\underline{w}$ is a reduced expression for an element in $I_0 \smallsetminus I_1$.

Of course the shift equivalence (1) induces an autoequivalence of the category $\mathscr{D}^{\oplus}_{\mathrm{BS},I_0,I_1}(\mathfrak{h}, W)$, which will be denoted similarly. If B, B' are in $\mathscr{D}^{\oplus}_{\mathrm{BS},I_0}(\mathfrak{h}, W)$, the

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left and right actions of R on $\mathrm{Hom}^{\bullet}_{\mathscr{D}^{\oplus}_{\mathrm{BS},I_0}(\mathfrak{h},W)}(B,B')$ descend to actions on

$$\mathrm{Hom}^{\bullet}_{\mathscr{D}^{\oplus}_{\mathrm{BS},I_0,I_1}(\mathfrak{h},W)}(B,B') := \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathscr{D}^{\oplus}_{\mathrm{BS},I_0,I_1}(\mathfrak{h},W)}(B,B'(n)).$$

Moreover, if $B = B_{\underline{v}}$ and $B' = B_{\underline{w}}$ where $\underline{v}, \underline{w}$ are reduced expressions for elements of I_0, then it follows from Lemma 4.2 that $\mathrm{Hom}^{\bullet}_{\mathscr{D}^{\oplus}_{\mathrm{BS},I_0,I_1}(\mathfrak{h},W)}(B_{\underline{v}},B_{\underline{w}})$ is free as a left and as a right graded R-module, and that the images of the light-leaves morphisms $\mathbb{LL}^{\underline{v},\underline{w}}_{x,\mathbf{f},\mathbf{e}}$ with $x \in I_0 \smallsetminus I_1$ form a graded basis of this space (both as a left and as a right R-module). More generally, this implies that for arbitrary B, B' in $\mathscr{D}^{\oplus}_{\mathrm{BS},I_0,I_1}(\mathfrak{h},W)$, the space $\mathrm{Hom}^{\bullet}_{\mathscr{D}^{\oplus}_{\mathrm{BS},I_0,I_1}(\mathfrak{h},W)}(B,B')$ is graded free both as a left and as a right R-module.

Lemma 4.3 *Up to canonical equivalence, the category $\mathscr{D}^{\oplus}_{\mathrm{BS},I_0,I_1}(\mathfrak{h},W)$ only depends on the locally closed subset $I_0 \smallsetminus I_1$.*

Proof Let $I := I_0 \smallsetminus I_1$. Then I_0 contains $\overline{I} := \{z \in W \mid \exists x \in I, z \leq x\}$, so that we have a natural inclusion of categories $\mathscr{D}^{\oplus}_{\mathrm{BS},\overline{I}}(\mathfrak{h},W) \subset \mathscr{D}^{\oplus}_{\mathrm{BS},I_0}(\mathfrak{h},W)$ that induces a functor

$$\mathscr{D}^{\oplus}_{\mathrm{BS},\overline{I},\overline{I}\smallsetminus I}(\mathfrak{h},W) \longrightarrow \mathscr{D}^{\oplus}_{\mathrm{BS},I_0,I_1}(\mathfrak{h},W).$$

The description of morphism spaces in $\mathscr{D}^{\oplus}_{\mathrm{BS},I_0,I_1}(\mathfrak{h},W)$ in terms of light-leaves morphisms considered above implies that this functor is fully faithful. By the remarks above, it is also essentially surjective, and hence an equivalence. ∎

From Lemma 4.3 it follows that it makes sense to define, for any locally closed subset $I \subset W$, the category $\mathscr{D}^{\oplus}_{\mathrm{BS},I}(\mathfrak{h},W)$ as

$$\mathscr{D}^{\oplus}_{\mathrm{BS},I}(\mathfrak{h},W) = \mathscr{D}^{\oplus}_{\mathrm{BS},I_0,I_1}(\mathfrak{h},W) = \mathscr{D}^{\oplus}_{\mathrm{BS},I_0}(\mathfrak{h},W) /\!\!/ \mathscr{D}^{\oplus}_{\mathrm{BS},I_1}(\mathfrak{h},W),$$

where $I_1 \subset I_0$ are any closed subsets of W such that $I = I_0 \smallsetminus I_1$. Of course, if I is closed, the category we obtain coincides with the category defined in Section 4.1. It is clear that the autoequivalences $\mathbb{D}_{I_0}$ and $\langle 1\rangle$ of $\mathscr{D}^{\oplus}_{\mathrm{BS},I_0}(\mathfrak{h},W)$ induce autoequivalences of $\mathscr{D}^{\oplus}_{\mathrm{BS},I}(\mathfrak{h},W)$ that will be denoted $\mathbb{D}_I$ and $\langle 1\rangle$, respectively.

4.3 The Case of a Singleton

In this subsection we consider the special case $I = \{w\}$ for $w \in W$. (This subset is obviously locally closed in W.) For any choice of a reduced expression $\underline{w}$ for w, we can consider the image of the corresponding object $B_{\underline{w}}$ in $\mathscr{D}^{\oplus}_{\mathrm{BS},\{w\}}(\mathfrak{h},W)$. If $\underline{w}'$ is another reduced expression for w, then $\underline{w}$ and $\underline{w}'$ can be related by a *rex move*, *i.e.*, a sequence of braid relations (meaning the replacement of a subword $(s,t,\dots)$ by the word $(t,s,\dots)$, where the words have length the order $m_{s,t}$ of st and their entries alternate between s and t). See [EW2, §4.2] for details. A morphism in $\mathscr{D}^{\oplus}_{\mathrm{BS}}(\mathfrak{h},W)$ is associated (by definition) with each such braid relation; composing these morphisms we obtain a *rex move morphism* $B_{\underline{w}} \to B_{\underline{w}'}$. By [EW2, Lemma 7.4, Lemma 7.5], the image of this morphism in $\mathscr{D}^{\oplus}_{\mathrm{BS},\{w\}}(\mathfrak{h},W)$ does not depend on the choice of rex move, and is an isomorphism. In particular, the images of $B_{\underline{w}}$ and $B_{\underline{w}'}$ in $\mathscr{D}^{\oplus}_{\mathrm{BS},\{w\}}(\mathfrak{h},W)$ are

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canonically isomorphic. Hence they define a canonical object in $\mathscr{D}^{\oplus}_{\mathrm{BS},\{w\}}(\mathfrak{h}, W)$ that will be denoted b_w.

Lemma 4.4 *There exists a canonical equivalence of categories*

$$\gamma\colon \mathscr{D}^{\oplus}_{\mathrm{BS},\{w\}}(\mathfrak{h}, W) \xrightarrow{\sim} \mathrm{Free}^{\mathrm{fg},\mathbb{Z}}(R)$$

such that $\gamma(b_w) = R$*. Under this equivalence, the autoequivalence* (1) *identifies with the* shift of grading *autoequivalence of* $\mathrm{Free}^{\mathrm{fg},\mathbb{Z}}(R)$ *defined by* $(M(1))^n = M^{n+1}$.

Proof It follows from the definition and the comments above that any object of $\mathscr{D}^{\oplus}_{\mathrm{BS},\{w\}}(\mathfrak{h}, W)$ is isomorphic to a direct sum of shifts of b_w. Moreover, since we have $\mathrm{End}^{\bullet}_{\mathscr{D}^{\oplus}_{\mathrm{BS},\{w\}}(\mathfrak{h},W)}(b_w) = R$ by Lemma 4.2, we deduce that the functor

$$\gamma := \mathrm{Hom}^{\bullet}_{\mathscr{D}^{\oplus}_{\mathrm{BS},\{w\}}(\mathfrak{h},W)}(b_w, -)$$

provides the desired equivalence. ∎

4.4 Closed and Open Inclusions

Let $I \subset W$ be a locally closed subset, and write $I = I_0 \smallsetminus I_1$ for some closed subsets $I_1 \subset I_0 \subset W$. Any subset $J \subset I$ that is closed as a subset of I can be written as $J_0 \smallsetminus (J_0 \cap I_1)$ for some closed subset $J_0 \subset I_0$. There exists a natural embedding

$$\mathscr{D}^{\oplus}_{\mathrm{BS},J_0}(\mathfrak{h}, W) \subset \mathscr{D}^{\oplus}_{\mathrm{BS},I_0}(\mathfrak{h}, W),$$

which induces a functor

$$\mathscr{D}^{\oplus}_{\mathrm{BS},J_0}(\mathfrak{h}, W) /\!\!/ \mathscr{D}^{\oplus}_{\mathrm{BS},J_0\cap I_1}(\mathfrak{h}, W) \longrightarrow \mathscr{D}^{\oplus}_{\mathrm{BS},I_0}(\mathfrak{h}, W) /\!\!/ \mathscr{D}^{\oplus}_{\mathrm{BS},I_1}(\mathfrak{h}, W).$$

The description of morphism spaces in terms of light-leaves morphisms in Section 4.2 shows that this functor is fully faithful. As explained in Section 4.2, the categories involved here do not depend on the choices of I_0 and J_0. It is clear that, under these identifications, the functor does not depend on these choices either; it will be denoted

$$(i^I_J)_*\colon \mathscr{D}^{\oplus}_{\mathrm{BS},J}(\mathfrak{h}, W) \longrightarrow \mathscr{D}^{\oplus}_{\mathrm{BS},I}(\mathfrak{h}, W).$$

It is clear that this functor satisfies $(i^I_J)_* \circ \mathbb{D}_J \cong \mathbb{D}_I \circ (i^I_J)_*$, and that this construction is compatible with composition of closed inclusions in the obvious way.

Now let $K \subset I$ be a subset that is open in the order topology on I. Let $J = I \smallsetminus K$ be the complementary closed subset, and write $J = J_0 \smallsetminus (J_0 \cap I_1)$ as above, so that $K = I_0 \smallsetminus K_1$, where $K_1 = J_0 \cup I_1$. Then by definition there exists a natural full functor

$$\mathscr{D}^{\oplus}_{\mathrm{BS},I_0}(\mathfrak{h}, W) /\!\!/ \mathscr{D}^{\oplus}_{\mathrm{BS},I_1}(\mathfrak{h}, W) \longrightarrow \mathscr{D}^{\oplus}_{\mathrm{BS},I_0}(\mathfrak{h}, W) /\!\!/ \mathscr{D}^{\oplus}_{\mathrm{BS},K_1}(\mathfrak{h}, W).$$

Once again, this functor does not depend on the choices of I_0 and J_0; it will be denoted

$$(i^I_K)^*\colon \mathscr{D}^{\oplus}_{\mathrm{BS},I}(\mathfrak{h}, W) \longrightarrow \mathscr{D}^{\oplus}_{\mathrm{BS},K}(\mathfrak{h}, W).$$

This functor satisfies $(i^I_K)^* \circ \mathbb{D}_I = \mathbb{D}_K \circ (i^I_K)^*$, and this construction is compatible with composition of open inclusions in the obvious way.

It is clear from this construction that if $J \subset I$ is closed, we have

$$(i^I_{I\smallsetminus J})^* \circ (i^I_J)_* = 0. \tag{4.1}$$

Example 4.5. Let $I \subset W$ be a locally closed subset and let $w \in I$ be a minimal element. Then the subset $\{w\} \subset I$ is closed. Let us fix a reduced expression $\underline{w}$ for w. If $\underline{x}$ and $\underline{y}$ are reduced expressions for elements of I, then by Lemma 4.2 (see also Section 4.2) the subsets $\{\mathbb{LL}^{\underline{x},\underline{w}}_{w,\mathbf{1},\mathbf{e}} : \mathbf{e} \in M(\underline{x}, w)\}$ and $\{\mathbb{LL}^{\underline{w},\underline{y}}_{w,\mathbf{f},\mathbf{1}} : \mathbf{f} \in M(\underline{y}, w)\}$, where $\mathbf{1}$ means the subexpression consisting only of 1's, form R-bases of the modules $\mathrm{Hom}^{\bullet}_{\mathscr{D}^{\oplus}_{\mathrm{BS},I}(\mathfrak{h},W)}(B_{\underline{x}}, B_{\underline{w}})$ and $\mathrm{Hom}^{\bullet}_{\mathscr{D}^{\oplus}_{\mathrm{BS},I}(\mathfrak{h},W)}(B_{\underline{w}}, B_{\underline{y}})$, respectively, both for the left and for the right actions. Moreover, composition induces a morphism

$$\mathrm{Hom}^{\bullet}_{\mathscr{D}^{\oplus}_{\mathrm{BS},I}(\mathfrak{h},W)}(B_{\underline{w}}, B_{\underline{y}}) \otimes_R \mathrm{Hom}^{\bullet}_{\mathscr{D}^{\oplus}_{\mathrm{BS},I}(\mathfrak{h},W)}(B_{\underline{x}}, B_{\underline{w}}) \longrightarrow \mathrm{Hom}^{\bullet}_{\mathscr{D}^{\oplus}_{\mathrm{BS},I}(\mathfrak{h},W)}(B_{\underline{x}}, B_{\underline{y}}),$$

where the right R-module structure on $\mathrm{Hom}^{\bullet}_{\mathscr{D}^{\oplus}_{\mathrm{BS},I}(\mathfrak{h},W)}(B_{\underline{w}}, B_{\underline{y}})$ and the left R-module structure on $\mathrm{Hom}^{\bullet}_{\mathscr{D}^{\oplus}_{\mathrm{BS},I}(\mathfrak{h},W)}(B_{\underline{x}}, B_{\underline{w}})$ are both given either by adding a box to the right of diagrams, or by adding a box to the left of diagrams. Considerations of the light-leaves basis from Section 4.2 also show that this morphism is injective for both choices of conventions for R-actions.

Remark 4.6. Let $I \subset W$ be a locally closed subset, and let $J \subset I$ be a subset that is both open and closed. Then from the definitions we see that $(i^I_J)^* \circ (i^I_J)_* = \mathrm{id}$.

Moreover, using the light-leaves basis for morphisms in $\mathscr{D}^{\oplus}_{\mathrm{BS},I}(\mathfrak{h}, W)$ (Section4.2), it is not difficult to check that for any B in $\mathscr{D}_{\mathrm{BS},J}(\mathfrak{h}, W)$ and B' in $\mathscr{D}^{\oplus}_{\mathrm{BS},I\smallsetminus J}(\mathfrak{h}, W)$ we have $\mathrm{Hom}_{\mathscr{D}^{\oplus}_{\mathrm{BS},I}(\mathfrak{h},W)}((i^I_J)_*B, (i^I_{I\smallsetminus J})_*B') = 0$. It follows that any object B of $\mathscr{D}^{\oplus}_{\mathrm{BS},I}(\mathfrak{h}, W)$ has a canonical decomposition $B \cong (i^I_J)_*B' \oplus (i^I_{I\smallsetminus J})_*B''$ with B' in $\mathscr{D}^{\oplus}_{\mathrm{BS},J}(\mathfrak{h}, W)$ and B'' in $\mathscr{D}^{\oplus}_{\mathrm{BS},I\smallsetminus J}(\mathfrak{h}, W)$, and that we have $B' = (i^I_J)^*B$ and $B'' = (i^I_{I\smallsetminus J})^*B$. From this we deduce that the pairs $((i^I_J)^*, (i^I_J)_*)$ and $((i^I_J)_*, (i^I_J)^*)$ are adjoint pairs of functors.

5 Recollement

We continue with the setting of Sections 3 and 4. Our goal in this rather technical section is to construct a recollement formalism (in the sense of [BBD, §1.4.3]) for the category $\mathsf{BE}(\mathfrak{h}, W)$ that will allow us to describe this category in terms of local versions associated with locally closed subsets of W.

5.1 The Biequivariant Category Associated With a Locally Closed Subset

If $I \subset W$ is a locally closed subset, we define the triangulated category $\mathsf{BE}_I(\mathfrak{h}, W)$ by setting $\mathsf{BE}_I(\mathfrak{h}, W) := K^{\mathrm{b}}\mathscr{D}^{\oplus}_{\mathrm{BS},I}(\mathfrak{h}, W)$. As for $\mathsf{BE}(\mathfrak{h}, W)$, this category admits "shift" autoequivalences $[n]$, $\langle n\rangle$, (n) defined as above (for $n \in \mathbb{Z}$). The contravariant autoequivalence $\mathbb{D}_I$ of $\mathscr{D}^{\oplus}_{\mathrm{BS},I}(\mathfrak{h}, W)$ also induces a (contravariant) autoequivalence of $\mathsf{BE}_I(\mathfrak{h}, W)$ that will be denoted similarly. By definition we have $\mathbb{D}_I \circ [n] = [-n] \circ \mathbb{D}_I$, $\mathbb{D}_I \circ \langle n\rangle = \langle -n\rangle \circ \mathbb{D}_I$, and $\mathbb{D}_I \circ (n) = (-n) \circ \mathbb{D}_I$.

If $J \subset I$ is a closed subset, then the functor $(i^I_J)_*$ defined in Section 4.4 induces a fully faithful functor from $\mathsf{BE}_J(\mathfrak{h}, W)$ to $\mathsf{BE}_I(\mathfrak{h}, W)$ that will also be denoted $(i^I_J)_*$. Whenever convenient, we will identify $\mathsf{BE}_J(\mathfrak{h}, W)$ with its image in $\mathsf{BE}_I(\mathfrak{h}, W)$, and omit the functor $(i^I_J)_*$.

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Similarly, if $K \subset I$ is an open subset, then the functor $(i_K^I)^*$ defined in Section 4.4 induces a functor from $\mathsf{BE}_I(\mathfrak{h}, W)$ to $\mathsf{BE}_K(\mathfrak{h}, W)$ that will also be denoted $(i_K^I)^*$. As in Section 4.4, we have

$$(i_J^I)_* \circ \mathbb{D}_J = \mathbb{D}_I \circ (i_J^I)_*, \quad (i_K^I)^* \circ \mathbb{D}_I = \mathbb{D}_K \circ (i_K^I)^*. \tag{5.1}$$

Note that the functors $(i_J^I)_*$ identify the category $\mathsf{BE}_I(\mathfrak{h}, W)$ with the inductive limit of the categories $\mathsf{BE}_J(\mathfrak{h}, W)$, for $J \subset I$ a finite closed subset. This observation will allow us to generalize some of our constructions below from finite subsets of W to arbitrary subsets.

5.2 Closed Embedding of a Singleton

In this subsection we fix a locally closed subset $I \subset W$ and a minimal element $w \in I$, so that $\{w\}$ is a closed subset of I. Our goal is to prove Lemma 5.1 below.

The statement of this lemma involves the "$*$" operation from [BBD, §1.3.9]. We recall the definition of this notation: if $\mathcal{D}$ is a triangulated category, and if $\mathcal{A}, \mathcal{B} \subset \mathcal{D}$ are two full subcategories, then $\mathcal{A} * \mathcal{B}$ denotes the strictly full subcategory of $\mathcal{D}$ whose objects X are those that fit into a distinguished triangle

$$A \longrightarrow X \longrightarrow B \xrightarrow{[1]}$$

with $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

Lemma 5.1 *The functor $(i_{I \smallsetminus \{w\}}^I)^*$ admits a left adjoint $(i_{I \smallsetminus \{w\}}^I)_!$ and a right adjoint $(i_{I \smallsetminus \{w\}}^I)_*$. Moreover, the adjunction morphisms*

$$(i_{I \smallsetminus \{w\}}^I)^* (i_{I \smallsetminus \{w\}}^I)_* \longrightarrow \mathrm{id} \quad \textit{and} \quad \mathrm{id} \longrightarrow (i_{I \smallsetminus \{w\}}^I)^* (i_{I \smallsetminus \{w\}}^I)_!$$

are isomorphisms, and we have

$$\mathsf{BE}_I(\mathfrak{h}, W) = (i_{I \smallsetminus \{w\}}^I)_! (\mathsf{BE}_{I \smallsetminus \{w\}}(\mathfrak{h}, W)) * (i_{\{w\}}^I)_* (\mathsf{BE}_{\{w\}}(\mathfrak{h}, W)),$$
$$\mathsf{BE}_I(\mathfrak{h}, W) = (i_{\{w\}}^I)_* (\mathsf{BE}_{\{w\}}(\mathfrak{h}, W)) * (i_{I \smallsetminus \{w\}}^I)_* (\mathsf{BE}_{I \smallsetminus \{w\}}(\mathfrak{h}, W)).$$

The proof of this lemma will use the following construction. We fix once and for all a reduced expression $\underline{w}$ for w. Then for any reduced expression $\underline{x}$ for an element in $I \smallsetminus \{w\}$, we consider the complex $B_{\underline{x}}^+$ given by

$$\cdots \longrightarrow 0 \longrightarrow B_{\underline{w}} \underline{\otimes}_R \operatorname{Hom}^\bullet_{\mathscr{D}^\oplus_{\mathrm{BS},I}(\mathfrak{h},W)}(B_{\underline{w}}, B_{\underline{x}}) \longrightarrow B_{\underline{x}} \to 0 \longrightarrow \cdots,$$

where $B_{\underline{w}} \underline{\otimes}_R \operatorname{Hom}^\bullet_{\mathscr{D}^\oplus_{\mathrm{BS},I}(\mathfrak{h},W)}(B_{\underline{w}}, B_{\underline{x}})$ is in cohomological degree -1, $B_{\underline{x}}$ is in cohomological degree 0, all the other terms are 0, and the only nontrivial differential is given by the morphism defined in (2.2). (In particular, the R-module structure on $\operatorname{Hom}^\bullet_{\mathscr{D}^\oplus_{\mathrm{BS},I}(\mathfrak{h},W)}(B_{\underline{w}}, B_{\underline{x}})$ that we consider here is as defined before (2.2).) Note that we have a canonical distinguished triangle

$$B_{\underline{x}} \longrightarrow B_{\underline{x}}^+ \longrightarrow B_{\underline{w}} \underline{\otimes}_R \operatorname{Hom}^\bullet_{\mathscr{D}^\oplus_{\mathrm{BS},I}(\mathfrak{h},W)}(B_{\underline{w}}, B_{\underline{x}})[1] \xrightarrow{[1]} \tag{5.2}$$

in $\mathsf{BE}_I(\mathfrak{h}, W)$.

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Lemma 5.2 *If $\underline{x}$ is a reduced expression for an element in $I \smallsetminus \{w\}$ and $\underline{y}$ is a reduced expression for an element in I, then for any $n, m \in \mathbb{Z}$, the functor $(i^I_{I\smallsetminus\{w\}})^*$ induces an isomorphism*

$$\mathrm{Hom}_{\mathsf{BE}_I(\mathfrak{h},W)}(B_{\underline{y}}, B^+_{\underline{x}}(m)[n]) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{BE}_{I\smallsetminus\{w\}}(\mathfrak{h},W)}((i^I_{I\smallsetminus\{w\}})^* B_{\underline{y}}, (i^I_{I\smallsetminus\{w\}})^* B^+_{\underline{x}}(m)[n]).$$

Moreover, these $\Bbbk$-modules are zero unless $\pi(\underline{y}) \neq w$ and $n = 0$, in which case they are isomorphic to $\mathrm{Hom}_{\mathscr{D}^{\oplus}_{\mathrm{BS},I\smallsetminus\{w\}}(\mathfrak{h},W)}(B_{\underline{y}}, B_{\underline{x}}(m))$.

Proof It is clear that in the morphism under consideration, the left-hand side vanishes unless $n \in \{-1, 0\}$ and the right-hand side vanishes unless $n = 0$, because $(i^I_{I\smallsetminus\{w\}})^* B^+_{\underline{x}} = (i^I_{I\smallsetminus\{w\}})^* B_{\underline{x}}$. In particular, the claim is obvious, unless $n \in \{-1, 0\}$.

From (5.2) we deduce an exact sequence

$$\begin{aligned}0 \longrightarrow \mathrm{Hom}_{\mathsf{BE}_I(\mathfrak{h},W)}(B_{\underline{y}}, B^+_{\underline{x}}(m)[-1]) \\ \longrightarrow \mathrm{Hom}_{\mathscr{D}^{\oplus}_{\mathrm{BS},I}(\mathfrak{h},W)}\big(B_{\underline{y}}, B_{\underline{w}} \otimes_R \mathrm{Hom}^{\bullet}_{\mathscr{D}^{\oplus}_{\mathrm{BS},I}(\mathfrak{h},W)}(B_{\underline{w}}, B_{\underline{x}})(m)\big) \\ \longrightarrow \mathrm{Hom}_{\mathscr{D}^{\oplus}_{\mathrm{BS},I}(\mathfrak{h},W)}(B_{\underline{y}}, B_{\underline{x}}(m)) \longrightarrow \mathrm{Hom}_{\mathsf{BE}_I(\mathfrak{h},W)}(B_{\underline{y}}, B^+_{\underline{x}}(m)) \longrightarrow 0.\end{aligned}$$

Now by definition (Section 2.2) the term on the middle line identifies with

$$\big(\mathrm{Hom}^{\bullet}_{\mathscr{D}^{\oplus}_{\mathrm{BS},I}(\mathfrak{h},W)}(B_{\underline{y}}, B_{\underline{w}}(m)) \otimes_R \mathrm{Hom}^{\bullet}_{\mathscr{D}^{\oplus}_{\mathrm{BS},I}(\mathfrak{h},W)}(B_{\underline{w}}, B_{\underline{x}})\big)^0,$$

and the differential to $\mathrm{Hom}_{\mathscr{D}^{\oplus}_{\mathrm{BS},I}(\mathfrak{h},W)}(B_{\underline{y}}, B_{\underline{x}}(m))$ identifies with the natural composition morphism. As explained in Example 4.5 this map is injective. It follows that $\mathrm{Hom}_{\mathsf{BE}_I(\mathfrak{h},W)}(B_{\underline{y}}, B^+_{\underline{x}}(m)[-1]) = 0$, proving the desired isomorphism in this case. The fact that our morphism is an isomorphism when $n = 0$ also follows from this exact sequence, together with the light-leaves basis considerations in Section 4.2. ■

Proof of Lemma 5.1 We will explain the construction of the functor $(i^I_{I\smallsetminus\{w\}})_*$ and prove that it satisfies the desired properties; then in view of (5.1) the functor

$$(i^I_{I\smallsetminus\{w\}})_! := \mathbb{D}_I \circ (i^I_{I\smallsetminus\{w\}})_* \circ \mathbb{D}_{I\smallsetminus\{w\}} \tag{5.3}$$

will also satisfy the corresponding properties.

Let $\mathsf{D}^+ \subset \mathsf{BE}_I(\mathfrak{h}, W)$ be the full graded, *i.e.*, stable by (1), triangulated subcategory generated by the objects $B^+_{\underline{x}}$ for all reduced expressions $\underline{x}$ for elements in $I \smallsetminus \{w\}$, and let $\iota\colon \mathsf{D}^+ \to \mathsf{BE}_I(\mathfrak{h}, W)$ be the inclusion. By Lemma 5.2 and using the five-lemma, it follows that the functor $(i^I_{I\smallsetminus\{w\}})^*$ induces an isomorphism

$$\mathrm{Hom}_{\mathsf{BE}_I(\mathfrak{h},W)}(Y, \iota X) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{BE}_{I\smallsetminus\{w\}}(\mathfrak{h},W)}((i^I_{I\smallsetminus\{w\}})^* Y, (i^I_{I\smallsetminus\{w\}})^* \iota X) \tag{5.4}$$

for all X in D^+ and Y in $\mathsf{BE}_I(\mathfrak{h}, W)$. In particular, this shows that the functor

$$(i^I_{I\smallsetminus\{w\}})^* \circ \iota$$

is fully faithful. Moreover, since this functor sends $B^+_{\underline{x}}(m)$ to $B_{\underline{x}}(m)$ for any reduced expression $\underline{x}$ of an element in $I \smallsetminus \{w\}$, and since these objects generate D^+

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and $\mathsf{BE}_{I\smallsetminus\{w\}}(\mathfrak{h}, W)$, respectively, as triangulated categories, we even obtain that

$$(i^I_{I\smallsetminus\{w\}})^* \circ \iota$$

is an equivalence of categories. This fact allows us to set

$$(i^I_{I\smallsetminus\{w\}})_* := \iota \circ \big((i^I_{I\smallsetminus\{w\}})^* \circ \iota \big)^{-1} \colon \mathsf{BE}_{I\smallsetminus\{w\}}(\mathfrak{h}, W) \longrightarrow \mathsf{BE}_I(\mathfrak{h}, W).$$

What remains to be proved is that this functor satisfies the desired properties.

By definition we have a canonical isomorphism $(i^I_{I\smallsetminus\{w\}})^* \circ (i^I_{I\smallsetminus\{w\}})_* \cong \mathrm{id}$. To prove that $(i^I_{I\smallsetminus\{w\}})_*$ is right adjoint to $(i^I_{I\smallsetminus\{w\}})^*$, we need to prove that the composition

$$\begin{aligned}\mathrm{Hom}_{\mathsf{BE}_I(\mathfrak{h},W)}(X, (i^I_{I\smallsetminus\{w\}})_* Y) &\xrightarrow{(i^I_{I\smallsetminus\{w\}})^*} \\ &\mathrm{Hom}_{\mathsf{BE}_{I\smallsetminus\{w\}}(\mathfrak{h},W)}\big((i^I_{I\smallsetminus\{w\}})^* X, (i^I_{I\smallsetminus\{w\}})^* (i^I_{I\smallsetminus\{w\}})_* Y \big) \\ &\qquad \cong \mathrm{Hom}_{\mathsf{BE}_{I\smallsetminus\{w\}}(\mathfrak{h},W)}((i^I_{I\smallsetminus\{w\}})^* X, Y)\end{aligned}$$

is an isomorphism for all X in $\mathsf{BE}_I(\mathfrak{h}, W)$ and Y in $\mathsf{BE}_{I\smallsetminus\{w\}}(\mathfrak{h}, W)$. In fact, this is clear from the isomorphism (5.4).

To conclude, it remains to prove that

$$\mathsf{BE}_I(\mathfrak{h}, W) = (i^I_{\{w\}})_*(\mathsf{BE}_{\{w\}}(\mathfrak{h}, W)) * (i^I_{I\smallsetminus\{w\}})_*(\mathsf{BE}_{I\smallsetminus\{w\}}(\mathfrak{h}, W)). \tag{5.5}$$

However, by construction we have

$$(i^I_{I\smallsetminus\{w\}})_* B_{\underline{x}} = B^+_{\underline{x}} \tag{5.6}$$

for any reduced expression $\underline{x}$ for an element in $I\smallsetminus\{w\}$. In view of the triangle (5.2) and the comments at the beginning of Section 4.3, it follows that the triangulated category $\mathsf{BE}_I(\mathfrak{h}, W)$ is generated by the essential images of the functors $(i^I_{I\smallsetminus\{w\}})_*$ and $(i^I_{\{w\}})_*$. Since there exists no nonzero morphism from an object of $(i^I_{\{w\}})_*(\mathsf{BE}_{\{w\}}(\mathfrak{h}, W))$ to an object of $(i^I_{I\smallsetminus\{w\}})_*(\mathsf{BE}_{I\smallsetminus\{w\}}(\mathfrak{h}, W))$ (by adjunction and the fact that $(i^I_{I\smallsetminus\{w\}})^* \circ (i^I_{\{w\}})_* = 0$, see (4.1)), we deduce (5.5). ■

Remark 5.3. The claims in Lemma 5.1 about the adjunction morphisms amount to saying that the functors $(i^I_{I\smallsetminus\{w\}})_*$ and $(i^I_{I\smallsetminus\{w\}})_!$ are fully faithful.

Example 5.4. Let $w \in W$ and $s \in S$ be such that $ws > w$. Then $\{w\}$ is closed in $\{w, ws\}$, and its open complement is $\{ws\}$. If $\underline{w}$ is a reduced expression for w, then there exist canonical distinguished triangles

$$B_{\underline{w}}\langle -1\rangle \longrightarrow \big(i^{\{w,ws\}}_{\{ws\}}\big)_! B_{\underline{ws}} \longrightarrow B_{\underline{ws}} \xrightarrow{[1]} \tag{5.7}$$

$$B_{\underline{ws}} \longrightarrow \big(i^{\{w,ws\}}_{\{ws\}}\big)_* B_{\underline{ws}} \longrightarrow B_{\underline{w}}\langle 1\rangle \xrightarrow{[1]} \tag{5.8}$$

in $\mathsf{BE}_{\{w,ws\}}(\mathfrak{h}, W)$. In fact, the R-module $\mathrm{Hom}^\bullet_{\mathscr{D}^\oplus_{\mathrm{BS},\{w,ws\}}(\mathfrak{h},W)}(B_{\underline{w}}, B_{\underline{ws}})$ is generated by

$\mathrm{id}_{B_{\underline{w}}} \star$  ,

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which has degree 1. Hence (5.8) is a special case of the triangle (5.2), and (5.7) is deduced by applying $\mathbb{D}_{\{w,ws\}}$ (see also (5.6)).

Below we will need the following technical result.

Lemma 5.5 *Let I and w be as in Lemma 5.1, and let $J \subset I$ be a closed subset containing w. Then there exist canonical isomorphisms*

$$(i^I_{I\smallsetminus\{w\}})_! \circ (i^{I\smallsetminus\{w\}}_{J\smallsetminus\{w\}})_* \cong (i^I_J)_* \circ (i^J_{J\smallsetminus\{w\}})_!, \quad (i^I_{I\smallsetminus\{w\}})_* \circ (i^{I\smallsetminus\{w\}}_{J\smallsetminus\{w\}})_* \cong (i^I_J)_* \circ (i^J_{J\smallsetminus\{w\}})_*.$$

Proof As for Lemma 5.1, we only prove the second isomorphism; the first one follows by composing on the left with $\mathbb{D}_I$ and on the right with $\mathbb{D}_{J\smallsetminus\{w\}}$ (see (5.1), (5.3)). We consider the subcategories $\mathsf{D}^+_I \subset \mathsf{BE}_I(\mathfrak{h}, W)$ and $\mathsf{D}^+_J \subset \mathsf{BE}_J(\mathfrak{h}, W)$ constructed in the proof of Lemma 5.1 (applied to the "ambient" locally closed subsets I and J, respectively), and the corresponding embeddings ι_I and ι_J. It is clear that the functor $(i^I_J)_* \circ \iota_J$ factors through a functor $(i^I_J)^+_*\colon \mathsf{D}^+_J \to \mathsf{D}^+_I$. It is clear also that $(i^I_{I\smallsetminus\{w\}})^* \circ (i^I_J)_* = (i^{I\smallsetminus\{w\}}_{J\smallsetminus\{w\}})_* \circ (i^J_{J\smallsetminus\{w\}})^*$. We deduce that

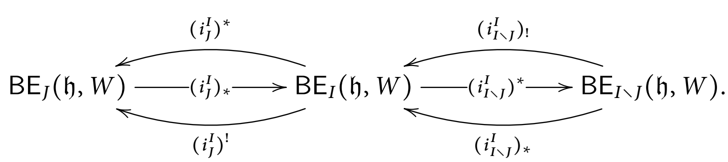
$$(i^I_{I\smallsetminus\{w\}})^* \circ \iota_I \circ (i^I_J)^+_* = (i^{I\smallsetminus\{w\}}_{J\smallsetminus\{w\}})_* \circ (i^J_{J\smallsetminus\{w\}})^* \circ \iota_J.$$

Composing on the left with $(i^I_{I\smallsetminus\{w\}})_* := \iota_I \circ \big((i^I_{I\smallsetminus\{w\}})^* \circ \iota_I\big)^{-1}$ and on the right with $\big((i^J_{J\smallsetminus\{w\}})^* \circ \iota_J\big)^{-1}$, we deduce the desired isomorphism. ■

5.3 Recollement

We now formulate and prove the main result of the section.

Proposition 5.6 *Let $I \subset W$ be a locally closed subset, and let $J \subset I$ be a finite closed subset. Then the functor $(i^I_{I\smallsetminus J})^*\colon \mathsf{BE}_I(\mathfrak{h}, W) \to \mathsf{BE}_{I\smallsetminus J}(\mathfrak{h}, W)$ admits a left adjoint $(i^I_{I\smallsetminus J})_!$ and a right adjoint $(i^I_{I\smallsetminus J})_*$. Similarly, the functor $(i^I_J)_*\colon \mathsf{BE}_J(\mathfrak{h}, W) \to \mathsf{BE}_I(\mathfrak{h}, W)$ admits a left adjoint $(i^I_J)^*$ and a right adjoint $(i^I_J)^!$. Together, these functors give a recollement diagram*



Proof We begin by showing, by induction on $|J|$, that

- the functor $(i^I_{I\smallsetminus J})_!$ exists;
- the adjunction morphism $\mathrm{id} \to (i^I_{I\smallsetminus J})^* \circ (i^I_{I\smallsetminus J})_!$ is an isomorphism;
- we have

$$\mathsf{BE}_I(\mathfrak{h}, W) = (i^I_{I\smallsetminus J})_!(\mathsf{BE}_{I\smallsetminus J}(\mathfrak{h}, W)) * (i^I_J)_*(\mathsf{BE}_J(\mathfrak{h}, W)). \tag{5.9}$$

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If $|J| = 1$, these assertions are part of the statement of Lemma 5.1. If $|J| > 1$, we pick $w \in J$ minimal. By induction the functors

$$(i_{I\smallsetminus J}^{I\smallsetminus\{w\}})^*\colon \mathsf{BE}_{I\smallsetminus\{w\}}(\mathfrak{h}, W) \longrightarrow \mathsf{BE}_{I\smallsetminus J}(\mathfrak{h}, W)$$

and

$$(i_{I\smallsetminus\{w\}}^{I})^*\colon \mathsf{BE}_{I}(\mathfrak{h}, W) \longrightarrow \mathsf{BE}_{I\smallsetminus\{w\}}(\mathfrak{h}, W)$$

admit left adjoints $(i_{I\smallsetminus J}^{I\smallsetminus\{w\}})_!$ and $(i_{I\smallsetminus\{w\}}^{I})_!$, respectively. Hence their composition, which is $(i_{I\smallsetminus J}^{I})^*$ (Section 4.4), also admits a left adjoint $(i_{I\smallsetminus J}^{I})_!$, and we have

$$(i_{I\smallsetminus J}^{I})_! = (i_{I\smallsetminus\{w\}}^{I})_! \circ (i_{I\smallsetminus J}^{I\smallsetminus\{w\}})_!.$$

From the corresponding claims for the embeddings $i_{I\smallsetminus J}^{I\smallsetminus\{w\}}$ and $i_{I\smallsetminus\{w\}}^{I}$ it is not difficult to deduce that the adjunction morphism

$$\mathrm{id} \to (i_{I\smallsetminus J}^{I})^* \circ (i_{I\smallsetminus J}^{I})_! \tag{5.10}$$

is an isomorphism. Finally, by induction we have

$$\mathsf{BE}_{I\smallsetminus\{w\}}(\mathfrak{h}, W) = (i_{I\smallsetminus J}^{I\smallsetminus\{w\}})_!(\mathsf{BE}_{I\smallsetminus J}(\mathfrak{h}, W)) * (i_{J\smallsetminus\{w\}}^{I\smallsetminus\{w\}})_*(\mathsf{BE}_{J\smallsetminus\{w\}}(\mathfrak{h}, W)),$$
$$\mathsf{BE}_{I}(\mathfrak{h}, W) = (i_{I\smallsetminus\{w\}}^{I})_!(\mathsf{BE}_{I\smallsetminus\{w\}}(\mathfrak{h}, W)) * (i_{\{w\}}^{I})_*(\mathsf{BE}_{\{w\}}(\mathfrak{h}, W)).$$

Using the associativity of the operation "$*$" [BBD, Lemme 1.3.10], Lemma 5.1, and Lemma 5.5, we deduce (5.9), which finishes the induction.

Now we prove the existence of the functor $(i_J^I)^*$ and construct a distinguished triangle

$$(i_{I\smallsetminus J}^{I})_!(i_{I\smallsetminus J}^{I})^*\mathscr{F} \longrightarrow \mathscr{F} \longrightarrow (i_J^I)_*(i_J^I)^*\mathscr{F} \xrightarrow{[1]} \tag{5.11}$$

for any $\mathscr{F}$ in $\mathsf{BE}_I(\mathfrak{h}, W)$. We first observe that both the functors $(i_J^I)_*$ and $(i_{I\smallsetminus J}^{I})_!$ are fully faithful (see Section 5.1 for $(i_J^I)_*$; for $(i_{I\smallsetminus J}^{I})_!$ this follows from the invertibility of (5.10).) Using (5.9), it then follows that for any $\mathscr{F} \in \mathsf{BE}_I(\mathfrak{h}, W)$ there exist unique objects $\mathscr{F}' \in \mathsf{BE}_{I\smallsetminus J}(\mathfrak{h}, W)$ and $\mathscr{F}'' \in \mathsf{BE}_J(\mathfrak{h}, W)$ and a unique distinguished triangle

$$(i_{I\smallsetminus J}^{I})_!\mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow (i_J^I)_*\mathscr{F}'' \xrightarrow{[1]}. \tag{5.12}$$

(Here, the uniqueness claims follow from [BBD, Proposition 1.1.9]). Since we have $(i_{I\smallsetminus J}^{I})^*(i_J^I)_* = 0$ and $(i_{I\smallsetminus J}^{I})^*(i_{I\smallsetminus J}^{I})_! \cong \mathrm{id}$ (see (4.1) and (5.10), respectively), we have a canonical isomorphism $(i_{I\smallsetminus J}^{I})^*\mathscr{F} \cong \mathscr{F}'$. We set $(i_J^I)^*\mathscr{F} := \mathscr{F}''$. Another application of [BBD, Proposition 1.1.9] shows that this defines a functor $(i_J^I)^*$. Then this functor is left adjoint to $(i_J^I)_*$ thanks to the distinguished triangle (5.11) and the fact that $(i_{I\smallsetminus J}^{I})^*(i_J^I)_* = 0$.

Finally, we remark that $(i_J^I)^*(i_J^I)_*\mathscr{G} \cong \mathscr{G}$ for all $\mathscr{G} \in \mathsf{BE}_J(\mathfrak{h}, W)$ by uniqueness of the distinguished triangle (5.12). Composing with the appropriate duality functors, from the existence of the functors $(i_{I\smallsetminus J}^{I})_!$ and $(i_J^I)^*$, we deduce the existence of the functors $(i_{I\smallsetminus J}^{I})_*$ and $(i_J^I)^!$ (see (5.1)), and from the properties we proved for the former functors, we deduce similar properties for the latter functors; this finishes the proof of the proposition. ∎

Remark 5.7. Once the recollement formalism is constructed, we see from [BBD, Proposition 1.4.5] that if $I = I_0 \smallsetminus I_1$ with $I_1 \subset I_0 \subset W$ closed subsets and I_0 finite, then the functor $(i_I^{I_0})^*$ identifies the category $\mathsf{BE}_I(\mathfrak{h}, W)$ with the Verdier quotient of $\mathsf{BE}_{I_0}(\mathfrak{h}, W)$ by the full triangulated subcategory $\mathsf{BE}_{I_1}(\mathfrak{h}, W)$. This remark provides an alternative perspective on $\mathsf{BE}_I(\mathfrak{h}, W)$, separate from that coming from Section 4.2.

Let us point out once again that in the setting of Proposition 5.6 we have canonical isomorphisms

$$\mathbb{D}_I \circ (i^I_{I\smallsetminus J})_! \cong (i^I_{I\smallsetminus J})_* \circ \mathbb{D}_{I\smallsetminus J} \quad \text{and} \quad \mathbb{D}_J \circ (i^I_J)^! \cong (i^I_J)^* \circ \mathbb{D}_I. \tag{5.13}$$

Also, our functors are compatible with composition of inclusions in the sense of the following lemma.

Lemma 5.8 *Let $I \subset W$ be a locally closed subset and let $J' \subset J \subset I$ be finite closed subsets. Then for $\dagger \in \{!, *\}$ we have canonical isomorphisms*

$$(i^I_J)^\dagger \circ (i^J_{J'})^\dagger \cong (i^I_{J'})^\dagger, \qquad (i^I_{I\smallsetminus J'})_\dagger \circ (i^{I\smallsetminus J'}_{I\smallsetminus J})_\dagger \cong (i^I_{I\smallsetminus J})_\dagger.$$

Proof The claim follows by adjunction from the corresponding properties for the functors $(i^I_J)_*$ and $(i^I_{I\smallsetminus J})^*$ (and the similar functors for the other embeddings); see Section 4.4. ∎

Remark 5.9. (1) Assume that I is a finite locally closed subset of W, and that $J \subset I$ is both open and closed. Then we have the naive functors $(i^I_J)_*$ and $(i^I_J)^*$ defined as in Section 5.1, and also the functors constructed (by adjunction) in Proposition 5.6, which we will denote provisionally $(i^I_J)_{(*)}$, $(i^I_J)_{(!)}$, $(i^I_J)^{(*)}$, and $(i^I_J)^{(!)}$. It follows from Remark 4.6 that we have canonical isomorphisms

$$(i^I_J)_{(*)} \cong (i^I_J)_{(!)} \cong (i^I_J)_* \quad \text{and} \quad (i^I_J)^{(*)} \cong (i^I_J)^{(!)} \cong (i^I_J)^*,$$

so that we can stop distinguishing these functors.

(2) We note for later use that if w is minimal in I, then for any B in $\mathscr{D}_{\mathrm{BS},I}(\mathfrak{h}, W)$ we have $(i^I_{\{w\}})^! B \cong b_w \underline{\otimes}_R \operatorname{Hom}^\bullet_{\mathscr{D}^\oplus_{\mathrm{BS},I}(\mathfrak{h},W)}(B_{\underline{w}}, B)$ (so that, in particular, $(i^I_{\{w\}})^! B$ is isomorphic to a complex concentrated in degree 0). In fact, it suffices to prove this isomorphism when $B = B_{\underline{x}}$ for $\underline{x}$, a reduced expression for an element in I. If this element is not w, then the isomorphism is obtained from the triangle (5.2). If now $\underline{x}$ is a reduced expression for w, then the isomorphism follows from the fact that $(i^I_{\{w\}})^!(i^I_{\{w\}})_* \cong \mathrm{id}$ (because $(i^I_{\{w\}})_*$ is fully faithful).

5.4 Pushforward and Pullback Under Locally Closed Inclusions

Our next goal is to define pullback and pushforward functors for any finite *locally closed* inclusion $J \subset I$, where I is locally closed in W.

Lemma 5.10 *Let I be a finite locally closed subset of W, let $J \subset I$ be a closed subset, let $K \subset I$ be an open subset, and let $L \subset J \cap K$ be a subset that is open in J and closed in K. Then for $\dagger \in \{!, *\}$ there exist canonical isomorphisms*

$$(i^I_K)_\dagger \circ (i^K_L)_* \cong (i^I_J)_* \circ (i^J_L)_\dagger, \qquad (i^K_L)^\dagger \circ (i^I_K)^* \cong (i^J_L)^* \circ (i^I_J)^\dagger.$$

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Proof We will show, by induction on $|I|$, the first isomorphism for $\dagger = !$. Then, as in the proof of [AR1, Lemma 2.6], the other isomorphisms follow by duality and adjunction.

We need to consider three cases. First we assume that $I = J = K$. Then L is open and closed in I, and the desired isomorphism follows from Remark 5.9 (1).

Now we assume $K \neq I$. Let $w \in I \smallsetminus K$ be minimal, so that $\{w\}$ is closed in $I \smallsetminus K$, and hence in I. Then $I' = I \smallsetminus \{w\}$ is open in I and $J' := J \cap I' = J \smallsetminus \{w\}$ is closed in I'. By induction we have $(i^{I'}_K)_! \circ (i^K_L)_* \cong (i^{I'}_{J'})_* \circ (i^{J'}_L)_!$, so to conclude, by Lemma 5.8 it suffices to prove that $(i^I_{I'})_! \circ (i^{I'}_{J'})_* \cong (i^I_J)_* \circ (i^J_{J'})_!$. If $w \in J$, then this isomorphism was proved in Lemma 5.5. If now $w \notin J$, then $J' = J$ and J is both open and closed in $J \cup \{w\}$. By Remark 5.9 (1), this implies that $(i^{J\cup\{w\}}_J)_* \cong (i^{J\cup\{w\}}_J)_!$, and then using Lemma 5.5, applied to $J \cup \{w\}$ instead of J, that

$$(i^I_{I'})_! \circ (i^{I'}_J)_* \cong (i^I_{J\cup\{w\}})_* \circ (i^{J\cup\{w\}}_J)_! \cong (i^I_{J\cup\{w\}})_* \circ (i^{J\cup\{w\}}_J)_* \cong (i^I_J)_*.$$

Finally, we consider the case $I = K$, but $J \neq I$. Then L is closed in I, and hence also in J, and by assumption it is also open in J. Hence by Remark 5.9 (1) we have

$$(i^J_L)_! \cong (i^J_L)_*,$$

and the desired isomorphism follows from the compatibility of pushforward functors (for closed embeddings) with composition. ∎

Lemma 5.10 allows us to define pullback and pushforward functors for any locally closed embedding when I is finite. More precisely, let $I \subset W$ be a finite locally closed subset, and let $J \subset I$ be a locally closed subset. Then we can write $J = J_0 \smallsetminus J_1$ with $J_1 \subset J_0 \subset I$ closed subsets. (Here, since J is fixed, J_1 is determined by J_0, and J_0 is determined by J_1.) By Lemma 5.10 we have a canonical isomorphism

$$(i^I_{J_0})_* \circ (i^{J_0}_J)_* \cong (i^I_{I\smallsetminus J_1})_* \circ (i^{I\smallsetminus J_1}_J)_*.$$

Moreover, we claim that these functors do not depend on the choice of J_0 or J_1 (up to canonical isomorphism). In fact, for any choice we have $J_0 \supset \overline{J}$, where

$$\overline{J} := \{w \in I \mid \exists x \in J,\ w \leq x\}.$$

Lemma 5.10 applied to the diagram

$$\begin{array}{ccc} J & \hookrightarrow & J_0 \\ \| & & \uparrow \\ J & \hookrightarrow & \overline{J} \end{array}$$

implies that $(i^{J_0}_J)_* \cong (i^{J_0}_{\overline{J}})_* (i^{\overline{J}}_J)_*$, from which we deduce that

$$(i^I_{J_0})_* \circ (i^{J_0}_J)_* \cong (i^I_{\overline{J}})_* \circ (i^{\overline{J}}_J)_*,$$

which clearly does not depend on J_0. These considerations show that it is legitimate to set $(i^I_J)_* := (i^I_{J_0})_* \circ (i^{J_0}_J)_*$. Similar arguments show that one can also set

$$(i^I_J)_! := (i^I_{J_0})_* \circ (i^{J_0}_J)_!, \quad (i^I_J)^* := (i^{I\smallsetminus J_1}_J)^* \circ (i^I_{I\smallsetminus J_1})^*, \quad (i^I_J)^! := (i^{I\smallsetminus J_1}_J)^! \circ (i^I_{I\smallsetminus J_1})^*,$$

i.e., that these functors do not depend on the choice of J_0 or J_1, and can be expressed in a way where open and closed embeddings play an opposite role. Moreover, the pairs $((i^I_J)_!, (i^I_J)^!)$ and $((i^I_J)^*, (i^I_J)_*)$ are adjoint pairs of functors.

In view of (5.1) and (5.13), we have canonical isomorphisms

$$\mathbb{D}_I \circ (i^I_J)_! \cong (i^I_J)_* \circ \mathbb{D}_J \quad \text{and} \quad \mathbb{D}_J \circ (i^I_J)^! \cong (i^I_J)^* \circ \mathbb{D}_I. \tag{5.14}$$

Moreover, since this is true for open and closed embeddings (by the axioms of recollement), the adjunction morphisms

$$(i^I_J)^* \circ (i^I_J)_* \longrightarrow \mathrm{id} \quad \text{and} \quad \mathrm{id} \longrightarrow (i^I_J)^! \circ (i^I_J)_! \tag{5.15}$$

are isomorphisms; in other words, the functors $(i^I_J)_*$ and $(i^I_J)_!$ are fully faithful (see in particular Remark 5.3). Finally, we note that

$$(i^I_J)_* = (i^I_J)_! \quad \text{if } J \subset I \text{ is closed} \tag{5.16}$$

and

$$(i^I_J)^! = (i^I_J)^* \quad \text{if } J \subset I \text{ is open.} \tag{5.17}$$

Remark 5.11. Recall that an adjoint of a triangulated functor is triangulated [N, Lemma 5.3.6]. Thus, all six functors in Proposition 5.6 are triangulated. Since the functors $(i^I_J)_*$, $(i^I_J)_!$, $(i^I_J)^*$, and $(i^I_J)^!$ defined above are all compositions of functors coming from Proposition 5.6, they are again triangulated.

These constructions are also compatible with composition in the sense of the following lemma.

Lemma 5.12 *Let $I \subset W$ be a finite locally closed subset, and let $J \subset I$ and $K \subset J$ be locally closed subsets. Then there exist canonical isomorphisms*

$$(i^I_J)_* \circ (i^J_K)_* \cong (i^I_K)_*, \qquad (i^I_J)_! \circ (i^J_K)_! \cong (i^I_K)_!$$
$$(i^J_K)^* \circ (i^I_J)^* \cong (i^I_K)^*, \qquad (i^J_K)^! \circ (i^I_J)^! \cong (i^I_K)^!.$$

Proof One can choose closed subsets $J_1 \subset J_0 \subset I$ and $K_1 \subset K_0 \subset I$ such that

$$J = J_0 \smallsetminus J_1, \quad K = K_0 \smallsetminus K_1, \quad J_1 \subset K_1 \subset K_0 \subset J_0.$$

(For instance, with $J_0 = \overline{J}$ and $K_0 = \overline{K} \cup (\overline{J} \smallsetminus J)$, these conditions are satisfied.) Then we have a diagram of embeddings

$$\begin{array}{ccccc} K & \xhookrightarrow{o} & K_0 \cap J & \xhookrightarrow{c} & J \\ & & \downarrow{\scriptstyle o} & & \downarrow{\scriptstyle o} \\ & & K_0 & \xhookrightarrow{c} & J_0 & \xhookrightarrow{c} & I \end{array}$$

where the arrows decorated with "o" are open embeddings, and those decorated with "c" are closed embeddings. (To justify the claim about the embedding $K \subset K_0 \cap J$, we observe that the complement of this embedding is $K_1 \cap J$, which is closed in $K_0 \cap J$. For the embedding $K_0 \cap J \subset K_0$, one simply observes that $K_0 \cap J = K_0 \smallsetminus J_1$.) Then the desired isomorphisms follow from Lemma 5.10, the compatibility of pushforward under closed embeddings, and pullback under open embeddings with composition (Section 4.4), and Lemma 5.8. ■

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6 Study of Standard and Costandard Objects

6.1 Generation of the Categories by Reduced Expressions

We begin with the following lemma. Recall the notion of rex moves [EW2, §4.2], [RW, §4.3] and the associated morphisms in $\mathscr{D}^{\oplus}_{\mathrm{BS}}(\mathfrak{h}, W)$.

Lemma 6.1 *Let $\underline{x}$ and $\underline{y}$ be reduced expressions for an element $w \in W$. Consider a rex move $\underline{x} \rightsquigarrow \underline{y}$, and denote by $f \colon B_{\underline{x}} \to B_{\underline{y}}$ the associated morphism. Then the cone of f belongs to $\mathsf{BE}_{\{<w\}}(\mathfrak{h}, W)$.*

Proof Consider also the reversed rex move $\underline{y} \rightsquigarrow \underline{x}$, and denote the associated morphism $g \colon B_{\underline{y}} \to B_{\underline{x}}$. Then by [EW2, Lemma 7.4, Lemma 7.5], there exists an object B in $\mathscr{D}^{\oplus}_{\mathrm{BS},\{<w\}}(\mathfrak{h}, W)$ and morphisms $h_1 \colon B_{\underline{x}} \to B$ and $h_2 \colon B \to B_{\underline{x}}$ such that $gf = \mathrm{id}_{B_{\underline{x}}} + h_2 \circ h_1$. Then we can consider the morphisms of complexes

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & B_{\underline{x}} & \xrightarrow{\;gf\;} & B_{\underline{x}} & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & & & {\scriptstyle -h_2}\big\uparrow\big\downarrow{\scriptstyle h_1} & & {\scriptstyle -h_2}\big\uparrow\big\downarrow{\scriptstyle h_1} & & & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & B & \xrightarrow{\;\mathrm{id}_B + h_1 h_2\;} & B & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

It is not difficult to check that the images of these morphisms are inverse isomorphisms in $\mathsf{BE}(\mathfrak{h}, W)$. In particular, the cone of gf belongs to $\mathsf{BE}_{\{<w\}}(\mathfrak{h}, W)$. Similar arguments show that the cone of fg belongs to $\mathsf{BE}_{\{<w\}}(\mathfrak{h}, W)$, and this implies that the image of f in the Verdier quotient $\mathsf{BE}(\mathfrak{h}, W)/\mathsf{BE}_{\{<w\}}(\mathfrak{h}, W)$ is an isomorphism, *i.e.*, that the image of the cone $\mathscr{C}$ of f in $\mathsf{BE}(\mathfrak{h}, W)/\mathsf{BE}_{\{<w\}}(\mathfrak{h}, W)$ is trivial. In view of [Kr, Proposition 4.6.2], this means that there exists an object $\mathscr{F}$ in $\mathsf{BE}_{\{<w\}}(\mathfrak{h}, W)$ such that the identity of $\mathscr{C}$ factors as a composition $\mathscr{C} \to \mathscr{F} \to \mathscr{C}$. We deduce that $(i^{\{\le w\}}_{\{<w\}})^* \mathscr{C} = 0$. By the recollement formalism (Proposition 5.6) it follows that $\mathscr{C}$ belongs to $\mathsf{BE}_{\{<w\}}(\mathfrak{h}, W)$, as desired. ∎

Let us denote by "$*$" the *Hecke product* on W [BM, §3]. (Recall in particular that this product is associative.) For an expression $\underline{w} = (s_1, \dots, s_r)$, we set

$$*\underline{w} := s_1 * \cdots * s_r \quad \in W.$$

Lemma 6.2 *For any expression $\underline{w}$, the object $B_{\underline{w}}$ belongs to $\mathsf{BE}_{\{\le *\underline{w}\}}(\mathfrak{h}, W)$.*

Proof We argue by induction on $\ell(\underline{w})$. Of course, the claim is obvious if $\underline{w}$ is a reduced expression, and, in particular, when $\ell(\underline{w}) = 0$. Now let $\underline{w}$ be a nonempty expression, and assume the claim is known for expressions of strictly smaller length. Write $\underline{w} = \underline{y}s$ for some $s \in S$; then by induction we know that $B_{\underline{y}} \in \mathsf{BE}_{\{\le *\underline{y}\}}(\mathfrak{h}, W)$. In view of the definition of $\mathsf{BE}_{\{\le *\underline{y}\}}(\mathfrak{h}, W)$, we therefore need to show that if $\underline{z}$ is a reduced expression for an element $z \le *\underline{y}$, then $B_{\underline{z}s} \in \mathsf{BE}_{\{\le *\underline{w}\}}(\mathfrak{h}, W)$.

If $\ell(\underline{z}) < \ell(\underline{y})$, then $\ell(\underline{z}s) < \ell(\underline{w})$; so by induction we know that $B_{\underline{z}s}$ belongs to $\mathsf{BE}_{\le *(\underline{z}s)}(\mathfrak{h}, W)$. On the other hand, by [BM, Proposition 3.1], we have $*(\underline{z}s) = z * s \le (*\underline{y}) * s = *\underline{w}$, and hence the desired claim follows in this case.

Assume now that $\ell(\underline{z}) = \ell(\underline{y})$; so that $z = *\underline{y}$ and $\underline{y}$ is a reduced expression for z. If $zs > z$, then $*\underline{w} = (*\underline{y})s$ and $\underline{z}s$ is a reduced expression for $*\underline{w}$; hence the claim is clear from definitions. Now assume that $zs < z$, so that $*\underline{w} = z$. Choose a reduced expression $\underline{z}'$ for z ending with s, and a rex move $\underline{z} \rightsquigarrow \underline{z}'$. By Lemma 6.1, the cone of the associated morphism $f\colon B_{\underline{z}} \to B_{\underline{z}'}$ belongs to $\mathsf{BE}_{\{<z\}}(\mathfrak{h}, W)$; as above, using the induction hypothesis, this implies that the cone of $f \star B_s$ belongs to $\mathsf{BE}_{\{\le *\underline{w}\}}(\mathfrak{h}, W)$. Since $B_{\underline{z}'} \star B_s \cong B_{\underline{z}'}(1) \oplus B_{\underline{z}'}(-1)$ by (3.2), such that $B_{\underline{z}'} \star B_s$ belongs to $\mathsf{BE}_{\{\le z\}}(\mathfrak{h}, W)$, and since $z = *\underline{w}$, we finally deduce that $B_{\underline{z}s}$ belongs to $\mathsf{BE}_{\{\le *\underline{w}\}}$, as desired. ■

Remark 6.3.

(1) Note that Lemma 6.2 implies in particular that the category $\mathsf{BE}(\mathfrak{h}, W)$ is generated (as a triangulated category) by the objects $B_{\underline{w}}$, where $\underline{w}$ is a reduced expression; in other words the canonical embedding $\mathsf{BE}_W(\mathfrak{h}, W) \to \mathsf{BE}(\mathfrak{h}, W)$ is an equivalence of categories. (Of course, this fact follows readily from [EW2, Theorem 6.26] when this result applies, *i.e.*, when $\Bbbk$ is a field or a complete local ring.) In the rest of the paper we will identify these categories without further notice.

(2) Statements closely related to Lemma 6.2 and the comment in (1) appear as [RW, Lemmas 5.23, 5.24]. But the proof in [RW] has a gap, since a variant of Lemma 6.1 is asserted without details. It turns out that the recollement formalism exactly provides the tools needed to fill this gap.

Below we will also use the following consequence of Lemma 6.2.

Corollary 6.4 *Let $I \subset W$ be a closed subset, and let $s \in S$ be such that I is stable under the map $x \mapsto xs$. Then the subcategory $\mathsf{BE}_I(\mathfrak{h}, W)$ of $\mathsf{BE}(\mathfrak{h}, W)$ is stable under right multiplication by B_s.*

Proof We need to prove that if $\underline{w}$ is a reduced expression for an element in I, then $B_{\underline{w}} \underline{\star} B_s = B_{\underline{w}s}$ belongs to $\mathsf{BE}_I(\mathfrak{h}, W)$. However Lemma 6.2 implies that this object belongs to $\mathsf{BE}_{\{\le *(\underline{w}s)\}}(\mathfrak{h}, W)$. Under our assumption $*(\underline{w}s) \in I$, so $\{\le *(\underline{w}s)\} \subset I$, and the claim follows. ■

6.2 Inclusions of Singletons

Let $I \subset W$ be a finite locally closed subset. Then for any $x \in I$, the subset $\{x\} \subset I$ is locally closed. Hence we can consider in particular the functors associated with this inclusion, which for simplicity will be denoted $(i_x^I)_*$, $(i_x^I)_!$, $(i_x^I)^*$, and $(i_x^I)^!$.

Lemma 6.5 *If $J \subset I$ is a closed subset and if $x \notin J$, then*

$$(i_x^I)^! \circ (i_J^I)_* = 0 \quad \textit{and} \quad (i_x^I)^* \circ (i_J^I)_* = 0.$$

Proof Using (5.14) and (5.16), the first equality follows from the second one by duality. And to prove the second equality, we remark that $(i_x^I)^* = (i_x^{I\smallsetminus J})^* \circ (i_{I\smallsetminus J}^I)^*$ by Lemma 5.12, so that $(i_x^I)^* \circ (i_J^I)_* = (i_x^{I\smallsetminus J})^* \circ (i_{I\smallsetminus J}^I)^* \circ (i_J^I)_* = 0$ by (4.1). ■

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Lemma 6.5 implies that if $x \neq y$ are both in I, we have

$$(i_x^I)^! \circ (i_y^I)_* = 0, \qquad (i_x^I)^* \circ (i_y^I)_! = 0, \tag{6.1}$$

because $(i_y^I)_* = (i^I_{\{z\in I \mid z\le y\}})_* \circ (i_y^{\{z\in I \mid z \le y\}})_*$ and similarly for $(i_y^I)_!$. On the other hand, for any $x \in I$ we have

$$(i_x^I)^! \circ (i_x^I)_* \cong \mathrm{id}, \qquad (i_x^I)^* \circ (i_x^I)_! \cong \mathrm{id}\,. \tag{6.2}$$

For the first isomorphism we remark that $(i_x^I)_* \cong (i^I_{\{z\in I \mid z\le x\}})_! \circ (i_x^{\{z\in I \mid z\le x\}})_*$ by Lemmas 5.12 and (5.16), and $(i_x^I)^! \cong (i_x^{\{z\in I \mid z\le x\}})^* \circ (i^I_{\{z\in I \mid z\le x\}})^!$ by Lemmas 5.12 and (5.17). Then the claim follows from the invertibility of the morphisms in (5.15).

6.3 Definition of Standard and Costandard Objects

Now recall the object b_w of $\mathscr{D}^{\oplus}_{\mathrm{BS},\{w\}}(\mathfrak{h}, W)$ defined in Section 4.3. Identifying this object with the complex concentrated in degree 0 and with 0-th term b_w, it can be considered as an object in $\mathsf{BE}_{\{w\}}(\mathfrak{h}, W)$. The corresponding *standard* and *costandard* objects in $\mathsf{BE}_I(\mathfrak{h}, W)$ are defined by $\Delta_w^I := (i_w^I)_! b_w$ and $\nabla_w^I := (i_w^I)_* b_w$. The main property of these objects is the following.

Lemma 6.6 *Let $I \subset W$ be a finite locally closed subset, and let $x, y \in I$. Then we have*

$$\mathrm{Hom}_{\mathsf{BE}_I(\mathfrak{h},W)}(\Delta_x^I, \nabla_y^I\langle n\rangle[m]) \cong \begin{cases} R^m = \mathrm{S}^{m/2}(V^*) & \text{if } x = y \text{ and } m = -n \in 2\mathbb{Z}_{\ge 0},\\ 0 & \text{otherwise.}\end{cases}$$

Proof This follows from adjunction, isomorphisms (6.1), (6.2), and Lemma 4.4. ∎

Example 6.7. (1) If w is minimal in I, then $\Delta_w^I = \nabla_w^I$ by (5.16), and this object is the image of $B_{\underline{w}}$ in $\mathsf{BE}_I(\mathfrak{h}, W)$, where $\underline{w}$ is any reduced expression for w. In particular, if I contains the neutral element $e \in W$, then $\Delta_e^I = \nabla_e^I$ is the image of $B_\varnothing$.

(2) Let $s \in S$. In view of Example 5.4, the complex $\Delta_s^{\{e,s\}}$ coincides with the complex

$$\cdots 0 \longrightarrow B_s \xrightarrow{\;s\;} B_\varnothing(1) \longrightarrow 0 \cdots,$$

where the nonzero terms are in degrees 0 and 1, respectively. Similarly, $\nabla_s^{\{e,s\}}$ is the complex

$$\cdots 0 \longrightarrow B_\varnothing(-1) \xrightarrow{\;s\;} B_s \longrightarrow 0 \cdots,$$

where the nonzero terms are in degrees -1 and 0, respectively. These complexes, in fact, describe Δ_s^I and ∇_s^I for any I containing e and s. In particular, our present notation is compatible with that used in [AMRW1, Example 4.2.2].

It will sometimes be convenient to have standard and costandard objects also when I is not finite. For a general I and any $w \in I$, we define Δ_w^I and ∇_w^I by

$$\Delta_w^I := (i_J^I)_* \Delta_w^J \quad \text{and} \quad \nabla_w^I := (i_J^I)_* \nabla_w^J,$$

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where $J \subset I$ is any finite closed subset containing w. It is easy to check that these objects do not depend on the choice of J, up to canonical isomorphism, and that Lemma 6.6 still holds in this generality. When $I = W$, we will sometimes omit the superscript in this notation.

6.4 First Properties

Lemma 6.8 *Let $I \subset W$ be a finite locally closed subset, and let $J \subset I$ be a locally closed subset. Then for any $w \in J$, we have $(i_J^I)_! \Delta_w^J \cong \Delta_w^I$ and $(i_J^I)_* \nabla_w^J \cong \nabla_w^I$, and for any $w \in I$, we have*

$$(i_J^I)^* \Delta_w^I \cong \begin{cases} \Delta_w^J & \text{if } w \in J, \\ 0 & \text{otherwise}, \end{cases} \qquad (i_J^I)^! \nabla_w^I \cong \begin{cases} \nabla_w^J & \text{if } w \in J, \\ 0 & \text{otherwise}. \end{cases}$$

Proof The first two isomorphisms follow from Lemma 5.12. For the other isomorphisms, we treat the case of $(i_J^I)^* \Delta_w^I$; the case of $(i_J^I)^! \nabla_w^I$ is similar. It suffices to prove these isomorphisms when J is either closed or open. First assume that J is closed. If $w \notin J$, then the desired vanishing follows from the first isomorphism in Lemma 6.5 and adjunction. If $w \in J$, then we have $(i_J^I)^* \Delta_w^I \cong (i_J^I)^* (i_J^I)_* \Delta_w^J$ by (5.16) and Lemma 5.12, and the claim follows from the invertibility of the first morphism in (5.15). Now assume that J is open. If $w \in J$, then using (5.17) we have

$$(i_J^I)^* \Delta_w^I \cong (i_J^I)^! (i_J^I)_! \Delta_w^J \cong \Delta_w^J.$$

And if $w \notin J$, then $(i_J^I)^* \Delta_w^I \cong (i_J^I)^* (i_{I \smallsetminus J}^I)_* \Delta_w^{I \smallsetminus J}$ and the desired vanishing holds by (4.1). ∎

Another important property of standard and costandard objects is provided by the following observation.

Lemma 6.9 *For any locally closed subset $I \subset W$, the category $\mathsf{BE}_I(\mathfrak{h}, W)$ is generated as a triangulated category by the objects of the form $\Delta_w^I(m)$ with $w \in I$ and $m \in \mathbb{Z}$, as well as by the objects of the form $\nabla_w^I(m)$ with $w \in I$ and $m \in \mathbb{Z}$.*

Proof We treat the case of the standard objects; the other case is similar or follows by duality. We can clearly assume that I is finite, and proceed by induction on $|I|$.

When $|I| = 1$, the lemma is clear from Lemma 4.4. Now assume $|I| > 1$, and choose $w \in I$ minimal. Then any object $\mathscr{F}$ in $\mathsf{BE}_I(\mathfrak{h}, W)$ fits in a distinguished triangle

$$(i_{I \smallsetminus \{w\}}^I)_! (i_{I \smallsetminus \{w\}}^I)^* \mathscr{F} \longrightarrow \mathscr{F} \longrightarrow (i_w^I)_* (i_w^I)^* \mathscr{F} \xrightarrow{[1]} .$$

By induction $(i_{I \smallsetminus \{w\}}^I)^* \mathscr{F}$ belongs to the triangulated subcategory of $\mathsf{BE}_{I \smallsetminus \{w\}}(\mathfrak{h}, W)$ generated by the objects $\Delta_x^{I \smallsetminus \{w\}}(m)$ with $x \in I \smallsetminus \{w\}$. Since $(i_{I \smallsetminus \{w\}}^I)_! \Delta_x^{I \smallsetminus \{w\}} \cong \Delta_x^I$ for such x (Lemma 6.8), we deduce that $(i_{I \smallsetminus \{w\}}^I)_! (i_{I \smallsetminus \{w\}}^I)^* \mathscr{F}$ belongs to the triangulated subcategory of $\mathsf{BE}_I(\mathfrak{h}, W)$ under consideration. It is easy to see that $(i_w^I)_* (i_w^I)^* \mathscr{F}$ belongs to the triangulated subcategory generated by the objects $\Delta_w^I(m) = \nabla_w^I(m)$, and the proof is complete. ∎

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6.5 Convolution of Standard and Costandard Objects

Lemma 6.10 *Let $w \in W$ and $s \in S$ be such that $ws > w$. Then there exist distinguished triangles*

$$\Delta_w\langle -1\rangle \longrightarrow \Delta_{ws} \longrightarrow \Delta_w \,\underline{\star}\, B_s \xrightarrow{[1]}, \quad \nabla_w \,\underline{\star}\, B_s \longrightarrow \nabla_{ws} \longrightarrow \nabla_w\langle 1\rangle \xrightarrow{[1]}$$

in $\mathsf{BE}(\mathfrak{h}, W)$*, in which the third arrows are generators of the free rank-1 $\Bbbk$-modules*

$$\mathrm{Hom}_{\mathsf{BE}(\mathfrak{h},W)}(\Delta_w \,\underline{\star}\, B_s, \Delta_w\langle -1\rangle[1]) \quad \textit{and} \quad \mathrm{Hom}_{\mathsf{BE}(\mathfrak{h},W)}(\nabla_w\langle 1\rangle, \nabla_w \,\underline{\star}\, B_s[1]),$$

respectively.

Proof We will construct the first triangle; the second one can then be obtained by duality or by similar arguments. We set $I := \{z \in W \mid z \leq ws\}$. In this triangle all the objects live in $\mathsf{BE}_I(\mathfrak{h}, W)$ (see Corollary 6.4 for the third term); therefore, we can perform all the computations in this subcategory. To simplify notation, we will also set $J := I \smallsetminus \{w, ws\}$, a closed subset of I.

Let $\underline{w}$ be a reduced expression for w, and recall the triangle constructed in Example 5.4. Applying the functor $(i^I_{\{w,ws\}})_!$, we deduce (Remark 5.11) a distinguished triangle

$$\Delta^I_w\langle -1\rangle \longrightarrow \Delta^I_{ws} \longrightarrow (i^I_{\{w,ws\}})_!(B_{\underline{ws}}) \xrightarrow{[1]} \tag{6.3}$$

in $\mathsf{BE}_I(\mathfrak{h}, W)$, where we write $B_{\underline{ws}}$ for the image of this object in the category

$$\mathsf{BE}_{\{w,ws\}}(\mathfrak{h}, W).$$

Hence, to conclude the construction of the triangle, it suffices to construct an isomorphism

$$(i^I_{\{w,ws\}})_!(B_{\underline{ws}}) \cong \Delta^I_w \,\underline{\star}\, B_s. \tag{6.4}$$

First we remark that

$$(i^I_J)^*(\Delta^I_w \,\underline{\star}\, B_s) = 0. \tag{6.5}$$

In fact, if $\mathscr{F}$ belongs to $\mathsf{BE}_J(\mathfrak{h}, W)$ we have

$$\begin{aligned}\mathrm{Hom}_{\mathsf{BE}_J(\mathfrak{h},W)}\big((i^I_J)^*(\Delta^I_w \,\underline{\star}\, B_s), \mathscr{F}\big) &\cong \mathrm{Hom}_{\mathsf{BE}_I(\mathfrak{h},W)}(\Delta^I_w \,\underline{\star}\, B_s, (i^I_J)_*\mathscr{F})\\ &\cong \mathrm{Hom}_{\mathsf{BE}_I(\mathfrak{h},W)}(\Delta^I_w, ((i^I_J)_*\mathscr{F}) \,\underline{\star}\, B_s).\end{aligned}$$

It follows from Corollary 6.4 that $((i^I_J)_*\mathscr{F}) \,\underline{\star}\, B_s$ belongs to the essential image of $\mathsf{BE}_J(\mathfrak{h}, W)$, and then from (4.1) we deduce that

$$\mathrm{Hom}_{\mathsf{BE}_J(\mathfrak{h},W)}((i^I_J)^*(\Delta^I_w \,\underline{\star}\, B_s), \mathscr{F}) = 0,$$

which implies (6.5).

From (6.5) we deduce that adjunction induces an isomorphism

$$(i^I_{\{w,ws\}})_!(i^I_{\{w,ws\}})^*(\Delta^I_w \,\underline{\star}\, B_s) \xrightarrow{\sim} \Delta^I_w \,\underline{\star}\, B_s.$$

Hence, to prove (6.4), it suffices to prove that

$$(i^I_{\{w,ws\}})^*(\Delta^I_w \,\underline{\star}\, B_s) \cong B_{\underline{ws}} \tag{6.6}$$

in $\mathsf{BE}_{\{w,ws\}}(\mathfrak{h}, W)$. However there exists a natural distinguished triangle

$$\Delta^I_w \to B_{\underline{w}} \longrightarrow (i^I_J)_*(i^I_J)^* B_{\underline{w}} \xrightarrow{[1]} .$$

Applying the functor $(i^I_{\{w,ws\}})^*(- \mathbin{\underline{\star}} B_s)$ and Corollary 6.4 once again, we deduce the isomorphism (6.6), and hence finally (6.4).

To conclude the proof, it remains to prove that the $\Bbbk$-module

$$\mathrm{Hom}_{\mathsf{BE}(\mathfrak{h},W)}\big((i^I_{\{w,ws\}})_!(B_{\underline{ws}}), \Delta^I_w\langle -1\rangle[1]\big)$$

is free of rank 1, and generated by the morphism appearing in (6.3). However, as noted after (5.15), the functor $(i^I_{\{w,ws\}})_!$ is fully faithful. Hence it suffices to prove the corresponding claim for $\mathrm{Hom}_{\mathsf{BE}_{\{w,ws\}}(\mathfrak{h},W)}(B_{\underline{ws}}, B_{\underline{w}}\langle -1\rangle[1])$. This claim is clear from the construction in Example 5.4. ■

The next proposition answers a question raised in [AMRW1, Remark 4.2.3].

Proposition 6.11 *Let $w \in W$.*

(i) *If $(s_1, \dots, s_r)$ is a reduced expression for w, then we have*

$$\Delta_w \cong \Delta_{s_1} \mathbin{\underline{\star}} \Delta_{s_2} \mathbin{\underline{\star}} \cdots \mathbin{\underline{\star}} \Delta_{s_r}, \qquad \nabla_w \cong \nabla_{s_1} \mathbin{\underline{\star}} \nabla_{s_2} \mathbin{\underline{\star}} \cdots \mathbin{\underline{\star}} \nabla_{s_r}.$$

(ii) *We have isomorphisms $\Delta_w \mathbin{\underline{\star}} \nabla_{w^{-1}} \cong \nabla_{w^{-1}} \mathbin{\underline{\star}} \Delta_w \cong B_\varnothing$.*

Proof We will prove the claims by induction on $\ell(w)$. We note here that (ii) holds when $\ell(w) = 1$ by [AMRW1, Lemma 4.2.4]. In particular, it follows that (ii) is a consequence of (i) (applied to w and w^{-1}).

Of course, if $\ell(w) = 0$, there is nothing to prove. Now let $w \in W \smallsetminus \{e\}$, and assume the claims are known for elements of length strictly smaller than that of w. We will prove the first isomorphism in (i) for w; the second one can be proved similarly or follows by duality, and as noted above (ii) will follow. Let $(s_1, \dots, s_r)$ be a reduced expression for w, and let $y := s_1 \cdots s_{r-1}$ and $s := s_r$, so that $w = ys$. Using (i) for y, which is known by induction, we know that $\Delta_y \cong \Delta_{s_1} \mathbin{\underline{\star}} \Delta_{s_2} \mathbin{\underline{\star}} \cdots \mathbin{\underline{\star}} \Delta_{s_{r-1}}$. Hence to conclude, it suffices to prove that $\Delta_w \cong \Delta_y \mathbin{\underline{\star}} \Delta_s$.

The special case of Lemma 6.10 for the neutral element e provides a distinguished triangle $B_\varnothing\langle -1\rangle \to \Delta_s \to B_s \xrightarrow{[1]}$ in which the third arrow is a generator of

$$\mathrm{Hom}_{\mathsf{BE}(\mathfrak{h},W)}(B_s, B_\varnothing\langle -1\rangle[1]),$$

a free rank-1 $\Bbbk$-module. Now (ii) for y implies that the functor

$$\Delta_y \mathbin{\underline{\star}} (-) \colon \mathsf{BE}(\mathfrak{h}, W) \longrightarrow \mathsf{BE}(\mathfrak{h}, W)$$

is an equivalence of triangulated categories with quasi-inverse $\nabla_{y^{-1}} \mathbin{\underline{\star}} (-)$. Hence applying this functor, we obtain a distinguished triangle

$$\Delta_y\langle -1\rangle \longrightarrow \Delta_y \mathbin{\underline{\star}} \Delta_s \longrightarrow \Delta_y \mathbin{\underline{\star}} B_s \xrightarrow{[1]}$$

in which the third arrow is a generator of $\mathrm{Hom}_{\mathsf{BE}(\mathfrak{h},W)}(\Delta_y \mathbin{\underline{\star}} B_s, \Delta_y\langle -1\rangle[1])$, a free rank-1 $\Bbbk$-module. Comparing with the triangle of Lemma 6.10 (now for y), we deduce the isomorphism $\Delta_w \cong \Delta_y \mathbin{\underline{\star}} \Delta_s$ as expected. ■

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Remark 6.12. (1) Proposition 6.11(i) shows that the objects Δ_w ($w \in W$) are generalizations of the *Rouquier complexes* [Rou] associated with canonical lifts of elements of W to the braid group of (W, S). More precisely, consider a reflection faithful representation V of (W, S) over $\Bbbk = \mathbb{R}$ as constructed by Soergel or arising from a symmetrizable Kac–Moody group [R2, Proposition 1.1]. Then, as explained in [EW2, Example 3.3(2)–(4)], there exists a natural realization $\mathfrak{h}$ of (W, S) with underlying vector space V. Moreover, by [EW2, Theorem 6.30] there exists a canonical equivalence of graded additive categories between the Karoubian envelope of $\mathscr{D}^{\oplus}_{\mathrm{BS}}(\mathfrak{h}, W)$ and the associated category of Soergel bimodules. Under the induced equivalence between bounded homotopy categories [AMRW1, Lemma 4.9.1], Proposition 6.11 shows that Δ_w corresponds to the Rouquier complex (as defined in [Rou, Proposition 9.4]) associated with the canonical lift of w to the braid group. From this point of view, Lemma 6.6 is a generalization of the main result of [LW].

(2) Proposition 6.11(i) suggests a different approach to our study, starting with a direct definition of standard and costandard objects. However, from such a definition it seems to be difficult (at least to the authors) to prove that such objects have the properties they ought to possess such as independence of the reduced expression, or Lemma 6.6.

6.6 Application

In this subsection we apply the results of this section to describe the split Grothendieck group of the additive category $\mathscr{D}^{\oplus}_{\mathrm{BS}}(\mathfrak{h}, W)$. If $\mathcal{A}$ is an essentially small triangulated category, resp., additive category, we denote by $[\mathcal{A}]_\Delta$, resp., $[\mathcal{A}]_\oplus$, the Grothendieck group of $\mathcal{A}$, resp., the split Grothendieck group of $\mathcal{A}$. Recall also the Hecke algebra $\mathcal{H}_{(W,S)}$ associated with the Coxeter system (W, S), where we follow the conventions of [So2]. With this notation introduced, we can state our result more precisely.

Theorem 6.13 *There exists a unique ring isomorphism* $\mathcal{H}_{(W,S)} \xrightarrow{\sim} [\mathscr{D}^{\oplus}_{\mathrm{BS}}(\mathfrak{h}, W)]_\oplus$ *sending* v *to* $[B_\varnothing(1)]$ *and* $\underline{H}_s = H_s + v$ *to* $[B_s]$*, for any* $s \in S$.

Remark 6.14. When $\Bbbk$ is a complete local ring, this result appears as [EW2, Corollary 6.27], where the result is stated in terms of the Karoubian hull of $\mathscr{D}^{\oplus}_{\mathrm{BS}}(\mathfrak{h}, W)$. However, it is easy to deduce from [EW2, Theorem 6.26] that under their assumption the natural functor from $\mathscr{D}^{\oplus}_{\mathrm{BS}}(\mathfrak{h}, W)$ to its Karoubian hull induces an isomorphism on split Grothendieck groups. The fact that our methods might allow one to generalize this result was suggested to one of us by G. Williamson.

The proof of Theorem 6.13 will use the following lemma that is the main result of [Ros].

Lemma 6.15 *For any essentially small additive category* $\mathcal{A}$*, the natural group morphism* $[\mathcal{A}]_\oplus \to [K^{\mathrm{b}}\mathcal{A}]_\Delta$ *is an isomorphism.*

Proof of Theorem 6.13 In view of Lemma 6.15, the natural morphism

$$[\mathscr{D}^{\oplus}_{\mathrm{BS}}(\mathfrak{h}, W)]_\oplus \longrightarrow [\mathsf{BE}(\mathfrak{h}, W)]_\Delta$$

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is an isomorphism. Moreover this morphism is clearly a ring morphism. Therefore, to prove the theorem we only have to prove that there exists a unique isomorphism

$$\mathcal{H}_{(W,S)} \xrightarrow{\sim} [\mathsf{BE}(\mathfrak{h}, W)]_\Delta$$

sending v to $[B_\varnothing(1)]$ and $H_s + v$ to $[B_s]$. Uniqueness is clear, since $\mathcal{H}_{(W,S)}$ is generated (as a ring) by v and the elements $H_s + v$ for $s \in S$.

To prove existence, we first remark that the classes of the standard objects $[\Delta_w(m)]$ form a $\mathbb{Z}$-basis of $[\mathsf{BE}(\mathfrak{h}, W)]_\Delta$. In fact, Lemma 6.9 and Remark 6.3(1) imply that these classes span $[\mathsf{BE}(\mathfrak{h}, W)]_\Delta$. On the other hand, assume for a contradiction that there exists a relation

$$\sum_{\substack{x\in Y_1\\ m\in\mathbb{Z}}} \lambda_{x,m} \cdot [\Delta_x(m)] = \sum_{\substack{y\in Y_2\\ n\in\mathbb{Z}}} \lambda_{y,n} \cdot [\Delta_y(n)]$$

for some disjoint finite subsets $Y_1, Y_2 \subset W$ (with $Y_1 \neq \varnothing$ and $\lambda_{x,m} \neq 0$ for at least one $x \in Y_1$ and $m \in \mathbb{Z}$) and some integers $\lambda_{y,n} \in \mathbb{Z}_{\geq 0}$ (with $\lambda_{x,m} = 0$ and $\lambda_{y,n} = 0$ for all but finitely many m's and n's). Then, if we set

$$\mathscr{X}_1 := \bigoplus_{\substack{x\in Y_1\\ m\in\mathbb{Z}}} \big(\Delta_x(m)\big)^{\oplus\lambda_{x,m}}, \quad \mathscr{X}_2 := \bigoplus_{\substack{y\in Y_2\\ n\in\mathbb{Z}}} \big(\Delta_y(n)\big)^{\oplus\lambda_{y,n}},$$

by [Th, Lemma 2.4] there exist objects $\mathscr{C}, \mathscr{C}', \mathscr{C}''$ and distinguished triangles

$$\mathscr{C} \oplus \mathscr{X}_1 \longrightarrow \mathscr{C}' \longrightarrow \mathscr{C}'' \xrightarrow{[1]}, \quad \mathscr{C} \oplus \mathscr{X}_2 \longrightarrow \mathscr{C}' \longrightarrow \mathscr{C}'' \xrightarrow{[1]}.$$

There exists a finite closed subset $I \subset W$ such that all the objects above belong to $\mathsf{BE}_I(\mathfrak{h}, W)$. Then choose $x \in Y_1$ such that $\lambda_{x,m} \neq 0$ for at least one m. Applying $(i_x^I)^*$ and using Lemma 6.8 we obtain distinguished triangles

$$(i_x^I)^*\mathscr{C} \oplus (i_x^I)^*\mathscr{X}_1 \longrightarrow (i_x^I)^*\mathscr{C}' \longrightarrow (i_x^I)^*\mathscr{C}'' \xrightarrow{[1]},$$
$$(i_x^I)^*\mathscr{C} \longrightarrow (i_x^I)^*\mathscr{C}' \longrightarrow (i_x^I)^*\mathscr{C}'' \xrightarrow{[1]}.$$

Hence the class of $(i_x^I)^*\mathscr{X}_1$ in $[\mathsf{BE}_{\{x\}}(\mathfrak{h}, W)]_\Delta$ vanishes. But Lemma 4.4 and Lemma 6.15 imply that the classes $[b_x(m)]$ with $m \in \mathbb{Z}$ form a basis of $[\mathsf{BE}_{\{x\}}(\mathfrak{h}, W)]_\Delta$, and by construction, the coefficient of $(i_x^I)^*\mathscr{X}_1$ on $[b_x(m)]$ is $\lambda_{x,m}$. One of these coefficients is nonzero, providing the desired contradiction.

We now prove that the assignment $v \mapsto [B_\varnothing(1)]$, $H_w \mapsto [\Delta_w]$ ($w \in W$) induces a ring morphism $\mathcal{H}_{(W,S)} \to [\mathsf{BE}(\mathfrak{h}, W)]_\Delta$. For this we need to prove that

$$([\Delta_s])^2 = [\Delta_e] + [\Delta_s(-1)] - [\Delta_s(1)] \tag{6.7}$$

for $s \in S$ and that for $x, y \in W$ such that $\ell(xy) = \ell(x) + \ell(y)$ we have

$$[\Delta_{xy}] = [\Delta_x \,\underline{\star}\, \Delta_y]. \tag{6.8}$$

Here (6.8) follows from Proposition 6.11, while (6.7) follows from the fact that $[\Delta_s] = [B_s] - [\Delta_e(1)]$ (Example 6.7(2)) and the isomorphism (3.2).

Finally we argue that our morphism $\mathcal{H}_{(W,S)} \to [\mathsf{BE}(\mathfrak{h}, W)]_\Delta$ is invertible because it sends a $\mathbb{Z}$-basis of $\mathcal{H}_{(W,S)}$ to a $\mathbb{Z}$-basis of $[\mathsf{BE}(\mathfrak{h}, W)]_\Delta$; moreover it sends v to $B_\varnothing(1)$ by definition, and $H_s + v$ to $[B_s]$ since, as already noticed above, we have $[\Delta_s] = [B_s] - [\Delta_e(1)]$ in $[\mathsf{BE}(\mathfrak{h}, W)]_\Delta$. ■

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Remark 6.16. Let us notice that, viewed as an isomorphism $\mathcal{H}_{(W,S)} \xrightarrow{\sim} [\mathsf{BE}(\mathfrak{h}, W)]_\Delta$, the isomorphism of Theorem 6.13 is very explicit: it sends H_w to $[\Delta_w]$.

7 The Perverse t-structure

Henceforth, we assume that $\Bbbk$ satisfies the assumptions of Section 2.3. The goal of this section is to endow the biequivariant category $\mathsf{BE}(\mathfrak{h}, W)$ with a bounded t-structure and investigate its heart.

7.1 t-structure for Categories Associated With Singletons

We start by considering singleton sets in analogy with [AR1, Lemmas 3.1, 3.18]. Recall the equivalence γ of Lemma 4.4. Passing to bounded homotopy categories, we obtain an equivalence $\mathsf{BE}_{\{w\}}(\mathfrak{h}, W) \cong K^{\mathrm{b}}\mathrm{Free}^{\mathrm{fg},\mathbb{Z}}(R)$. Composing with the equivalence of Lemma 2.1, we deduce an equivalence of triangulated categories

$$\mathsf{BE}_{\{w\}}(\mathfrak{h}, W) \cong D^{\mathrm{b}}\mathrm{Mod}^{\mathrm{fg},\mathbb{Z}}(R). \tag{7.1}$$

Here the autoequivalence $\langle 1 \rangle$ of $\mathsf{BE}_{\{w\}}(\mathfrak{h}, W)$ corresponds to the autoequivalence of $D^{\mathrm{b}}\mathrm{Mod}^{\mathrm{fg},\mathbb{Z}}(R)$ sending a complex $(M^n, d^n)_{n\in\mathbb{Z}}$ to the complex $(M^n(1), -d^n)_{n\in\mathbb{Z}}$, where (1) is as in Lemma 4.4. This autoequivalence will also be denoted $\langle 1 \rangle$.

Now let us recall the linear Koszul duality construction of [AR2, Section 4] (see also [MR] for a slightly different and more general construction). Let Λ be the differential graded algebra defined as the exterior algebra of the free $\Bbbk$-module V placed in degree -1 with trivial differential. We will consider Λ as a $\mathbb{Z}$-graded dg-algebra (sometimes called a differential graded graded (dgg) algebra or a $\mathbb{G}_{\mathbf{m}}$-equivariant dg-algebra), where V is in degree -2 for this new grading. Then composing the Koszul duality equivalence of [AR2, Theorem 4.1] with the regrading equivalence denoted ξ in [AR2, §4.2], we obtain an equivalence of triangulated categories

$$D^{\mathrm{b}}\mathrm{Mod}^{\mathrm{fg},\mathbb{Z}}(R) \xrightarrow{\sim} D^{\mathrm{fg}}_{\mathbb{Z}}(\Lambda),$$

where the right-hand side is the derived category of $\mathbb{Z}$-graded Λ-dg-modules whose cohomology is finitely generated over $\Bbbk$.[5] Composing with (7.1), we deduce an equivalence

$$\mathsf{BE}_{\{w\}}(\mathfrak{h}, W) \xrightarrow{\sim} D^{\mathrm{fg}}_{\mathbb{Z}}(\Lambda). \tag{7.2}$$

Since Λ is concentrated in nonpositive cohomological degrees, the right-hand side has a canonical t-structure defined by

$$\left(D^{\mathrm{fg}}_{\mathbb{Z}}(\Lambda)\right)^{\leq 0} = \{M \in D^{\mathrm{fg}}_{\mathbb{Z}}(\Lambda) \mid \mathsf{H}^{>0}(M) = 0\},$$
$$\left(D^{\mathrm{fg}}_{\mathbb{Z}}(\Lambda)\right)^{\geq 0} = \{M \in D^{\mathrm{fg}}_{\mathbb{Z}}(\Lambda) \mid \mathsf{H}^{<0}(M) = 0\}.$$

The *perverse t-structure* on $\mathsf{BE}_{\{w\}}(\mathfrak{h}, W)$, denoted

$$\left({}^p\mathsf{BE}_{\{w\}}(\mathfrak{h}, W)^{\leq 0}, {}^p\mathsf{BE}_{\{w\}}(\mathfrak{h}, W)^{\geq 0}\right),$$

[5] In [AR2], for simplicity this claim is stated only when $\Bbbk$ is a field. But the same arguments apply in the present generality; see [MR] for similar constructions.

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is defined as the transport of the t-structure on $D^{\mathrm{fg}}_{\mathbb{Z}}(\Lambda)$ considered above along the equivalence (7.2). It can be checked from the definitions that, under this equivalence, the autoequivalence $\langle 1\rangle$ of $\mathsf{BE}_{\{w\}}(\mathfrak{h}, W)$ corresponds to the autoequivalence of the category $D^{\mathrm{fg}}_{\mathbb{Z}}(\Lambda)$ sending a $\mathbb{Z}$-graded dg-module M to the same dg-module, with degree-j part (for the "extra" $\mathbb{Z}$-grading) the degree-$(j-1)$ part of M. The latter equivalence is clearly t-exact; hence, so is the autoequivalence $\langle 1\rangle$ on $\mathsf{BE}_{\{w\}}(\mathfrak{h}, W)$.

It is also clear that the object b_w considered in Section 6.3 belongs to the heart of the perverse t-structure on $\mathsf{BE}_{\{w\}}(\mathfrak{h}, W)$. In fact, this object characterizes the t-structure in the following sense.

Lemma 7.1 *The subcategory* ${}^p\mathsf{BE}_{\{w\}}(\mathfrak{h}, W)^{\leq 0}$ *is generated under extensions by the objects* $b_w\langle m\rangle[n]$ *with* $m \in \mathbb{Z}$ *and* $n \in \mathbb{Z}_{\geq 0}$.

Proof The equivalence (7.2) sends b_w to $\Bbbk$ (the trivial Λ-dg-module, concentrated in degree 0). Hence the statement amounts to the claim that $(D^{\mathrm{fg}}_{\mathbb{Z}}(\Lambda))^{\leq 0}$ is generated by the dg-modules that are concentrated in one cohomological degree $n \leq 0$, and free as a graded $\Bbbk$-module. However, using truncation functors, we see that any object of $(D^{\mathrm{fg}}_{\mathbb{Z}}(\Lambda))^{\leq 0}$ is an extension of dg-modules concentrated in one cohomological degree n, where n varies in $\mathbb{Z}_{\leq 0}$. Choosing finite free resolutions of these $\Bbbk$-modules, which exist under our assumptions, we obtain the desired claim. ∎

7.2 Definition of the t-structure

We are now ready to introduce our main definition, following [AR1, Definition 3.18].

Definition 7.2 Let I be a finite locally closed subset of W. The *perverse t-structure* on $\mathsf{BE}_I(\mathfrak{h}, W)$ is the bounded t-structure given by

$$ {}^p\mathsf{BE}_I(\mathfrak{h}, W)^{\leq 0} = \left\{ \mathscr{F} \in \mathsf{BE}_I(\mathfrak{h}, W) \mid \forall w \in I,\ (i^I_w)^*(\mathscr{F}) \in {}^p\mathsf{BE}_w(\mathfrak{h}, W)^{\leq 0} \right\}, $$
$$ {}^p\mathsf{BE}_I(\mathfrak{h}, W)^{\geq 0} = \left\{ \mathscr{F} \in \mathsf{BE}_I(\mathfrak{h}, W) \mid \forall w \in I,\ (i^I_w)^!(\mathscr{F}) \in {}^p\mathsf{BE}_w(\mathfrak{h}, W)^{\geq 0} \right\}. $$

Here, the fact that this pair of subcategories indeed forms a bounded t-structure follows from the general theory of recollement [BBD, Théorème 1.4.10] together with Lemma 5.12.

Lemma 7.3 *The following functors are t-exact:*

(i) $\langle 1\rangle$;
(ii) $(i^I_J)_*$ *for* $I \subset W$ *a finite locally closed subset and* $J \subset I$ *a closed subset;*
(iii) $(i^I_K)^*$ *for* $I \subset W$ *a finite locally closed subset and* $K \subset I$ *an open subset.*

Proof The case of $\langle 1\rangle$ is an immediate consequence of the special case when I is a singleton, which was justified in Section 7.1, and the case of $(i^I_K)^*$ follows from Lemma 5.12 and (5.17). To justify the exactness of $(i^I_J)_*$, it suffices to prove that for $w \in I$ we have

$$ (i^I_w)^*(i^I_J)_* \cong \begin{cases} (i^J_w)^* & \text{if } w \in J, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad (i^I_w)^!(i^I_J)_* \cong \begin{cases} (i^J_w)^! & \text{if } w \in J, \\ 0 & \text{otherwise.} \end{cases} $$

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Here the isomorphisms on the right-hand side follow from those on the left-hand side by duality. For the left-hand side, if $w \in J$, then $(i^I_w)^*(i^I_J)_* \cong (i^J_w)^*(i^I_J)^*(i^I_J)_* \cong (i^J_w)^*$ by Lemma 5.12 and the invertibility of the first morphism in (5.15). If $w \notin J$, then the claim follows from Lemma 6.5. ■

Using Lemma 7.3, the definition of the perverse t-structure can be generalized to any locally closed subset $I \subset W$ as follows. By definition, $\mathsf{BE}_I(\mathfrak{h}, W)$ is the direct limit of the categories $\mathsf{BE}_J(\mathfrak{h}, W)$ for $J \subset I$ a finite closed subset (for the embeddings $(i^{J'}_J)_* \colon \mathsf{BE}_J(\mathfrak{h}, W) \to \mathsf{BE}_{J'}(\mathfrak{h}, W)$ for $J \subset J' \subset I$ closed subsets). Since under these embeddings we have

$$^p\mathsf{BE}_J(\mathfrak{h}, W)^{\leq 0} = \mathsf{BE}_J(\mathfrak{h}, W) \cap {}^p\mathsf{BE}_{J'}(\mathfrak{h}, W)^{\leq 0},$$
$$^p\mathsf{BE}_J(\mathfrak{h}, W)^{\geq 0} = \mathsf{BE}_J(\mathfrak{h}, W) \cap {}^p\mathsf{BE}_{J'}(\mathfrak{h}, W)^{\geq 0}$$

(see in particular Lemma 7.3), we can define $^p\mathsf{BE}_I(\mathfrak{h}, W)^{\leq 0}$ and $^p\mathsf{BE}_I(\mathfrak{h}, W)^{\geq 0}$ as the direct limits of the categories $^p\mathsf{BE}_J(\mathfrak{h}, W)^{\leq 0}$ and $^p\mathsf{BE}_J(\mathfrak{h}, W)^{\geq 0}$, respectively, for J running over finite closed subsets of I. It is clear that Lemma 7.3 then also holds without the assumption that I is finite.

Let us immediately note the following consequence of the existence of the perverse t-structure, in view of the main result of [LC].

Corollary 7.4 *For any locally closed subset $I \subset W$, the category $\mathsf{BE}_I(\mathfrak{h}, W)$ is Karoubian.*

The subcategory $^p\mathsf{BE}_I(\mathfrak{h}, W)^{\leq 0}$ can be described in more concrete terms as follows.

Lemma 7.5 *Let $I \subset W$ be a locally closed subset.*

(i) *The subcategory $^p\mathsf{BE}_I(\mathfrak{h}, W)^{\leq 0}$ is generated under extensions by the objects $\Delta^I_w\langle m\rangle[n]$ with $w \in W$, $m \in \mathbb{Z}$, and $n \in \mathbb{Z}_{\geq 0}$.*

(ii) *The subcategory $^p\mathsf{BE}_I(\mathfrak{h}, W)^{\geq 0}$ contains the objects $\nabla^I_w\langle m\rangle[n]$ with $w \in W$, $m \in \mathbb{Z}$, and $n \in \mathbb{Z}_{\leq 0}$.*

Proof We can assume that I is finite. Observe first that (6.1) and (6.2) imply that Δ^I_w belongs to $\mathsf{BE}_I(\mathfrak{h}, W)^{\leq 0}$ and that ∇^I_w belongs to $\mathsf{BE}_I(\mathfrak{h}, W)^{\geq 0}$.

Since $\langle 1\rangle$ is a t-exact equivalence (Lemma 7.3), we deduce the containments

$$\langle \Delta^I_w\langle m\rangle[n] : w \in W, m \in \mathbb{Z}, n \in \mathbb{Z}_{\geq 0}\rangle_{\mathrm{ext}} \subset {}^p\mathsf{BE}_I(\mathfrak{h}, W)^{\leq 0},$$
$$\langle \nabla^I_w\langle m\rangle[n] : w \in W, m \in \mathbb{Z}, n \in \mathbb{Z}_{\leq 0}\rangle_{\mathrm{ext}} \subset {}^p\mathsf{BE}_I(\mathfrak{h}, W)^{\geq 0},$$

where the left-hand side denotes the subcategory generated under extensions by the objects indicated.

We prove the reverse containment by induction on $|I|$. If $|I| = 1$, then the desired claim was proved in Lemma 7.1. Then for a general I, choose $w \in I$ maximal, so that $\{w\}$ is open, and for $\mathscr{F}$ in $^p\mathsf{BE}_I(\mathfrak{h}, W)^{\leq 0}$, consider the distinguished triangle

$$(i^I_w)_!(i^I_w)^*\mathscr{F} \to \mathscr{F} \longrightarrow (i^I_{I\smallsetminus\{w\}})_!(i^I_{I\smallsetminus\{w\}})^*\mathscr{F} \xrightarrow{[1]} .$$

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From the definitions we see that $(i^I_w)^*\mathscr{F}$ belongs to ${}^p\mathsf{BE}_{\{w\}}(\mathfrak{h}, W)^{\leq 0}$ and $(i^I_{I\smallsetminus\{w\}})^*$ belongs to ${}^p\mathsf{BE}_{I\smallsetminus\{w\}}(\mathfrak{h}, W)^{\leq 0}$. Using induction and Lemma 6.8, we deduce that $\mathscr{F}$ belongs to $\langle \Delta^I_w\langle m\rangle[n] : w \in W, m \in \mathbb{Z}, n \in \mathbb{Z}_{\geq 0}\rangle_{\mathrm{ext}}$, as desired. ∎

Remark 7.6. When $\Bbbk$ is a field, Lemma 7.5 can be made symmetric: in that case, ${}^p\mathsf{BE}_I(\mathfrak{h}, W)^{\geq 0}$ is generated under extensions by the objects $\nabla^I_w\langle m\rangle[n]$ with $w \in W$, $m \in \mathbb{Z}$, and $n \in \mathbb{Z}_{\leq 0}$ (and as a consequence, the functor $\mathbb{D}_I$ is t-exact). But this statement is not true for general coefficients. Indeed, it can fail already when I is a singleton.

7.3 Standard and Costandard Objects Are Perverse

The heart of the t-structure on $\mathsf{BE}_I(\mathfrak{h}, W)$ constructed in Section 7.2 will be denoted $\mathsf{P}^{\mathsf{BE}}_I(\mathfrak{h}, W)$. (When $I = W$, the subscript will sometimes be omitted.) The objects which belong to this heart will be called *perverse*.

Our next goal is to show that standard and costandard objects are perverse.

Lemma 7.7 *If $w \in W$, then we have the following.*

(i) *The functors $(-) \,\underline{\star}\, \nabla_w$, $\nabla_w \,\underline{\star}\, (-) \colon \mathsf{BE}(\mathfrak{h}, W) \to \mathsf{BE}(\mathfrak{h}, W)$ are right t-exact with respect to the perverse t-structure;*

(ii) *the functors $(-) \,\underline{\star}\, \Delta_w$, $\Delta_w \,\underline{\star}\, (-) \colon \mathsf{BE}(\mathfrak{h}, W) \to \mathsf{BE}(\mathfrak{h}, W)$ are left t-exact with respect to the perverse t-structure.*

Proof (i) We prove the right exactness of $\nabla_w \,\underline{\star}\, (-)$; the other functor can be treated similarly. In view of Proposition 6.11 we can assume that $\ell(\underline{w}) = 1$, *i.e.*, that $\underline{w} = (s)$ for some $s \in S$. Then Lemma 7.5 shows that to conclude, it suffices to prove that for any $w \in W$, we have $\nabla_s \,\underline{\star}\, \Delta_w \in {}^p\mathsf{BE}(\mathfrak{h}, W)^{\leq 0}$. If $sw < w$, then $\nabla_s \,\underline{\star}\, \Delta_w \cong \Delta_{sw}$ by Proposition 6.11, so the claim is clear in this case. If $sw > w$, then we use the triangles of Lemma 6.10 (for $w = e$) to deduce distinguished triangles

$$B_s \,\underline{\star}\, \Delta_w \longrightarrow \nabla_s \,\underline{\star}\, \Delta_w \longrightarrow \Delta_w\langle 1\rangle \xrightarrow{[1]}, \qquad \Delta_w\langle -1\rangle \longrightarrow \Delta_s \,\underline{\star}\, \Delta_w \longrightarrow B_s \,\underline{\star}\, \Delta_w \xrightarrow{[1]}.$$

In the second triangle, the second term is isomorphic to Δ_{sw} by Proposition 6.11, so that the third term belongs to ${}^p\mathsf{BE}(\mathfrak{h}, W)^{\leq 0}$. Once this information is known, the first triangle shows that $\nabla_s \,\underline{\star}\, \Delta_w$ belongs to ${}^p\mathsf{BE}(\mathfrak{h}, W)^{\leq 0}$, as desired.

(ii) The left exactness of our functors follows from the right-exactness of their inverses (proved in (i)) in view of [KS, Corollary 10.1.18]. ∎

Proposition 7.8 *If $w, y \in W$, then $\Delta_w \,\underline{\star}\, \nabla_y$ and $\nabla_y \,\underline{\star}\, \Delta_w$ are perverse. In particular, Δ_w and ∇_w belong to $\mathsf{P}^{\mathsf{BE}}(\mathfrak{h}, W)$.*

Proof Lemma 7.5 (i) and Lemma 7.7 (i) imply that $\Delta_w \,\underline{\star}\, \nabla_y$ belongs to the subcategory ${}^p\mathsf{BE}(\mathfrak{h}, W)^{\leq 0}$, and Lemma 7.5 (ii) and Lemma 7.7 (ii) imply that $\Delta_w \,\underline{\star}\, \nabla_y$ belongs to ${}^p\mathsf{BE}(\mathfrak{h}, W)^{\geq 0}$. Hence this object is perverse. Similar considerations show that $\nabla_y \,\underline{\star}\, \Delta_w$ is perverse. The final claims are obtained by setting $y = e$. ∎

Once Proposition 7.8 is established, its final claim can be extended to the categories $\mathsf{BE}_I(\mathfrak{h}, W)$, as follows.

Corollary 7.9 *For any locally closed subset $I \subset W$ and any $w \in I$, the objects Δ^I_w and ∇^I_w are perverse.*

Proof We can assume that I is finite. Choose some finite closed subset $J \subset W$ containing I and in which I is open. Then, since the functor $(i^W_J)_*$ is t-exact (Lemma 7.3) and does not kill any object (since it is fully faithful), we see that for any $w \in W$ the object Δ^J_w belongs to $\mathsf{P}^{\mathsf{BE}}_J(\mathfrak{h}, W)$. Then, since the functor $(i^J_I)^*$ is t-exact (Lemma 7.3), we obtain the desired claim. ∎

8 The Case of Field Coefficients

In this section we assume that $\Bbbk$ is a field.

8.1 Simple Perverse Objects

In the present setting where $\Bbbk$ is a field, the recollement formalism provides a description of the simple objects in $\mathsf{P}^{\mathsf{BE}}_I(\mathfrak{h}, W)$ [BBD, Proposition 1.4.26]. More precisely, for any $w \in I$, by Lemma 6.6 there exists (up to scalar) a unique nonzero morphism $\Delta^I_w \to \nabla^I_w$. If we denote the image of this morphism (in the abelian category $\mathsf{P}^{\mathsf{BE}}_I(\mathfrak{h}, W)$) by $\mathscr{L}^I_w$, then $\mathscr{L}^I_w$ is simple, and the isomorphism classes of simple objects in $\mathsf{P}^{\mathsf{BE}}_I(\mathfrak{h}, W)$ are in bijection with $I \times \mathbb{Z}$ via the map $(w, n) \mapsto \mathscr{L}^I_w\langle n\rangle$. Moreover, the same proof in [BBD, Théorème 4.3.1(i)] shows that $\mathsf{P}^{\mathsf{BE}}_I(\mathfrak{h}, W)$ is a finite length category. With this in hand, for any closed subset $J \subset I$ one can identify $\mathsf{P}^{\mathsf{BE}}_J(\mathfrak{h}, W)$ as the Serre subcategory of $\mathsf{P}^{\mathsf{BE}}_I(\mathfrak{h}, W)$ generated by the simple objects $\mathscr{L}^I_w\langle n\rangle$ with $n \in \mathbb{Z}$ and $w \in J$; in this setting we will sometimes consider $\mathscr{L}^I_w$ as an object of $\mathsf{P}^{\mathsf{BE}}_J(\mathfrak{h}, W)$. As usual, when $I = W$, we sometimes omit it from the notation.

By the general recollement formalism, the object $\mathscr{L}^I_w$ is characterized by the conditions that it belongs to $\mathsf{BE}_{\{\leq w\}\cap I}(\mathfrak{h}, W)$, that

$$(i^{\{\leq w\}\cap I}_w)^* \mathscr{L}^I_w \cong b_w, \tag{8.1}$$

and that for any $y \in I$ such that $y < w$ we have

$$(i^{\{\leq w\}\cap I}_y)^* \mathscr{L}^I_w \in {}^p\mathsf{BE}_{\{y\}}(\mathfrak{h}, W)^{\leq -1}, \quad (i^{\{\leq w\}\cap I}_y)^! \mathscr{L}^I_w \in {}^p\mathsf{BE}_{\{y\}}(\mathfrak{h}, W)^{\geq 1}; \tag{8.2}$$

see [BBD, Corollaire 1.4.24]. From this characterization, we deduce in particular that if $J \subset I$ is an open subset containing w, then we have $(i^I_J)^* \mathscr{L}^I_w \cong \mathscr{L}^J_w$.

Example 8.1. When $w = s$, it is easy to check that B_s satisfies conditions (8.1) and (8.2). Therefore, $\mathscr{L}_s \cong B_s$.

8.2 More Properties of Standard and Costandard Objects

It is easy to see that $\mathscr{L}_w$ is the head of Δ_w and the socle of ∇_w. Let us record the following fact about the other possible composition factors of these objects.

Lemma 8.2 *If $w \in W$, all the composition factors of the kernel of the surjection $\Delta_w \twoheadrightarrow \mathscr{L}_w$ and the cokernel of the embedding $\mathscr{L}_w \hookrightarrow \nabla_w$ are of the form $\mathscr{L}_v\langle n\rangle$ with $n \in \mathbb{Z}$ and $v \in W$, which satisfies $v < w$.*

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Proof By definition, Δ_w and ∇_w belong to $\mathsf{P}^{\mathsf{BE}}_{\{\leq w\}}(\mathfrak{h}, W)$. Moreover, the image of the canonical morphism $\Delta_w \to \nabla_w$ under $(i_w^{\{\leq w\}})^*$ is the identity map of b_w. Hence the kernel of the surjection $\Delta_w \twoheadrightarrow \mathscr{L}_w$ and the cokernel of the embedding of $\mathscr{L}_w \hookrightarrow \nabla_w$ are annihilated by $(i_w^{\{\leq w\}})^*$, so they belong to $\mathsf{BE}_{\{<w\}}(\mathfrak{h}, W)$. Since they are perverse, they in fact belong to $\mathsf{P}^{\mathsf{BE}}_{\{<w\}}(\mathfrak{h}, W)$, which finishes the proof. ∎

We will now prove the following claim, which is an analogue of a well-known result in the usual category $\mathscr{O}$ (see [Hu, §§4.1–4.2] for an algebraic proof and [BBM, §2.1] for a geometric approach).

Proposition 8.3 *Let $w \in W$.*

(i) *The socle of Δ_w is isomorphic to $\mathscr{L}_e\langle -\ell(w)\rangle$, and the cokernel of the inclusion $\mathscr{L}_e\langle -\ell(w)\rangle \hookrightarrow \Delta_w$ has no composition factor of the form $\mathscr{L}_e\langle n\rangle$.*

(ii) *The head of ∇_w is isomorphic to $\mathscr{L}_e\langle \ell(w)\rangle$, and the kernel of the surjection $\nabla_w \twoheadrightarrow \mathscr{L}_e\langle \ell(w)\rangle$ has no composition factor of the form $\mathscr{L}_e\langle n\rangle$.*

The proof of Proposition 8.3 will exploit two lemmas.

Lemma 8.4 *Let $w \in W$ and $s \in S$ be such that $ws > w$. Then $\Delta_w \,\underline{\star}\, B_s$ is perverse, and there exists a short exact sequence $\Delta_w\langle -1\rangle \hookrightarrow \Delta_{ws} \twoheadrightarrow \Delta_w \,\underline{\star}\, B_s$ in $\mathsf{P}^{\mathsf{BE}}(\mathfrak{h}, W)$.*

Proof Recall the first distinguished triangle in Lemma 6.10. By Proposition 7.8 the first two terms in this triangle belong to $\mathsf{P}^{\mathsf{BE}}(\mathfrak{h}, W)$, so the third term must lie in ${}^p\mathsf{BE}(\mathfrak{h}, W)^{\leq 0}$. On the other hand, by Example 8.1, B_s belongs to $\mathsf{P}^{\mathsf{BE}}(\mathfrak{h}, W)$, so Lemma 7.7(ii) tells us that $\Delta_w \,\underline{\star}\, B_s$ belongs to ${}^p\mathsf{BE}(\mathfrak{h}, W)^{\geq 0}$. We conclude that $\Delta_w \,\underline{\star}\, B_s$ in fact belongs to $\mathsf{P}^{\mathsf{BE}}(\mathfrak{h}, W)$, and that the triangle under consideration is a short exact sequence in $\mathsf{P}^{\mathsf{BE}}(\mathfrak{h}, W)$. ∎

The following lemma is more subtle; its proof will be given in Section 8.5.

Lemma 8.5 *Let $w \in W$ and $s \in S$ be such that $ws > w$. Then all the composition factors of $\Delta_w \,\underline{\star}\, B_s$ are of the form $\mathscr{L}_y\langle n\rangle$ with $ys < y$.*

Proof of Proposition 8.3 We will prove (i) by induction on $\ell(w)$; then (ii) follows by duality.

If $w = e$, we have $\Delta_e \cong \mathscr{L}_e$; see Example 6.7 (1). Thus, there is nothing to prove in this case. Now let $w \in W \smallsetminus \{e\}$, and assume the claim is known for elements $y \in W$ with $\ell(y) < \ell(w)$. Choose $s \in S$ such that $ws < w$, and consider the exact sequence $\Delta_{ws}\langle -1\rangle \hookrightarrow \Delta_w \twoheadrightarrow \Delta_{ws} \,\underline{\star}\, B_s$ provided by Lemma 8.4. By induction we know that there exists an embedding $\mathscr{L}_e\langle -\ell(w)\rangle \hookrightarrow \Delta_{ws}\langle -1\rangle$ whose cokernel has no composition factor of the form $\mathscr{L}_e\langle n\rangle$. On the other hand, Lemma 8.5 ensures that $\Delta_{ws} \,\underline{\star}\, B_s$ has no composition factor of this form either. Hence, we obtain an embedding $\mathscr{L}_e\langle -\ell(w)\rangle \hookrightarrow \Delta_w$ whose cokernel has no composition factor of the form $\mathscr{L}_e\langle n\rangle$. To finish the proof, it suffices to show that Δ_w has no subobject of the form $\mathscr{L}_x\langle n\rangle$ with $x \neq e$.

Assume for a contradiction that there exists an injective morphism $\mathscr{L}_x\langle n\rangle \hookrightarrow \Delta_w$ with $x \neq e$. Using induction we see that this morphism does not factor through $\Delta_{ws}\langle -1\rangle$; hence its composition with the surjection $\Delta_w \twoheadrightarrow \Delta_{ws} \,\underline{\star}\, B_s$ is nonzero. In view of Lemma 8.5, this implies that $xs < x$, and hence that $\Delta_x \cong \Delta_{xs} \underline{\star} \Delta_s$ (Proposition 6.11). Proposition 6.11 also shows that the functor

$$(8.3) \qquad (-) \underline{\star} \Delta_s \colon \mathsf{BE}(\mathfrak{h}, W) \longrightarrow \mathsf{BE}(\mathfrak{h}, W)$$

is invertible; in particular, it induces an isomorphism

$$\mathrm{Hom}_{\mathsf{P^{BE}}(\mathfrak{h},W)}(\Delta_{xs}\langle n\rangle, \Delta_{ws}) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{P^{BE}}(\mathfrak{h},W)}(\Delta_x\langle n\rangle, \Delta_w).$$

By induction we know that any nonzero subobject of Δ_{ws} must admit $\mathscr{L}_e\langle -\ell(ws)\rangle$ as a composition factor. Applying the induction hypothesis to Δ_{xs} also, we deduce that any nonzero morphism $\Delta_{xs}\langle n\rangle \to \Delta_{ws}$ must be injective. Since (8.3) is left t-exact (see Lemma 7.7 (ii)), we finally obtain that any nonzero morphism $\Delta_x\langle n\rangle \to \Delta_w$ is injective. However, since by assumption there exists an embedding $\mathscr{L}_x\langle n\rangle \hookrightarrow \Delta_w$, we can construct a nonzero and noninjective morphism of this form as the composition $\Delta_x\langle n\rangle \twoheadrightarrow \mathscr{L}_x\langle n\rangle \hookrightarrow \Delta_w$, where the first morphism is the natural one. This provides the desired contradiction. ∎

For completeness we record the following consequence of Proposition 8.3.

Proposition 8.6 *Let $w, y \in W$. Then*

$$\dim \mathrm{Hom}_{\mathsf{BE}(\mathfrak{h},W)}(\Delta_w, \Delta_y\langle n\rangle) = \begin{cases} 1 & \text{if } w \le y \text{ and } n = \ell(y) - \ell(w), \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, if $w \le y$, any nonzero morphism $\Delta_w \to \Delta_y\langle \ell(y) - \ell(w)\rangle$ is injective.

Proof If $w \nleq y$, then the Hom-space under consideration vanishes by adjunction and Lemma 6.5.

Assume now that $w \le y$ and set $m = \ell(y) - \ell(w)$. If $f \colon \Delta_w \to \Delta_y\langle n\rangle$ is a nonzero morphism, its image must admit $\mathscr{L}_e\langle n - \ell(y)\rangle$ as a composition factor; therefore its kernel cannot contain the socle of Δ_w. This means that the kernel is trivial, and f is injective. Moreover, we must have $n - \ell(y) = -\ell(w)$, *i.e.*, $n = m$.

To conclude, it remains to show that $\dim \mathrm{Hom}_{\mathsf{BE}(\mathfrak{h},W)}(\Delta_w, \Delta_y\langle m\rangle) = 1$ (where, as above, we assume that $w \le y$ and set $m = \ell(y) - \ell(w)$). We proceed by induction on $\ell(y)$, the case $\ell(y) = 0$ being obvious. Assume that $\ell(y) > 0$ and choose $s \in S$ such that $ys < y$. If $ws < w$, then as in the proof of Proposition 8.3 we have

$$\mathrm{Hom}_{\mathsf{P^{BE}}(\mathfrak{h},W)}(\Delta_w, \Delta_y\langle m\rangle) \cong \mathrm{Hom}_{\mathsf{P^{BE}}(\mathfrak{h},W)}(\Delta_{ws}, \Delta_{ys}\langle m\rangle),$$

and the result follows from the induction hypothesis. If now $ws > w$, then the exact sequence of Lemma 8.4, applied to ys, induces an exact sequence of $\Bbbk$-vector spaces

$$0 \longrightarrow \mathrm{Hom}_{\mathsf{P^{BE}}(\mathfrak{h},W)}(\Delta_w, \Delta_{ys}\langle m-1\rangle) \longrightarrow \mathrm{Hom}_{\mathsf{P^{BE}}(\mathfrak{h},W)}(\Delta_w, \Delta_y\langle m\rangle) \\ \longrightarrow \mathrm{Hom}_{\mathsf{P^{BE}}(\mathfrak{h},W)}(\Delta_w, \Delta_{ys} \underline{\star} B_s\langle m\rangle).$$

Here the last space must vanish, because $\Delta_{ys} \underline{\star} B_s\langle m\rangle$ does not admit $\mathscr{L}_w$ as a composition factor (Lemma 8.5). We deduce that

$$\mathrm{Hom}_{\mathsf{P^{BE}}(\mathfrak{h},W)}(\Delta_w, \Delta_{ys}\langle m-1\rangle) \cong \mathrm{Hom}_{\mathsf{P^{BE}}(\mathfrak{h},W)}(\Delta_w, \Delta_y\langle m\rangle),$$

and again the desired result follows from the induction hypothesis. ∎

By duality we have analogous properties for costandard objects.

Proposition 8.7 *Let $w, y \in W$. Then*

$$\dim \operatorname{Hom}_{\mathsf{BE}(\mathfrak{h},W)}(\nabla_y\langle n\rangle, \nabla_w) = \begin{cases} 1 & \text{if } w \leq y \text{ and } n = \ell(w) - \ell(y), \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, if $w \leq y$, any nonzero morphism $\nabla_y\langle \ell(w) - \ell(y)\rangle \to \nabla_w$ is surjective. ∎

8.3 A Category Attached to a Simple Reflection.

The goal of Sections 8.3–8.5 is to prove Lemma 8.5. These results will not be used in the rest of the paper. Most of our constructions could be performed for general coefficients; but for simplicity we continue to assume that $\mathbb{k}$ is a field.

We denote by $\mathsf{BE}(\mathfrak{h}, W|s)$ the full triangulated subcategory of $\mathsf{BE}(\mathfrak{h}, W)$ generated by the image of the functor $(-) \underline{\star} B_s$ and we fix $s \in S$. Our first objective is to endow this category with the same kind of structure (local versions, recollement, and perverse t-structure) as for $\mathsf{BE}(\mathfrak{h}, W)$.

Let $W^s = \{w \in W \mid ws < w\}$. A locally closed subset $I \subset W$ is said to be *right s-stable* if $w \in I$ implies $ws \in I$.

Recall from Corollary 6.4 that if I is closed and right s-stable, then the full subcategory $\mathsf{BE}_I(\mathfrak{h}, W)$ of $\mathsf{BE}(\mathfrak{h}, W)$ is stable under the functor $(-) \underline{\star} B_s$. If $I \subset W$ is now *locally* closed and finite, one can write $I = I_0 \smallsetminus I_1$ with $I_1 \subset I_0 \subset W$ finite, closed, and right s-stable. By Remark 5.7, the category $\mathsf{BE}_I(\mathfrak{h}, W)$ identifies with the Verdier quotient $\mathsf{BE}_{I_0}(\mathfrak{h}, W)/\mathsf{BE}_{I_1}(\mathfrak{h}, W)$. Then the functor $(-) \underline{\star} B_s \colon \mathsf{BE}_{I_0}(\mathfrak{h}, W) \to \mathsf{BE}_{I_0}(\mathfrak{h}, W)$ induces an endofunctor of $\mathsf{BE}_I(\mathfrak{h}, W)$ that will also be denoted $(-) \underline{\star} B_s$. This functor is clearly self-adjoint.

In this setting, we define $\mathsf{BE}_I(\mathfrak{h}, W|s)$ to be the full triangulated subcategory of $\mathsf{BE}_I(\mathfrak{h}, W)$ generated by the image of the functor $(-) \underline{\star} B_s$.

Lemma 8.8 *Let $w \in W^s$, and let $\underline{w}$ and $\underline{w}'$ be two reduced expressions for w. The images of $B_{\underline{w}}$ and $B_{\underline{w}'}$ in $\mathsf{BE}_{\{ws,w\}}(\mathfrak{h}, W)$ are canonically isomorphic.*

Proof We will use the calculations from Example 5.4. Let us rewrite the triangle (5.7) as

$$b_{ws}\langle -1\rangle \longrightarrow \Delta_w^{\{ws,w\}} \longrightarrow B_{\underline{w}} \xrightarrow{[1]} .$$

There is another version of this triangle in which the third term is replaced by $B_{\underline{w}'}$. We claim that there exist unique vertical maps p and q making the following diagram commute.

$$\begin{array}{ccccccc} b_{ws}\langle -1\rangle & \longrightarrow & \Delta_w^{\{ws,w\}} & \longrightarrow & B_{\underline{w}} & \xrightarrow{[1]} & \\ {\scriptstyle p}\downarrow & & \| & & \downarrow{\scriptstyle q} & & \\ b_{ws}\langle -1\rangle & \longrightarrow & \Delta_w^{\{ws,w\}} & \longrightarrow & B_{\underline{w}'} & \xrightarrow{[1]} & \end{array}$$

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According to [BBD, Proposition 1.1.9], the existence and uniqueness of p and q would follow if we knew the following two claims:

$$\mathrm{Hom}_{\mathsf{BE}_{\{ws,w\}}(\mathfrak{h},W)}(b_{ws}\langle -1\rangle, B_{\underline{w}'}) = 0,$$
$$\mathrm{Hom}_{\mathsf{BE}_{\{ws,w\}}(\mathfrak{h},W)}(b_{ws}\langle -1\rangle, B_{\underline{w}'}[-1]) = 0.$$

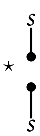
The first one is obvious for degree reasons. The second one is equivalent to the vanishing of $\mathrm{Hom}(b_{ws}, B_{\underline{w}'}(-1))$. As we observed in Example 5.4, the R-module

$$\mathrm{Hom}^{\bullet}_{\mathscr{D}^{\oplus}_{\mathrm{BS},\{ws,w\}}(\mathfrak{h},W)}(b_{ws}, B_{\underline{w}'})$$

is generated in degree 1; in particular, it contains no nonzero element of degree -1, as desired.

The same reasoning with the roles of $\underline{w}$ and $\underline{w}'$ reversed leads to a similar diagram with vertical maps in the opposite directions. Using the uniqueness of the various vertical maps, one concludes that p and q are isomorphisms, as desired. ∎

Henceforth, for $w \in W^s$, we set $b^s_w = (i^{\{\leq w\}}_{\{ws,w\}})^* B_{\underline{w}}$ for any reduced expression $\underline{w}$ for w. (By Lemma 8.8, this definition is independent of the choice of $\underline{w}$.) Choosing for $\underline{w}$ a reduced expression of the form $\underline{y}s$ (with $\underline{y}$ a reduced expression for ws), light-leaves considerations show that the R-module $\mathrm{Hom}^{\bullet}_{\mathscr{D}^{\oplus}_{\mathrm{BS},\{ws,w\}}(\mathfrak{h},W)}(b^s_w, b^s_w)$ is free of rank two, and generated by the identity (of degree 0) and the degree-2 morphism

$\mathrm{id}_{B_{\underline{y}}} \star$  .

In the course of the proof of Lemma 8.8, we saw that there are distinguished triangles

$$(8.4) \qquad b_{ws}\langle -1\rangle \longrightarrow \Delta^{\{ws,w\}}_w \longrightarrow b^s_w \xrightarrow{[1]}, \qquad b^s_w \longrightarrow \nabla^{\{ws,w\}}_w \longrightarrow b_{ws}\langle 1\rangle \xrightarrow{[1]}.$$

Lemma 8.9 *For any $w \in W^s$, the triangulated category $\mathsf{BE}_{\{ws,w\}}(\mathfrak{h}, W|s)$ is generated by the objects of the form $b^s_w(m)$ with $m \in \mathbb{Z}$.*

Proof The category $\mathsf{BE}_{\{ws,w\}}(\mathfrak{h}, W)$ is clearly generated by the objects of the form $b_{ws}(m)$ and $b^s_w(m)$. It follows that $\mathsf{BE}_{\{ws,w\}}(\mathfrak{h}, W|s)$ is generated by the objects

$$b_{ws}(m) \,\underline{\star}\, B_s \cong b^s_w(m) \quad \text{and} \quad b^s_w(m) \,\underline{\star}\, B_s \cong b^s_w(m+1) \oplus b^s_w(m-1),$$

where the latter isomorphism follows from (3.2). ∎

8.4 Recollement

We now show that the categories $\mathsf{BE}_I(\mathfrak{h}, W|s)$ (with I right s-stable) satisfy the same recollement formalism as the categories $\mathsf{BE}_I(\mathfrak{h}, W)$.

Proposition 8.10 *Let $I \subset W$ be a finite locally closed right s-stable subset and let $J \subset I$ be a closed right s-stable subset. Then the restriction of the functors from Proposition 5.6*

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gives a recollement diagram

$$\begin{array}{ccccc} & \xleftarrow{\;(i^I_J)^*\;} & & \xleftarrow{\;(i^I_{I\smallsetminus J})_!\;} & \\ \mathsf{BE}_J(\mathfrak{h}, W|s) & \xrightarrow{\;(i^I_J)_*\;} & \mathsf{BE}_I(\mathfrak{h}, W|s) & \xrightarrow{\;(i^I_{I\smallsetminus J})^*\;} & \mathsf{BE}_{I\smallsetminus J}(\mathfrak{h}, W|s). \\ & \xleftarrow[\;(i^I_J)^!\;]{} & & \xleftarrow[\;(i^I_{I\smallsetminus J})_*\;]{} & \end{array}$$

Proof We need to show that the six functors from Proposition 5.6 take the subcategory generated by $(-) \underline{\star} B_s$ to the subcategory generated by $(-) \underline{\star} B_s$. This is obvious for $(i^I_J)_*$ and $(i^I_{I\smallsetminus J})^*$.

Now we consider the functors $(i^I_J)^*$ and $(i^I_{I\smallsetminus J})_!$. Let $\mathscr{G} \in \mathsf{BE}_I(\mathfrak{h}, W)$, and set $\mathscr{F} := \mathscr{G} \underline{\star} B_s$. Then we have a distinguished triangle

$$(i^I_{I\smallsetminus J})_!(i^I_{I\smallsetminus J})^*\mathscr{F} \longrightarrow \mathscr{F} \longrightarrow (i^I_J)_*(i^I_J)^*\mathscr{F} \xrightarrow{[1]} . \tag{8.5}$$

On the other hand, we can also form the distinguished triangle

$$\big((i^I_{I\smallsetminus J})_!(i^I_{I\smallsetminus J})^*\mathscr{G}\big) \underline{\star} B_s \longrightarrow \mathscr{G} \underline{\star} B_s \longrightarrow \big((i^I_J)_*(i^I_J)^*\mathscr{G}\big) \underline{\star} B_s \xrightarrow{[1]} . \tag{8.6}$$

We claim that the triangles (8.5) and (8.6) are canonically isomorphic. This would follow from [BBD, Proposition 1.1.9] if we knew that

$$\mathrm{Hom}_{\mathsf{BE}_I(\mathfrak{h},W)}((i^I_{I\smallsetminus J})_!(i^I_{I\smallsetminus J})^*\mathscr{F}, ((i^I_J)_*(i^I_J)^*\mathscr{G}) \underline{\star} B_s[n]) = 0, \tag{8.7}$$

$$\mathrm{Hom}_{\mathsf{BE}_I(\mathfrak{h},W)}(((i^I_{I\smallsetminus J})_!(i^I_{I\smallsetminus J})^*\mathscr{G}) \underline{\star} B_s, (i^I_J)_*(i^I_J)^*\mathscr{F}[n]) = 0, \tag{8.8}$$

for all $n \in \mathbb{Z}$. (Actually, we only need this for $n \in \{0, -1\}$.) Now $\big((i^I_J)_*(i^I_J)^*\mathscr{G}\big) \underline{\star} B_s$ belongs to $\mathsf{BE}_J(\mathfrak{h}, W)$, so (8.7) holds by adjunction and basic properties of recollement. For (8.8), because $(-) \underline{\star} B_s$ is self-adjoint, we have

$$\begin{aligned} &\mathrm{Hom}_{\mathsf{BE}_I(\mathfrak{h},W)}\big(((i^I_{I\smallsetminus J})_!(i^I_{I\smallsetminus J})^*\mathscr{G}) \underline{\star} B_s, (i^I_J)_*(i^I_J)^*\mathscr{F}[n]\big) \\ &\qquad\cong \mathrm{Hom}_{\mathsf{BE}_I(\mathfrak{h},W)}\big((i^I_{I\smallsetminus J})_!(i^I_{I\smallsetminus J})^*\mathscr{G}, ((i^I_J)_*(i^I_J)^*\mathscr{F}) \underline{\star} B_s[n]\big). \end{aligned}$$

This vanishes by the same reasoning as above.

This result implies that for any $\mathscr{H}$ in $\mathsf{BE}_{I\smallsetminus J}(\mathfrak{h}, W)$ we have

$$(i^I_{I\smallsetminus J})_!(\mathscr{H} \underline{\star} B_s) \cong (i^I_{I\smallsetminus J})_!(\mathscr{H}) \underline{\star} B_s.$$

We deduce that $(i^I_{I\smallsetminus J})_!$ sends $\mathsf{BE}_{I\smallsetminus J}(\mathfrak{h}, W|s)$ to $\mathsf{BE}_I(\mathfrak{h}, W|s)$. Similarly, since the functor $(i^I_J)_*$ is fully faithful and commutes with the functors $(-) \underline{\star} B_s$, we obtain that

$$(i^I_J)^*(\mathscr{G} \underline{\star} B_s) \cong (i^I_J)^*(\mathscr{G}) \underline{\star} B_s.$$

Again, this implies that $(i^I_J)^*$ sends $\mathsf{BE}_I(\mathfrak{h}, W|s)$ to $\mathsf{BE}_J(\mathfrak{h}, W|s)$.

The analogous claims for $(i^I_J)^!$ and $(i^I_{I\smallsetminus J})_*$ can be proved similarly, or deduced by duality, which finishes the proof. ∎

Now let $I \subset W$ and $J \subset I$ be finite locally closed right s-stable subsets. In view of Proposition 8.10, we can also define the pushforward and pullback functors $(i^I_J)^*$, $(i^I_J)^!$, $(i^I_J)_*$, $(i^I_J)_!$ for the categories $\mathsf{BE}_J(\mathfrak{h}, W|s)$ and $\mathsf{BE}_I(\mathfrak{h}, W|s)$ as the restriction of those in Section 5.4. Then these functors also satisfy the properties of Lemma 5.12.

8.5 The Perverse t-structure

We will denote by C the full subcategory of $\mathsf{BE}_{\{ws,w\}}(\mathfrak{h}, W|s)$ whose objects are direct sums of objects of the form $b_w^s\langle n\rangle$, $n \in \mathbb{Z}$.

Lemma 8.11 *Let $w \in W^s$. Then if we set*

$$
\begin{aligned}
{}^p\mathsf{BE}_{\{ws,w\}}(\mathfrak{h}, W|s)^{\leq 0} &:= {}^p\mathsf{BE}_{\{ws,w\}}(\mathfrak{h}, W)^{\leq 0} \cap \mathsf{BE}_{\{ws,w\}}(\mathfrak{h}, W|s),\\
{}^p\mathsf{BE}_{\{ws,w\}}(\mathfrak{h}, W|s)^{\geq 0} &:= {}^p\mathsf{BE}_{\{ws,w\}}(\mathfrak{h}, W)^{\geq 0} \cap \mathsf{BE}_{\{ws,w\}}(\mathfrak{h}, W|s),
\end{aligned}
$$

the pair $({}^p\mathsf{BE}_{\{ws,w\}}(\mathfrak{h}, W|s)^{\leq 0}, {}^p\mathsf{BE}_{\{ws,w\}}(\mathfrak{h}, W|s)^{\geq 0})$ is a t-structure on the category $\mathsf{BE}_{\{ws,w\}}(\mathfrak{h}, W|s)$, whose heart is C.

Proof We claim that C is an *admissible abelian* subcategory of $\mathsf{BE}_{\{ws,w\}}(\mathfrak{h}, W|s)$ in the sense of [BBD, Definition 1.2.5]. It can be checked from the triangles in (8.4) that b_w^s lies in $\mathsf{P}^{\mathsf{BE}}_{\{w,ws\}}(\mathfrak{h}, W)$ (and thus that C is a subcategory of $\mathsf{P}^{\mathsf{BE}}_{\{w,ws\}}(\mathfrak{h}, W)$). It follows immediately that $\mathrm{Hom}_{\mathsf{BE}_{\{w,ws\}}(\mathfrak{h},W|s)}(b_w^s, b_w^s\langle n\rangle[m]) = 0$ if $m < 0$. Hence, C satisfies [BBD, §1.2.0]. On the other hand, to check that any morphism in C is admissible, we must check that $[\mathsf{C}] * [\mathsf{C}[1]] \subset [\mathsf{C}[1]] * [\mathsf{C}]$, as explained in [BBD, Exemple 1.3.11(ii)]. However, the objects whose class belongs to $[\mathsf{C}]*[\mathsf{C}[1]]$ are exactly the cones of morphisms in C. From the remarks in Section 8.3 we see that such a morphism is a direct sum of morphisms of the form $b_w^s \to 0$, $0 \to b_w^s$, or $b_w^s \xrightarrow{\mathrm{id}} b_w^s$. It is easily checked that the class of the cone of such morphisms belongs to $[\mathsf{C}[1]] * [\mathsf{C}]$, and the claim follows.

Since $\mathrm{Hom}_{\mathsf{BE}_{\{w,ws\}}(\mathfrak{h},W|s)}(\mathscr{F}, \mathscr{G}[1]) = 0$ for $\mathscr{F}, \mathscr{G}$ in C, this subcategory is also stable under extensions. Since C generates $\mathsf{BE}_{\{w,ws\}}(\mathfrak{h}, W|s)$ as a triangulated category (Lemma 8.9), applying [BBD, Proposition 1.3.13], we obtain a t-structure

$$
\left({}^p\mathsf{BE}_{\{ws,w\}}(\mathfrak{h}, W|s)^{\leq 0}, {}^p\mathsf{BE}_{\{ws,w\}}(\mathfrak{h}, W|s)^{\geq 0}\right)
$$

on $\mathsf{BE}_{\{w,ws\}}(\mathfrak{h}, W|s)$ whose nonnegative, resp., nonpositive, part is generated under extensions by the objects of the form $\mathscr{F}[n]$ with $\mathscr{F}$ in $\mathcal{C}$ and $n \leq 0$, resp., $n \geq 0$.

To conclude, it remains to prove that

$$
\begin{aligned}
{}^p\mathsf{BE}_{\{ws,w\}}(\mathfrak{h}, W|s)^{\leq 0} &= {}^p\mathsf{BE}_{\{ws,w\}}(\mathfrak{h}, W)^{\leq 0} \cap \mathsf{BE}_{\{ws,w\}}(\mathfrak{h}, W|s),\\
{}^p\mathsf{BE}_{\{ws,w\}}(\mathfrak{h}, W|s)^{\geq 0} &= {}^p\mathsf{BE}_{\{ws,w\}}(\mathfrak{h}, W)^{\geq 0} \cap \mathsf{BE}_{\{ws,w\}}(\mathfrak{h}, W|s).
\end{aligned}
$$

First we noted above that $\mathsf{C} \subset \mathsf{P}^{\mathsf{BE}}_{\{w,ws\}}(\mathfrak{h}, W)$, so each left-hand side above is contained in the corresponding right-hand side. Now, let $\mathscr{F} \in {}^p\mathsf{BE}_{\{ws,w\}}(\mathfrak{h}, W)^{\leq 0} \cap \mathsf{BE}_{\{ws,w\}}(\mathfrak{h}, W|s)$. Consider the truncation triangle $\tau_{\leq 0}(\mathscr{F}) \to \mathscr{F} \to \tau_{>0}(\mathscr{F}) \xrightarrow{[1]}$ for the t-structure we have just constructed on $\mathsf{BE}_{\{ws,w\}}(\mathfrak{h}, W|s)$. From the containments we have already proved, we see that this triangle identifies with the truncation triangle for the perverse t-structure on $\mathsf{BE}_{\{ws,w\}}(\mathfrak{h}, W)$. From our assumption we deduce that $\tau_{>0}(\mathscr{F}) = 0$, or in other words, that $\mathscr{F}$ belongs to ${}^p\mathsf{BE}_{\{ws,w\}}(\mathfrak{h}, W|s)^{\leq 0}$. The remaining equality can be proved similarly. ∎

For any finite locally closed right s-stable subset $I \subset W$, we now set

$$\begin{aligned}
&{}^p\mathsf{BE}_I(\mathfrak{h}, W|s)^{\leq 0} \\
&\quad = \left\{ \mathscr{F} \in \mathsf{BE}_I(\mathfrak{h}, W|s) \mid \forall w \in I \cap W^s \ (i^I_{\{ws,w\}})^*(\mathscr{F}) \in {}^p\mathsf{BE}_{\{ws,w\}}(\mathfrak{h}, W|s)^{\leq 0} \right\}, \\
&{}^p\mathsf{BE}_I(\mathfrak{h}, W|s)^{\geq 0} \\
&\quad = \left\{ \mathscr{F} \in \mathsf{BE}_I(\mathfrak{h}, W|s) \mid \forall w \in I \cap W^s, \ (i^I_{\{ws,w\}})^!(\mathscr{F}) \in {}^p\mathsf{BE}_{\{ws,w\}}(\mathfrak{h}, W|s)^{\geq 0} \right\}.
\end{aligned}$$

The recollement formalism ensures that this defines a t-structure on $\mathsf{BE}_I(\mathfrak{h}, W|s)$, which we will call the *perverse t-structure*. The same arguments as in the proof of Lemma 8.11 show that we have

$$\begin{aligned}
{}^p\mathsf{BE}_I(\mathfrak{h}, W|s)^{\leq 0} &= {}^p\mathsf{BE}_I(\mathfrak{h}, W)^{\leq 0} \cap \mathsf{BE}_I(\mathfrak{h}, W|s), \\
{}^p\mathsf{BE}_I(\mathfrak{h}, W|s)^{\geq 0} &= {}^p\mathsf{BE}_I(\mathfrak{h}, W)^{\geq 0} \cap \mathsf{BE}_I(\mathfrak{h}, W|s).
\end{aligned}$$

In particular, the heart of this t-structure is $\mathsf{P}^{\mathsf{BE}}_I(\mathfrak{h}, W) \cap \mathsf{BE}_I(\mathfrak{h}, W|s)$. As for

$$\mathsf{P}^{\mathsf{BE}}_I(\mathfrak{h}, W),$$

any object in this abelian category has finite length.

We now investigate the simple objects in this heart. Once again by the recollement formalism [BBD, Proposition 1.4.26], these objects can be classified as follows. For any $w \in W^s \cap I$, there exists a unique simple object $\mathscr{L}^{I,s}_w$ in $\mathsf{P}^{\mathsf{BE}}_I(\mathfrak{h}, W) \cap \mathsf{BE}_I(\mathfrak{h}, W|s)$ that belongs to $\mathsf{BE}_{\{\leq w\} \cap I}(\mathfrak{h}, W|s)$ and satisfies $(i^I_{\{w,ws\}})^* \mathscr{L}^{I,s}_w \cong b^s_w$. Moreover, any simple object in $\mathsf{P}^{\mathsf{BE}}_I(\mathfrak{h}, W) \cap \mathsf{BE}_I(\mathfrak{h}, W|s)$ is (up to isomorphism) of the form $\mathscr{L}^{I,s}_w \langle n \rangle$ for $w \in W^s \cap I$ and $n \in \mathbb{Z}$.

Lemma 8.12 *For any $w \in W^s \cap I$, we have $\mathscr{L}^{I,s}_w \cong \mathscr{L}^I_w$.*

Proof We will show that $\mathscr{L}^{I,s}_w$ satisfies the properties (8.1)–(8.2) that characterize $\mathscr{L}^I_w$.

First, by definition we have $(i^I_{\{ws,w\}})^* \mathscr{L}^{I,s}_w \cong b^s_w$. Using the triangles in (8.4) we deduce that $(i^I_w)^* \mathscr{L}^{I,s}_w \cong b_w$, so that $\mathscr{L}^{I,s}_w$ satisfies (8.1), and that

$$(i^I_{ws})^* \mathscr{L}^{I,s}_w \cong b_{ws} \langle -1 \rangle [1], \quad (i^I_{ws})^! \mathscr{L}^{I,s}_w \cong b_{ws} \langle 1 \rangle [-1].$$

Hence $\mathscr{L}^{I,s}_w$ satisfies (8.2) for $y = ws$. Now if $y \in I \cap \{< w\}$ and $y \neq ws$, by the analogue of (8.2) for $\mathscr{L}^{I,s}_w$ we have

$$(i^{\{\leq w\} \cap I}_{\{ys,y\}})^* \mathscr{L}^{I,s}_w \in {}^p\mathsf{BE}_{\{ys,y\}}(\mathfrak{h}, W|s)^{\leq -1} \subset {}^p\mathsf{BE}_{\{ys,y\}}(\mathfrak{h}, W)^{\leq -1}.$$

Therefore, $(i^{\{\leq w\} \cap I}_y)^* \mathscr{L}^{I,s}_w \in {}^p\mathsf{BE}_{\{y\}}(\mathfrak{h}, W)^{\leq -1}$. One proves similarly that

$$(i^{\{\leq w\} \cap I}_y)^! \mathscr{L}^{I,s}_w \in {}^p\mathsf{BE}_{\{y\}}(\mathfrak{h}, W)^{\geq 1},$$

which concludes the proof. ∎

We can finally prove Lemma 8.5.

Proof of Lemma 8.5 By Corollary 6.4, $\Delta_w \underline{\star} B_s$ belongs to $\mathsf{BE}_{\{\leq ws\}}(\mathfrak{h}, W|s)$, and Lemma 8.4 ensures that it also belongs to $\mathsf{P}^{\mathsf{BE}}(\mathfrak{h}, W)$. Therefore, it belongs to the heart of the perverse t-structure on $\mathsf{BE}_{\{\leq ws\}}(\mathfrak{h}, W|s)$. By Lemma 8.12, any finite filtration of this object (in the abelian category given by the heart of this t-structure) with simple subquotients can also be viewed as a finite filtration with simple subquotients in $\mathsf{P}^{\mathsf{BE}}(\mathfrak{h}, W)$, and as such these subquotients are of the form $\mathscr{L}_y\langle n\rangle$ with $ys < y$. ■

8.6 Description of Some Simple Objects

Under the present assumption that $\Bbbk$ is a field, [EW2, Theorem 6.26] provided a description of the isomorphism classes of indecomposable objects in the Karoubian envelope $\mathscr{D}(\mathfrak{h}, W)$ of $\mathscr{D}^{\oplus}_{\mathrm{BS}}(\mathfrak{h}, W)$: for any $w \in W$ there exists a unique indecomposable object B_w (up to isomorphism) which is a direct summand of $B_{\underline{w}}$ for any reduced expression $\underline{w}$ for w, but which is not isomorphic to a direct summand of an object of the form $B_{\underline{v}}(n)$ with $n \in \mathbb{Z}$ and $\underline{v}$ an expression such that $\ell(\underline{v}) < \ell(w)$. Moreover, the assignment $(w, n) \mapsto B_w(n)$ induces a bijection between $W \times \mathbb{Z}$ and the set of isomorphism classes of indecomposable objects in $\mathscr{D}(\mathfrak{h}, W)$. The natural functor $\mathsf{BE}(\mathfrak{h}, W) \to K^{\mathrm{b}}\mathscr{D}(\mathfrak{h}, W)$ is an equivalence of triangulated categories [AMRW1, Lemma 4.9.1]; in particular, using this identification we can see the objects B_w as living in $\mathsf{BE}(\mathfrak{h}, W)$.

Recall the ring isomorphism

$$\mathcal{H}_{(W,S)} \xrightarrow{\sim} [\mathsf{BE}(\mathfrak{h}, W)]_{\Delta} \tag{8.9}$$

constructed in Section 6.6. Recall also the Kazhdan–Lusztig basis $(\underline{H}_w : w \in W)$ considered in [So2]. We conclude this section with the following claim, which provides a description of $\mathscr{L}_w$ in a favorable situation.

Proposition 8.13 *Let $w \in W$, and assume that the image of $\underline{H}_w$ under (8.9) is the class of B_w. Then $B_w \cong \mathscr{L}_w$.*

Remark 8.14. The assumption in Proposition 8.13 is always satisfied if $\ell(w) \leq 2$, or if W is finite and w is the longest element in W. In the latter case, this property follows from the fact that we have $B_w \star B_s \cong B_w(1) \oplus B_w(-1)$ for any $s \in S$ [So2, Proposition 2.9]. See [JW] for more examples of situations when this condition is satisfied or not satisfied, (in the case when (W, S) is crystallographic).

Another setting where this assumption is known (for any $w \in W$) is the one considered in Remark 6.12 (1). Namely, the equivalence between $\mathscr{D}(\mathfrak{h}, W)$ and the category of Soergel bimodules considered in this remark sends B_w to the indecomposable Soergel bimodule B_w attached to w. In view of this identification, the condition in Proposition 8.13 becomes Soergel's conjecture for V, which was proved in [EW1].

Proof of Proposition 8.13 Recall the characterization of $\mathscr{L}_w$ given by (8.1)–(8.2). We will show that the object B_w satisfies the first condition in (8.2); the second one can be either proved similarly or deduced by duality; (8.1) is easy and left to the reader.

Let $y \in W$ be such that $y < w$. Writing $\{y\}$ as an intersection $\{\leq y\} \cap I$ where $I \subset \{\leq w\}$ is open and y is minimal in I and using Remark 5.9 (2), we see that $(i_y^{\{\leq w\}})^* B_w$ is isomorphic to an object of the form $\bigoplus_{m\in\mathbb{Z}} (b_y(m))^{\oplus \lambda_m}$ for some coefficients $\lambda_m \in \mathbb{Z}_{\geq 0}$ (with $\lambda_m = 0$ for all but finitely many m's). In terms of this decomposition, the class of this object in $[\mathsf{BE}_{\{y\}}(\mathfrak{h}, W)]$ is then $\sum_{m\in\mathbb{Z}} \lambda_m \cdot [b_y(m)] = \sum_{m\in\mathbb{Z}} v^m \lambda_m \cdot [b_y]$.

On the other hand, the coefficient of $\underline{H}_w$ on H_y (in the standard basis) belongs to $v\mathbb{Z}[v]$. Hence our assumption implies that $\lambda_m = 0$ for $m \le 0$, so that

$$(i_y^{\{\le w\}})^* B_w \cong \bigoplus_{m\in\mathbb{Z}_{>0}} (b_y\langle -m\rangle[m])^{\oplus\lambda_m}.$$

Here the right-hand side belongs to ${}^p\mathsf{BE}_{\{y\}}(\mathfrak{h}, W)^{\le -1}$, and the desired claim is proved. ■

9 The Right-equivariant Category

Recall the categories $\overline{\mathscr{D}}^{\oplus}_{\mathrm{BS}}(\mathfrak{h}, W)$ and $\mathsf{RE}(\mathfrak{h}, W)$ introduced in Section 3.4. The goal of the present section is to briefly indicate how most of the results considered so far adapt to these categories, allowing us to define the category $\mathsf{P}^{\mathsf{RE}}(\mathfrak{h}, W)$ of right- equivariant perverse objects.

9.1 Diagrammatic Categories Attached to Locally Closed Subsets

In Sections 9.1–9.2, $\Bbbk$ is an arbitrary integral domain.

Let $I \subset W$ be a closed subset. We define $\overline{\mathscr{D}}^{\oplus}_{\mathrm{BS},I}(\mathfrak{h}, W)$ to be the full subcategory of $\overline{\mathscr{D}}^{\oplus}_{\mathrm{BS}}(\mathfrak{h}, W)$ whose objects are direct sums of objects of the form $\overline{B}_{\underline{w}}(n)$ with $\underline{w}$ a reduced expression for an element in I. The autoequivalence (1) of $\mathscr{D}^{\oplus}_{\mathrm{BS}}(\mathfrak{h}, W)$ induces an autoequivalence of $\overline{\mathscr{D}}^{\oplus}_{\mathrm{BS}}(\mathfrak{h}, W)$ that in turn restricts to an autoequivalence of $\overline{\mathscr{D}}^{\oplus}_{\mathrm{BS},I}(\mathfrak{h}, W)$. All of these autoequivalences will be denoted similarly, and we will use the notation $\mathrm{Hom}^\bullet$ in these categories with the same conventions as in (3.1).

If $J \subset I \subset W$ are closed subsets, then there exists a natural embedding

$$(i_J^I)_* : \overline{\mathscr{D}}^{\oplus}_{\mathrm{BS},J}(\mathfrak{h}, W) \longrightarrow \overline{\mathscr{D}}^{\oplus}_{\mathrm{BS},I}(\mathfrak{h}, W).$$

If $I \subset W$ is a locally closed subset, and if we write $I = I_0 \smallsetminus I_1$ with $I_1 \subset I_0 \subset W$ closed, then we set $\overline{\mathscr{D}}^{\oplus}_{\mathrm{BS},I}(\mathfrak{h}, W) := \overline{\mathscr{D}}^{\oplus}_{\mathrm{BS},I_0}(\mathfrak{h}, W) /\!\!/ \overline{\mathscr{D}}^{\oplus}_{\mathrm{BS},I_1}(\mathfrak{h}, W)$, where the symbol "$/\!\!/$" has the same meaning as in Section 4.2. The natural functor $\mathscr{D}^{\oplus}_{\mathrm{BS}}(\mathfrak{h}, W) \to \overline{\mathscr{D}}^{\oplus}_{\mathrm{BS}}(\mathfrak{h}, W)$ restricts to a functor $\mathscr{D}^{\oplus}_{\mathrm{BS},I_0}(\mathfrak{h}, W) \to \overline{\mathscr{D}}^{\oplus}_{\mathrm{BS},I_0}(\mathfrak{h}, W)$ that in turn induces a functor $\mathscr{D}^{\oplus}_{\mathrm{BS},I}(\mathfrak{h}, W) \to \overline{\mathscr{D}}^{\oplus}_{\mathrm{BS},I}(\mathfrak{h}, W)$. From the definitions we see that this functor, which we will denote $M \mapsto \overline{M}$, induces an isomorphism

$$\Bbbk \otimes_R \mathrm{Hom}^\bullet_{\mathscr{D}^{\oplus}_{\mathrm{BS},I}(\mathfrak{h},W)}(M, N) \xrightarrow{\sim} \mathrm{Hom}^\bullet_{\overline{\mathscr{D}}^{\oplus}_{\mathrm{BS},I}(\mathfrak{h},W)}(\overline{M}, \overline{N}).$$

Using this, the considerations of Section 4.2 show that the category $\overline{\mathscr{D}}^{\oplus}_{\mathrm{BS},I}(\mathfrak{h}, W)$ does not depend on the choice of I_0 and I_1 (up to canonical equivalence), and that the morphism spaces in this category are free of finite rank over $\Bbbk$.

If $w \in W$, we denote by $\overline{b}_w$ the image of b_w in $\overline{\mathscr{D}}^{\oplus}_{\mathrm{BS},\{w\}}(\mathfrak{h}, W)$. Lemma 4.4 also implies the following claim.

Lemma 9.1 *There exists a canonical equivalence of categories*

$$\overline{\gamma} : \overline{\mathscr{D}}^{\oplus}_{\mathrm{BS},\{w\}}(\mathfrak{h}, W) \xrightarrow{\sim} \mathrm{Free}^{\mathrm{fg},\mathbb{Z}}(\Bbbk)$$

such that $\overline{\gamma}(\overline{b}_w) = \Bbbk$. *Under this equivalence, the autoequivalence* $\langle 1 \rangle$ *identifies with the "shift of grading" autoequivalence of* $\mathrm{Free}^{\mathrm{fg},\mathbb{Z}}(\Bbbk)$ *defined by* $(M\langle 1 \rangle)^n = M^{n+1}$.

Finally, if $J \subset I$ is a closed, resp., open, subset, then there exists a natural functor $(i^I_J)_* : \overline{\mathscr{D}}^{\oplus}_{\mathrm{BS},J}(\mathfrak{h}, W) \to \overline{\mathscr{D}}^{\oplus}_{\mathrm{BS},I}(\mathfrak{h}, W)$, resp., $(i^I_J)^* : \overline{\mathscr{D}}^{\oplus}_{\mathrm{BS},I}(\mathfrak{h}, W) \to \overline{\mathscr{D}}^{\oplus}_{\mathrm{BS},J}(\mathfrak{h}, W)$. We also have a duality functor $\mathbb{D}_I$ on $\overline{\mathscr{D}}^{\oplus}_{\mathrm{BS},I}(\mathfrak{h}, W)$, and compatibility properties similar to those stated in Section 4.4.

9.2 Right-equivariant Categories Attached to Locally Closed Subsets and Recollement

If $I \subset W$ is a locally closed subset, we set $\mathsf{RE}_I(\mathfrak{h}, W) := K^{\mathrm{b}}\overline{\mathscr{D}}^{\oplus}_{\mathrm{BS},I}(\mathfrak{h}, W)$. All the constructions of Section 5.1 adapt to this setting, and we obtain functors that will be denoted by the same symbol as in the case of $\mathsf{BE}_I(\mathfrak{h}, W)$. We also have a natural forgetful functor $\mathsf{For}^{\mathsf{BE}}_{\mathsf{RE}} : \mathsf{BE}_I(\mathfrak{h}, W) \to \mathsf{RE}_I(\mathfrak{h}, W)$.

Proposition 9.2 *Let* $I \subset W$ *be a locally closed subset, and let* $J \subset I$ *be a finite closed subset. Then the functor* $(i^I_{I\smallsetminus J})^* : \mathsf{RE}_I(\mathfrak{h}, W) \to \mathsf{RE}_{I\smallsetminus J}(\mathfrak{h}, W)$ *admits a left adjoint* $(i^I_{I\smallsetminus J})_!$ *and a right adjoint* $(i^I_{I\smallsetminus J})_*$. *Similarly, the functor* $(i^I_J)_* : \mathsf{RE}_J(\mathfrak{h}, W) \to \mathsf{RE}_I(\mathfrak{h}, W)$ *admits a left adjoint* $(i^I_J)^*$ *and a right adjoint* $(i^I_J)^!$. *Together, these functors give a recollement diagram*

$$\mathsf{RE}_J(\mathfrak{h}, W) \xrightarrow{(i^I_J)_*} \mathsf{RE}_I(\mathfrak{h}, W) \xrightarrow{(i^I_{I\smallsetminus J})^*} \mathsf{RE}_{I\smallsetminus J}(\mathfrak{h}, W).$$

with $(i^I_J)^*, (i^I_J)^! : \mathsf{RE}_I(\mathfrak{h}, W) \to \mathsf{RE}_J(\mathfrak{h}, W)$ and $(i^I_{I\smallsetminus J})_!, (i^I_{I\smallsetminus J})_* : \mathsf{RE}_{I\smallsetminus J}(\mathfrak{h}, W) \to \mathsf{RE}_I(\mathfrak{h}, W)$.

Proof The proof is identical to that of Proposition 5.6, starting with the case $|J| = 1$ and then using induction on $|J|$. Details are left to the reader. In the case where $J = \{w\}$, we replace the complex $B^+_{\underline{x}}$ by its image $\overline{B}^+_{\underline{x}}$ in $\mathsf{RE}_I(\mathfrak{h}, W)$, which fits into a distinguished triangle

$$\overline{B}_{\underline{x}} \to \overline{B}^+_{\underline{x}} \to \overline{B}_{\underline{w}} \,\underline{\otimes}_{\Bbbk}\, \mathrm{Hom}^{\bullet}_{\overline{\mathscr{D}}^{\oplus}_{\mathrm{BS},I}(\mathfrak{h}, W)}(\overline{B}_{\underline{w}}, \overline{B}_{\underline{x}})[1] \xrightarrow{[1]},$$

where the third term is defined in the natural way. ■

Starting with Proposition 9.2, we can define as in Section 5.4 the functors $(i^I_J)^*$, $(i^I_J)^!$, $(i^I_J)_*$, $(i^I_J)_!$ for any locally closed embedding $J \subset I$ of finite subsets of W. These functors satisfy the appropriate analogue of Lemma 5.12. Moreover, there exist canonical isomorphisms

$$(i^I_J)^* \circ \mathsf{For}^{\mathsf{BE}}_{\mathsf{RE}} \cong \mathsf{For}^{\mathsf{BE}}_{\mathsf{RE}} \circ (i^I_J)^*, \quad (i^I_J)^! \circ \mathsf{For}^{\mathsf{BE}}_{\mathsf{RE}} \cong \mathsf{For}^{\mathsf{BE}}_{\mathsf{RE}} \circ (i^I_J)^!,$$
$$(i^I_J)_* \circ \mathsf{For}^{\mathsf{BE}}_{\mathsf{RE}} \cong \mathsf{For}^{\mathsf{BE}}_{\mathsf{RE}} \circ (i^I_J)_*, \quad (i^I_J)_! \circ \mathsf{For}^{\mathsf{BE}}_{\mathsf{RE}} \cong \mathsf{For}^{\mathsf{BE}}_{\mathsf{RE}} \circ (i^I_J)_!,$$

where in each case the functor on the left-hand side is defined for RE categories, while the functor on the right-hand side is defined for BE categories. In fact, it suffices

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to prove these isomorphisms when J is either open or closed in I. In this case, the claim is either obvious or follows from the construction. For instance, we observe that in the construction for Lemma 5.1 and its counterpart for the RE categories, the subcategories $\mathsf{D}^+ \subset \mathsf{BE}_I(\mathfrak{h}, W)$ and $\overline{\mathsf{D}}^+ \subset \mathsf{RE}_I(\mathfrak{h}, W)$ satisfy $\mathsf{For}^{\mathsf{BE}}_{\mathsf{RE}}(\mathsf{D}^+) \subset \overline{\mathsf{D}}^+$.

9.3 Perverse t-structure

Henceforth we assume that $\Bbbk$ satisfies the conditions of Section 2.3. Then, as in Lemma 2.1, we have a canonical equivalence of triangulated categories

$$K^{\mathrm{b}}\mathrm{Free}^{\mathrm{fg},\mathbb{Z}}(\Bbbk) \xrightarrow{\sim} D^{\mathrm{b}}\mathrm{Mod}^{\mathrm{fg},\mathbb{Z}}(\Bbbk). \tag{9.1}$$

This gives rise to an equivalence $\mathsf{RE}_{\{w\}}(\mathfrak{h}, W) \cong D^{\mathrm{b}}\mathrm{Mod}^{\mathrm{fg},\mathbb{Z}}(\Bbbk)$ analogous to (7.1). However, there is also a different equivalence, described in the following lemma, which has no direct analogue in the setting of $\mathsf{BE}_{\{w\}}(\mathfrak{h}, W)$.

Lemma 9.3 *Let $w \in W$. There exists an equivalence of triangulated categories*

$$\mathsf{RE}_{\{w\}}(\mathfrak{h}, W) \xrightarrow{\sim} D^{\mathrm{b}}\mathrm{Mod}^{\mathrm{fg},\mathbb{Z}}(\Bbbk)$$

such that the autoequivalence $\langle 1 \rangle$ of $\mathsf{RE}_{\{w\}}(\mathfrak{h}, W)$ corresponds to the autoequivalence $D^{\mathrm{b}}((-1))$ of $D^{\mathrm{b}}\mathrm{Mod}^{\mathrm{fg},\mathbb{Z}}(\Bbbk)$, where (-1) is the inverse of the shift of grading autoequivalence of $\mathrm{Mod}^{\mathrm{fg},\mathbb{Z}}(\Bbbk)$ defined as in Lemma 4.4.

Proof We consider the composition

$$\mathsf{RE}_{\{w\}}(\mathfrak{h}, W) \xrightarrow[\sim]{K^{\mathrm{b}}(\overline{\gamma})} K^{\mathrm{b}}\mathrm{Free}^{\mathrm{fg},\mathbb{Z}}(\Bbbk) \xrightarrow[\sim]{(9.1)} D^{\mathrm{b}}\mathrm{Mod}^{\mathrm{fg},\mathbb{Z}}(\Bbbk) \xrightarrow[\sim]{\zeta} D^{\mathrm{b}}\mathrm{Mod}^{\mathrm{fg},\mathbb{Z}}(\Bbbk),$$

where ζ is the equivalence sending a complex $(M^{n,m})_{n\in\mathbb{Z}}$ of graded $\Bbbk$-modules (where $M^{n,m}$ means the part in cohomological degree n and internal degree m) to the complex with $\zeta(M)^{n,m} := M^{n-m,m}$ and the same differential. It is straightforward to check that this equivalence has the required property with respect to the functor $\langle 1 \rangle$. ■

We now define the perverse t-structure on $\mathsf{RE}_{\{w\}}(\mathfrak{h}, W)$ as the transport of the tautological t-structure on $D^{\mathrm{b}}\mathrm{Mod}^{\mathrm{fg},\mathbb{Z}}(\Bbbk)$ under the equivalence of Lemma 9.3. Under the equivalence (7.1) and that induced by the equivalence of Lemma 9.1, the functor $\mathsf{For}^{\mathsf{BE}}_{\mathsf{RE}}$ corresponds to the functor $\Bbbk \otimes^L_R (-)$. Hence, in view of [AR2, Proposition 4.4], we deduce that $\mathsf{For}^{\mathsf{BE}}_{\mathsf{RE}} \colon \mathsf{BE}_{\{w\}}(\mathfrak{h}, W) \to \mathsf{RE}_{\{w\}}(\mathfrak{h}, W)$ is t-exact for the perverse t-structures; more precisely, an object $\mathscr{F}$ of $\mathsf{BE}_{\{w\}}(\mathfrak{h}, W)$ is perverse if and only if $\mathsf{For}^{\mathsf{BE}}_{\mathsf{RE}}(\mathscr{F})$ is perverse in $\mathsf{RE}_{\{w\}}(\mathfrak{h}, W)$.

Once the perverse t-structure is defined on $\mathsf{RE}_{\{w\}}(\mathfrak{h}, W)$ for any $w \in W$, as in Section 7.2, using recollement we can define a perverse t-structure on $\mathsf{RE}_I(\mathfrak{h}, W)$ for any locally closed subset $I \subset W$. The heart of this t-structure will be denoted by $\mathsf{P}^{\mathsf{RE}}_I(\mathfrak{h}, W)$. The remarks above show that $\mathsf{For}^{\mathsf{BE}}_{\mathsf{RE}} \colon \mathsf{BE}_I(\mathfrak{h}, W) \to \mathsf{RE}_I(\mathfrak{h}, W)$ is t-exact for the perverse t-structures; in fact an object $\mathscr{F}$ of $\mathsf{BE}_I(\mathfrak{h}, W)$ is perverse if and only if $\mathsf{For}^{\mathsf{BE}}_{\mathsf{RE}}(\mathscr{F})$ is perverse in $\mathsf{RE}_I(\mathfrak{h}, W)$.

We define the standard and costandard objects in $\mathsf{RE}_I(\mathfrak{h}, W)$ by

$$\overline{\Delta}^I_w := \mathsf{For}^{\mathsf{BE}}_{\mathsf{RE}}(\Delta^I_w), \qquad \overline{\nabla}^I_w := \mathsf{For}^{\mathsf{BE}}_{\mathsf{RE}}(\nabla^I_w).$$

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Corollary 7.9 and the t-exactness of $\mathsf{For}^{\mathsf{BE}}_{\mathsf{RE}}$ imply that these objects belong to

$$\mathsf{P}^{\mathsf{RE}}_{I}(\mathfrak{h}, W).$$

Moreover, the same proof as for Lemma 6.6 shows that for $x, y \in I$ we have

$$\mathrm{Hom}_{\mathsf{RE}_I(\mathfrak{h},W)}(\overline{\Delta}^I_x, \overline{\nabla}^I_y\langle n\rangle[m]) \cong \begin{cases} \Bbbk & \text{if } x = y \text{ and } m = n = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{9.2}$$

9.4 Right-equivariant and Biequivariant Perverse Sheaves

The goal of this subsection is to prove the following claim.

Proposition 9.4 *Let $I \subset W$ be a locally closed subset. Then the functor*

$$\mathsf{P}^{\mathsf{BE}}_{I}(\mathfrak{h}, W) \longrightarrow \mathsf{P}^{\mathsf{RE}}_{I}(\mathfrak{h}, W)$$

obtained by restricting $\mathsf{For}^{\mathsf{BE}}_{\mathsf{RE}}$ to perverse objects is fully faithful.

The proof of this proposition will rely on the following lemma from homological algebra. We regard [AMRW1, §3.2] R as a $\mathbb{Z}$-graded dg-algebra[6] with generators in bidegree $(2, 2)$. Consider also $\Lambda := \mathrm{Sym}(V^*(-2)[1])$, so that Λ is the exterior algebra of V^*, regarded as a bigraded ring with generators in bidegree $(1, 2)$. For any $\mathbb{Z}$-graded R-dg-module M, the bigraded $\Bbbk$-module $\Lambda \otimes_{\Bbbk} M$ admits a natural structure of $\mathbb{Z}$-graded $\Bbbk$-dg-module, with (Koszul-type) differential given by

$$d((r_1 \wedge \cdots \wedge r_k) \otimes m) = \sum_{j=1}^{k} (-1)^{j-1} \cdot (r_1 \wedge \cdots \wedge \widehat{r_j} \wedge \cdots \wedge r_k) \otimes (r_j \cdot m) + (-1)^k \cdot (r_1 \wedge \cdots \wedge r_k) \otimes d(m).$$

It is clear that the assignment $m \mapsto 1 \otimes m$ defines a morphism of $\mathbb{Z}$-graded dg-modules $M \to \Lambda \otimes_{\Bbbk} M$.

Lemma 9.5 *Assume that $\mathsf{H}^{<0}(M) = 0$. Then the morphism $M \to \Lambda \otimes_{\Bbbk} M$ induces an isomorphism $\mathsf{H}^0(M) \xrightarrow{\sim} \mathsf{H}^0(\Lambda \otimes_{\Bbbk} M)$.*

Proof Let us fix a basis $(e_1, \dots, e_m)$ of V^*. Then any homogeneous element x of $\Lambda \otimes_{\Bbbk} M$ can be written uniquely as a sum

$$x = \sum_{1 \le i_1 < \cdots < i_k \le m} (e_{i_1} \wedge \cdots \wedge e_{i_k}) \otimes m_{i_1,\dots,i_k} \tag{9.3}$$

with $m_{i_1,\dots,i_k}$ homogeneous, of cohomological degree $\deg(x) - k$.

First we prove that our morphism is surjective. For this, assume that x has cohomological degree 0 and that $d(x) = 0$, and choose a sequence $\underline{i} := (i_1, \dots, i_k)$ with k maximal such that $m_{i_1,\dots,i_k} \neq 0$. If $k = 0$, then the class of x belongs to the image of our morphism. Otherwise, let $y := x - (e_{i_1} \wedge \cdots \wedge e_{i_k}) \otimes m_{\underline{i}}$. Then $d(x)$ equals

$$d(y) + \sum_{j=1}^{k} (-1)^{j-1} \cdot (e_{i_1} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge e_{i_k}) \otimes (e_{i_j} \cdot m_{\underline{i}}) + (-1)^k \cdot (e_{i_1} \wedge \cdots \wedge e_{i_k}) \otimes d(m_{\underline{i}}).$$

[6] In what follows, the "internal" grading will play no role, and hence can be forgotten.

Since $d(x) = 0$, the maximality of k implies that $d(m_{\underline{i}}) = 0$. Now $m_{\underline{i}}$ has strictly negative cohomological degree, so by our assumption there exists $n_{\underline{i}}$ in M such that $m_{\underline{i}} = d(n_{\underline{i}})$. We then set $x' := y - \sum_{j=1}^{k} (-1)^{k+j-1} \cdot (e_{i_1} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge e_{i_k}) \otimes (e_{i_j} \cdot n_{\underline{i}})$. The element $x - x'$ equals

$$(e_{i_1} \wedge \cdots \wedge e_{i_k}) \otimes m_{\underline{i}} + \sum_{j=1}^{k} (-1)^{k+j-1} \cdot (e_{i_1} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge e_{i_k}) \otimes (e_{i_j} \cdot n_{\underline{i}}) \\ = d\big((-1)^k \cdot (e_{i_1} \wedge \cdots \wedge e_{i_k}) \otimes n_{\underline{i}} \big).$$

Hence $d(x') = 0$, and x and x' have the same image in $\mathsf{H}^0(\Lambda \otimes_{\Bbbk} M)$. Repeating this procedure if necessary, we can decrease the number of terms in the decomposition (9.3) attached to sequences of length k, and then decrease the maximal length of a sequence $\underline{j}$ such that $m_{\underline{j}} \neq 0$, and obtain finally that the image of x in $\mathsf{H}^0(\Lambda \otimes_{\Bbbk} M)$ belongs to the image of $\mathsf{H}^0(M)$.

Now we prove the injectivity of our morphism. Let $x \in M^0$ be such that $d_M(x) = 0$, and assume that the image of x in $\mathsf{H}^0(\Lambda \otimes_{\Bbbk} M)$ vanishes, or in other words that $x = d(y)$ for some y in $(\Lambda \otimes_{\Bbbk} M)^{-1}$. Write y as in (9.3). If we assume that there exists a sequence $\underline{i} := (i_1, \ldots, i_k)$ with $k > 0$ such that $m_{i_1,\ldots,i_k} \neq 0$, then we choose such a sequence with k maximal. The fact that $d(y) = x$ implies that $d(m_{\underline{i}}) = 0$, and, as above, $y = y' + d(z)$ for some z in $(\Lambda \otimes_{\Bbbk} M)^{-2}$. Then $x = d(y')$, and repeating this procedure if necessary, we obtain that the class of x vanishes in $\mathsf{H}^0(M)$. ∎

Proof of Proposition 9.4 In this proof we assume the reader has some familiarity with the constructions of [AMRW1, Chapter 4].

As in [AMRW1], one can define the notion of a $\mathscr{D}^{\oplus}_{\mathsf{BS},I}(\mathfrak{h}, W)$-sequence and, for any such sequences $\mathscr{F}$ and $\mathscr{G}$, consider the bigraded $\Bbbk$-module $\underline{\mathrm{Hom}}_{\mathsf{BE},I}(\mathscr{F}, \mathscr{G})$. Then as in [AMRW1, §4.2] one can describe the category $\mathsf{BE}_I(\mathfrak{h}, W)$ as the category of pairs $(\mathscr{F}, \delta)$ with δ in $\underline{\mathrm{Hom}}_{\mathsf{BE},I}(\mathscr{F}, \mathscr{F})^{(1,0)}$ which satisfies $\delta \circ \delta = 0$, with appropriately defined morphisms. As in [AMRW1, §4.3], one also has a similar description for $\mathsf{RE}_I(\mathfrak{h}, W)$, replacing $\underline{\mathrm{Hom}}_{\mathsf{BE},I}(\mathscr{F}, \mathscr{G})$ with $\Bbbk \otimes_R \underline{\mathrm{Hom}}_{\mathsf{BE},I}(\mathscr{F}, \mathscr{G})$. Finally, replacing $\underline{\mathrm{Hom}}_{\mathsf{BE},I}(\mathscr{F}, \mathscr{G})$ with $\Lambda \otimes_{\Bbbk} \underline{\mathrm{Hom}}_{\mathsf{BE},I}(\mathscr{F}, \mathscr{G})$, one obtains the category $\mathsf{LM}_I(\mathfrak{h}, W)$ of left-monodromic complexes [AMRW1, §4.4]. With this notation, the functor $\mathsf{For}^{\mathsf{BE}}_{\mathsf{RE}}$ factors as a composition

$$\mathsf{BE}_I(\mathfrak{h}, W) \xrightarrow{\mathsf{For}^{\mathsf{BE}}_{\mathsf{LM}}} \mathsf{LM}_I(\mathfrak{h}, W) \xrightarrow{\mathsf{For}^{\mathsf{LM}}_{\mathsf{RE}}} \mathsf{RE}_I(\mathfrak{h}, W).$$

Moreover, as in [AMRW1, Theorem 4.6.2], the functor $\mathsf{For}^{\mathsf{LM}}_{\mathsf{RE}}$ is an equivalence of categories.

Now let $\mathscr{F}, \mathscr{G}$ be in $\mathsf{P}^{\mathsf{BE}}_I(\mathfrak{h}, W)$. Then the morphisms from $\mathscr{F}$ to shifts of $\mathscr{G}$ can be computed as the cohomology of the complex of $\mathbb{Z}$-graded R-dg-modules

$$\underline{\mathrm{Hom}}_{\mathsf{BE},I}(\mathscr{F}, \mathscr{G}).$$

Since $\mathscr{F}$ and $\mathscr{G}$ belong to the heart of a t-structure, this complex has no negative cohomology. On the other hand, with the same conventions as above, the complex

$$\Lambda \otimes_{\Bbbk} \underline{\mathrm{Hom}}_{\mathsf{BE},I}(\mathscr{F}, \mathscr{G})$$

computes the morphisms from $\mathsf{For}^{\mathsf{BE}}_{\mathsf{LM}}(\mathscr{F})$ to shifts of $\mathsf{For}^{\mathsf{BE}}_{\mathsf{LM}}(\mathscr{G})$. Hence Lemma 9.5 shows that $\mathsf{For}^{\mathsf{BE}}_{\mathsf{LM}}$ induces an isomorphism

$$\mathrm{Hom}_{\mathsf{BE}_I(\mathfrak{h},W)}(\mathscr{F},\mathscr{G}) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{LM}_I(\mathfrak{h},W)}(\mathsf{For}^{\mathsf{BE}}_{\mathsf{LM}}(\mathscr{F}), \mathsf{For}^{\mathsf{BE}}_{\mathsf{LM}}(\mathscr{G})).$$

Since $\mathsf{For}^{\mathsf{LM}}_{\mathsf{RE}}$ is an equivalence of categories, the claim of the proposition follows. ■

9.5 The Case of Field Coefficients

In this subsection we assume that $\Bbbk$ is a field. In this case, as in Section 8.1, the recollement formalism provides a description of the simple objects in the abelian category $\mathsf{P}^{\mathsf{RE}}_I(\mathfrak{h},W)$. In fact, t-exactness of $\mathsf{For}^{\mathsf{BE}}_{\mathsf{RE}}$ implies that, up to isomorphism, these simple objects are exactly the objects $\overline{\mathscr{L}}^I_w\langle n\rangle$ with $(w,n) \in W \times \mathbb{Z}$, where $\overline{\mathscr{L}}^I_w := \mathsf{For}^{\mathsf{BE}}_{\mathsf{RE}}(\mathscr{L}^I_w)$.

The following result is the main reason that motivates the generalization of our constructions to the RE categories. It uses the concept of (graded) *highest weight category* due Cline–Parshall–Scott; see [AR1, Definition A.1] for the definition we want to use, except that we replace Axiom (1) by the weaker condition that for any $s \in \mathscr{S}$ the set $\{t \in \mathscr{S} \mid t \le s\}$ is finite.

Theorem 9.6 *Let $I \subset W$ be a locally closed subset. The category $\mathsf{P}^{\mathsf{RE}}_I(\mathfrak{h},W)$ is a graded highest weight category with weight poset $(I,\le)$, normalized standard objects $(\overline{\Delta}^I_w : w \in I)$, and normalized costandard objects $(\overline{\nabla}^I_w : w \in I)$.*

Proof The first axiom is obviously satisfied. For the second one, we observe that the surjection $\overline{\Delta}^I_w \twoheadrightarrow \overline{\mathscr{L}}^I_w$ and the injection $\overline{\mathscr{L}}^I_w \hookrightarrow \overline{\nabla}^I_w$ induce an embedding

$$\mathrm{Hom}_{\mathsf{P}^{\mathsf{RE}}_I(\mathfrak{h},W)}(\overline{\mathscr{L}}^I_w, \overline{\mathscr{L}}^I_w\langle n\rangle) \hookrightarrow \mathrm{Hom}_{\mathsf{P}^{\mathsf{RE}}_I(\mathfrak{h},W)}(\overline{\Delta}^I_w, \overline{\nabla}^I_w\langle n\rangle);$$

then the desired claim follows from (9.2). To check the third axiom, we consider $J \subset I$ closed and $w \in J$ maximal. Then $\overline{\Delta}^I_w$ belongs to (the essential image of) $\mathsf{P}^{\mathsf{RE}}_J(\mathfrak{h},W)$, and if M belongs to $\mathsf{P}^{\mathsf{RE}}_J(\mathfrak{h},W)$ we have

$$\mathrm{Ext}^1_{\mathsf{P}^{\mathsf{RE}}_J(\mathfrak{h},W)}(\overline{\Delta}^I_w, M) \cong \mathrm{Hom}_{\mathsf{RE}_J(\mathfrak{h},W)}(\overline{\Delta}^J_w, M[1]) \cong \mathrm{Hom}_{\mathsf{RE}_{\{w\}}(\mathfrak{h},W)}(\overline{b}_w, (i^J_w)^* M[1]),$$

which vanishes since $(i^J_w)^* M$ is a perverse object. One checks similarly that

$$\mathrm{Ext}^1_{\mathsf{P}^{\mathsf{RE}}_J(\mathfrak{h},W)}(M, \overline{\nabla}^I_w) = 0.$$

The fourth axiom follows from Lemma 8.2. Finally, the fifth axiom in the definition of highest weight categories follows from (9.2) and [BGS, Lemma 3.2.4]. ■

Once Theorem 9.6 is established, one can consider the *tilting* objects in the highest weight category $\mathsf{P}^{\mathsf{RE}}_I(\mathfrak{h},W)$, *i.e.*, the objects that admit both a standard filtration and a costandard filtration [AR1, Definition A.2]. As in [AR1] we will use the notation $(\mathscr{T} : \overline{\Delta}^I_w\langle n\rangle)$ (or $(\mathscr{T} : \overline{\nabla}^I_w\langle n\rangle)$) for multiplicities of standard (or costandard) objects in a standard (or costandard) filtration. The indecomposable tilting objects are classified in the following way. For any $w \in W$, there exists a unique, up to isomorphism, indecomposable tilting object $\mathscr{T}^I_w$ in $\mathsf{P}^{\mathsf{RE}}_I(\mathfrak{h}, W)$ that satisfies $(\mathscr{T}^I_w : \overline{\Delta}^I_w) = 1$ and $((\mathscr{T}^I_w : \overline{\Delta}^I_x\langle n\rangle) \neq 0 \Rightarrow x \leq w)$. Moreover, the assignment $(w, n) \mapsto \mathscr{T}^I_w\langle n\rangle$ induces a bijection between $I \times \mathbb{Z}$ and the set of isomorphism classes of indecomposable tilting objects. (See [AR1, Proposition A.4], and [R1, Theorem 7.14] for more details in the ungraded setting.) By uniqueness, we have $\mathbb{D}_I(\mathscr{T}^I_w) \cong \mathscr{T}^I_w$. Moreover there exists a surjection

$$\mathscr{T}^I_w \twoheadrightarrow \overline{\nabla}^I_w, \tag{9.4}$$

resp., an embedding $\overline{\Delta}^I_w \hookrightarrow \mathscr{T}^I_w$, whose kernel, resp., cokernel, admits a costandard, resp., standard, filtration.

The study of such objects is particularly important in view of the following result, which follows from [AR1, Lemma A.5, Lemma A.6]. Here we denote by $\mathsf{Tilt}^{\mathsf{RE}}_I(\mathfrak{h}, W)$ the full subcategory of $\mathsf{P}^{\mathsf{RE}}_I(\mathfrak{h}, W)$ consisting of tilting objects.

Theorem 9.7 *The natural functors $K^{\mathrm{b}}\mathsf{Tilt}^{\mathsf{RE}}_I(\mathfrak{h}, W) \to D^{\mathrm{b}}\mathsf{P}^{\mathsf{RE}}_I(\mathfrak{h}, W) \to \mathsf{RE}_I(\mathfrak{h}, W)$ are equivalences of triangulated categories.*

We conclude this section by noting that, with the theory we developed here in hand, the results obtained in [AMRW1, §§10.5–10.7] generalize to the present setting. In particular, the tilting objects in $\mathsf{P}^{\mathsf{RE}}(\mathfrak{h}, W)$ can be produced via a Bott–Samelson type construction.

10 Ringel Duality and the Big Tilting Object

In this section we assume that W is finite, and denote by w_0 the longest element of W.

10.1 Ringel Duality

By Proposition 6.11 (ii), the functor $\mathsf{R} := (-) \,\underline{\star}\, \Delta_{w_0} : \mathsf{RE}(\mathfrak{h}, W) \to \mathsf{RE}(\mathfrak{h}, W)$ is an equivalence of triangulated categories, with quasi-inverse

$$\mathsf{R}^{-1} := (-) \,\underline{\star}\, \nabla_{w_0} : \mathsf{RE}(\mathfrak{h}, W) \longrightarrow \mathsf{RE}(\mathfrak{h}, W).$$

Lemma 10.1 *For any $w \in W$, we have $\mathsf{R}(\overline{\nabla}_x) \simeq \overline{\Delta}_{xw_0}$.*

Proof The desired isomorphism follows from the sequence of isomorphisms

$$\begin{aligned}\mathsf{R}(\overline{\nabla}_x) = \overline{\nabla}_x \,\underline{\star}\, \Delta_{w_0} &\cong \mathsf{For}^{\mathsf{BE}}_{\mathsf{RE}}(\nabla_x \,\underline{\star}\, \Delta_{w_0}) \\ &\cong \mathsf{For}^{\mathsf{BE}}_{\mathsf{RE}}(\nabla_x \,\underline{\star}\, \Delta_{x^{-1}} \,\underline{\star}\, \Delta_{xw_0}) \cong \mathsf{For}^{\mathsf{BE}}_{\mathsf{RE}}(\Delta_{xw_0}) \cong \overline{\Delta}_{xw_0},\end{aligned}$$

where the first isomorphism is a special case of (3.3), and the second and third ones follow from Proposition 6.11. ∎

10.2 Projective and Tilting Perverse Objects

Henceforth, we assume that $\Bbbk$ is a field. Then the category $\mathsf{P}^{\mathsf{RE}}(\mathfrak{h}, W)$ has enough projective and injective objects, and any projective, resp., injective, object admits a standard, resp., costandard, filtration [AR1, Theorem A.3]. For $x \in W$, we will denote by $\mathscr{P}_x$, resp., $\mathscr{I}_x$, the projective cover, resp., injective hull, of $\overline{\mathscr{L}}_x$. Recall the reciprocity formula

$$(\mathscr{P}_x : \overline{\Delta}_y\langle n\rangle) = [\overline{\nabla}_y\langle n\rangle : \overline{\mathscr{L}}_x], \tag{10.1}$$

where $x, y \in W$ and $n \in \mathbb{Z}$; see remarks after [BGS, Theorem 3.2.1].

It is a direct consequence of Lemma 10.1 that if $\mathscr{M}$ is a perverse object that admits a costandard filtration, then $\mathsf{R}(\mathscr{M})$ is also perverse, and it admits a standard filtration; moreover, we have

$$(\mathsf{R}(\mathscr{M}) : \overline{\Delta}_{xw_0}\langle n\rangle) = (\mathscr{M} : \overline{\nabla}_x\langle n\rangle), \tag{10.2}$$

for all $x \in W$ and $n \in \mathbb{Z}$.

Proposition 10.2 *For any $x \in W$, we have*

$$\mathsf{R}(\mathscr{T}_x) \cong \mathscr{P}_{xw_0}, \qquad \mathsf{R}(\mathscr{I}_x) \cong \mathscr{T}_{xw_0}.$$

Proof Let $\mathscr{T}$ be a tilting object in $\mathsf{P}^{\mathsf{RE}}(\mathfrak{h}, W)$. Then $\mathsf{R}(\mathscr{T})$ is perverse by the comments before the proposition. We claim that $\mathsf{R}(\mathscr{T})$ is projective. In fact, by Lemma 10.1, for $y \in W$ and $n, m \in \mathbb{Z}$, we have

$$\mathrm{Hom}_{\mathsf{RE}(\mathfrak{h},W)}(\mathsf{R}(\mathscr{T}), \overline{\Delta}_y\langle n\rangle[m]) \cong \mathrm{Hom}_{\mathsf{RE}(\mathfrak{h},W)}(\mathscr{T}, \overline{\nabla}_{yw_0}\langle n\rangle[m]).$$

Since $\mathscr{T}$ admits a standard filtration, (9.2) implies that this vector space vanishes unless $m = 0$. Using the analogue of Lemma 7.5 for the right-equivariant category, we deduce that $\mathrm{Hom}_{\mathsf{RE}(\mathfrak{h},W)}(\mathsf{R}(\mathscr{T}), \mathscr{M}) = 0$ for any $\mathscr{M}$ in ${}^p\mathsf{RE}(\mathfrak{h}, W)^{<0}$. In particular, this shows that $\mathrm{Ext}^1_{\mathsf{P}^{\mathsf{RE}}(\mathfrak{h},W)}(\mathsf{R}(\mathscr{T}), \mathscr{N}) = 0$ for any $\mathscr{N}$ in $\mathsf{P}^{\mathsf{RE}}(\mathfrak{h}, W)$, and hence that $\mathsf{R}(\mathscr{T})$ is projective, as claimed.

If now $\mathscr{T} = \mathscr{T}_x$, then, since R is an equivalence of categories, $\mathsf{R}(\mathscr{T}_x)$ is indecomposable. Moreover, the kernel of the natural surjection $\mathscr{T}_x \twoheadrightarrow \overline{\nabla}_x$ admits a costandard filtration; hence its image $\mathsf{R}(\mathscr{T}_x) \to \overline{\Delta}_{xw_0}$ under R is surjective. This shows that $\mathsf{R}(\mathscr{T}_x)$ surjects to $\mathscr{L}_{xw_0}$, and hence that $\mathsf{R}(\mathscr{T}_x) \cong \mathscr{P}_{xw_0}$.

Very similar arguments show that $\mathsf{R}^{-1}(\mathscr{T}_{xw_0})$ belongs to $\mathsf{P}^{\mathsf{RE}}(\mathfrak{h}, W)$, is indecomposable, and injective therein, and contains $\mathscr{L}_x$ as a simple subobject. Therefore, as above, we have $\mathsf{R}^{-1}(\mathscr{T}_{xw_0}) \cong \mathscr{I}_x$, which concludes the proof. ∎

10.3 The Big Tilting Object

The following theorem is an analogue of a well-known result in category $\mathscr{O}$ that is the starting point of the Soergel-theoretic analysis of this category.

Theorem 10.3 *There exist isomorphisms*

$$\mathscr{T}_{w_0} \cong \mathscr{P}_e\langle \ell(w_0)\rangle \cong \mathscr{I}_e\langle -\ell(w_0)\rangle. \tag{10.3}$$

Moreover, for any $x \in W$, we have

$$(\mathscr{T}_{w_0} : \overline{\nabla}_x\langle -n\rangle) = (\mathscr{T}_{w_0} : \overline{\Delta}_x\langle n\rangle) = \begin{cases} 1 & \text{if } n = \ell(xw_0), \\ 0 & \text{otherwise.} \end{cases} \tag{10.4}$$

Proof First we note that $(\mathscr{P}_e : \overline{\Delta}_x\langle -n\rangle)$ is 1 if $n = \ell(x)$ and 0 otherwise by the reciprocity formula (10.1) and Proposition 8.3. Then, using (10.2), Proposition 10.2, and the fact that $\mathbb{D}(\mathscr{T}_{w_0}) \cong \mathscr{T}_{w_0}$, we obtain (10.4). In particular, we deduce that for all $x \in W$, we have

$$\dim_{\Bbbk} \operatorname{Hom}_{\mathsf{P}^{\mathrm{RE}}(\mathfrak{h},W)}(\mathscr{T}_{w_0}, \overline{\nabla}_x\langle n\rangle) = \begin{cases} 1 & \text{if } n = \ell(xw_0), \\ 0 & \text{otherwise.} \end{cases} \tag{10.5}$$

We now claim that any nonzero morphism $f : \mathscr{T}_{w_0} \to \overline{\nabla}_x\langle \ell(xw_0)\rangle$ is surjective. In fact, since the corresponding Hom-space is one-dimensional, it suffices to prove that there exists a surjective morphism from $\mathscr{T}_{w_0}$ to $\overline{\nabla}_x\langle \ell(xw_0)\rangle$. Such a morphism is provided by the composition $\mathscr{T}_{w_0} \twoheadrightarrow \overline{\nabla}_{w_0} \twoheadrightarrow \overline{\nabla}_x\langle \ell(xw_0)\rangle$, where the first morphism is given by (9.4) and the second one is provided by Proposition 8.7.

This claim implies that $\mathscr{T}_{w_0}$ has no quotient of the form $\mathscr{L}_y\langle n\rangle$ with $y \neq e$, since otherwise we would obtain a nonzero and nonsurjective morphism $\mathscr{T}_{w_0} \to \overline{\nabla}_y\langle n\rangle$ as the composition $\mathscr{T}_{w_0} \twoheadrightarrow \mathscr{L}_y\langle n\rangle \hookrightarrow \overline{\nabla}_y\langle n\rangle$. In view of (10.5), we deduce that the head of $\mathscr{T}_{w_0}$ is $\mathscr{L}_e\langle \ell(w_0)\rangle$, and hence that there exists a surjective morphism

$$\mathscr{P}_e\langle \ell(w_0)\rangle \longrightarrow\!\!\!\!\!\rightarrow \mathscr{T}_{w_0}.$$

Since these objects have the same length, namely, the sum of the lengths of all objects $\overline{\Delta}_x$ with $x \in W$, this surjection must be an isomorphism, which proves the first isomorphism in (10.3). The second isomorphism follows by duality. ■

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