THE CONSTRUCTION OF PERFECT AND EXTREME FORMS

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1. Introduction. Let

$$f(\mathbf{x}) = f(x_1, x_2, \ldots, x_n) = \sum_i \sum_j a_{ij} x_i x_j \qquad (a_{ij} = a_{ji})$$

be a positive quadratic form of determinant D, and let M be the minimum of $f(\mathbf{x})$ for integral $\mathbf{x} \neq \mathbf{0}$. Then $f(\mathbf{x})$ assumes the value M for a finite number of integral vectors $\mathbf{x} = \pm\mathbf{m}_k$ $(k = 1, \ldots, s)$ called the *minimal vectors*.

The form $f(\mathbf{x})$ is said to be *perfect* if it is uniquely determined by its minimum M and its minimal vectors, i.e. if the s equations

$$f(\mathbf{m}_k) = \sum_i \sum_j a_{ij} m_{ki} m_{kj} = M \qquad (k = 1, \ldots, s)$$

uniquely determine the $\frac{1}{2}n(n+1)$ distinct coefficients a_{ij} of f.

The perfect forms are of interest because they include all extreme forms, i.e. those for which the ratio $M/D^{1/n}$ is a local maximum, and hence all absolutely extreme forms for which $M/D^{1/n}$ assumes its greatest value γ_n.

Although a large number of perfect forms are now known (see in particular **2**; **4**; **9**), no really satisfactory method of construction of these forms has been found. Voronoi (**10**) devised an algorithm by which all perfect forms in a given number of variables may be found; this has been successfully applied for $n \leqslant 6$ (**1**; **10**), but the analysis for $n = 6$ is so long and detailed that there is little hope that the method will be of use for larger n. Minkowski (**7**) considered a fundamental region in the space of the coefficients a_{ij}, in which all the extreme forms appear as edges, but the practical difficulties in his method appear to be insurmountable for large n.

More recently, Barnes (**2**) has given two methods of constructing perfect forms, each proceeding from a known form and producing a new form by extending the range of values and increasing the dimension respectively. However, from the tables of new forms in (**9**) it is apparent that only certain classes of forms can be obtained by either of these methods.

For this reason I describe here a method of constructing forms with large numbers of minimal vectors. Theorem 2.1 gives a useful means for proving that many of these forms are perfect and extreme. Part of the proof of this depends on Voronoi's criterion (**10**):

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A perfect form $f(\mathbf{x})$ is extreme if and only if it is eutactic, i.e. if its adjoint $F(\mathbf{x})$ is expressible as

$$F(\mathbf{x}) = \sum_{k=1}^{s} \rho_k (\mathbf{m}'_k \mathbf{x})^2, \qquad \rho_k > 0,$$

where $\mathbf{m}_1, \ldots, \mathbf{m}_s$ are the minimal vectors of $f(\mathbf{x})$.

In this paper the method will be used to obtain very simply many of the known forms, and to demonstrate some of their analogues in higher dimensions. It is also possible to obtain large numbers of new forms in this way, as for example the forms $R_n\ (r_1, \ldots, r_k)$ in **(9)**. However, the applications here are not intended to be exhaustive, but rather to illustrate the general method.

2. Method—Elimination of the minimum. Let $\phi_{r_1}, \ldots, \phi_{r_k}$ be perfect forms of dimensions $r_1, \ldots, r_k$, and determinants $D_1, \ldots, D_k$ respectively. Without loss of generality we may assume that the forms ϕ_{r_t} $(1 \leqslant t \leqslant k)$ have a common minimum M; we denote the minimal vectors of ϕ_{r_t} by $\mathbf{m}_j^{(t)}$ $(j = 1, \ldots, s_t; t = 1, \ldots, k)$.

We place one restriction on the choice of the forms ϕ_{r_t}: if for some integral $\mathbf{x} \neq \mathbf{0}$, $\phi_{r_t}(\mathbf{x}) \neq M$, then $\phi_{r_t}(\mathbf{x}) \geqslant 2M$. Let

$$\mathbf{x} = \mathbf{u}_j^{(t)} \qquad (j = 1, \ldots, u_t;\ t = 1, \ldots, k)$$

be the integral sets (if any) for which $\phi_{r_t}(\mathbf{x}) = 2M$.

We now consider the n-dimensional form

$$f = \phi_{r_1} + \phi_{r_2} + \ldots + \phi_{r_k}, \qquad \text{where } \sum_{t=1}^{k} r_t = n. \tag{2.1}$$

For $k \geqslant 2$, f is disconnected and hence cannot be perfect (for instance all minimal vectors satisfy $x_1 x_n = 0$). We now restrict the variables of f to lie on a sublattice Λ of the integral lattice Γ, where Λ is chosen in such a way that no vector $\mathbf{m}_j^{(t)}$ $(1 \leqslant j \leqslant s_t,\ 1 \leqslant t \leqslant k)$ lies on Λ. We represent the form f with lattice Λ by $(f; \Lambda)$.

If the determinant of Λ is $d(\Lambda)$, then the determinant of $(f; \Lambda)$ is given by

$$D(f; \Lambda) = [d(\Lambda)]^2 \prod_{t=1}^{k} D_t. \tag{2.2}$$

Clearly for any Λ chosen in this way, $M(f; \Lambda) \geqslant 2M$. We shall only consider forms $(f; \Lambda)$ having $M(f; \Lambda) = 2M$, this value being attained for

(i) all vectors $\mathbf{u}_j^{(t)}$ $(1 \leqslant j \leqslant u_t, 1 \leqslant t \leqslant k)$ for which $\mathbf{u}_j^{(t)} \in \Lambda$;

(ii) all vectors $\mathbf{m}_i^{(t_1)} \pm \mathbf{m}_j^{(t_2)}$ $(1 \leqslant i \leqslant s_{t_1},\ 1 \leqslant j \leqslant s_{t_2},\ 1 \leqslant t_1 \leqslant t_2 \leqslant r_k)$ which lie on Λ.

Following **(4)**, for a form h having minimum M and determinant D we set

$$\Delta = \Delta(h) = (2/M)^n D.$$

Then

$$\Delta_t = \Delta(\phi_{r_t}) = (2/M)^{r_t} D_t,$$

and from (2.2)

(2.3)
$$\Delta(f; \Lambda) = \left(\frac{2}{2M}\right)^n [d(\Lambda)]^2 \prod_{t=1}^{k} D_t$$
$$= \frac{1}{2^n} [d(\Lambda)]^2 \prod_{t=1}^{k} \Delta_t.$$

The choice of the lattice Λ is now somewhat arbitrary, but in practice we find that best results are obtained by choosing the index of Λ in Γ as small as possible (i.e. make $d(\Lambda)$ as small as possible).

For any given class of forms, the following result is often useful.

THEOREM 2.1. *Let $(f; \Lambda)$ be the form (2.1) with variables lying on some sublattice Λ of the integral lattice. Then if the forms*

$$(f_{ij}; \Lambda) = (\phi_{r_i} + \phi_{r_j}; \Lambda) \qquad (1 \leqslant i < j \leqslant k)$$

are perfect, so is $(f; \Lambda)$; and if $(f_{ij}; \Lambda)$ $(1 \leqslant i < j \leqslant k)$ are extreme, so is $(f; \Lambda)$.

Proof. By construction, each minimal vector of $(f; \Lambda)$ occurs as a minimal vector of just one of the $(f_{ij}; \Lambda)$, except for those vectors (if any) for which $\phi_{r_t} = 2M$ $(1 \leqslant t \leqslant k)$; these occur as minimal vectors of precisely $k - 1$ of the $(f_{ij}; \Lambda)$.

(i) If now the forms $(f_{ij}; \Lambda)$ $(1 \leqslant i < j \leqslant k)$ are each uniquely determined by their minimal vectors, then

$$((k-1)f; \Lambda) = \left(\sum_{i<j} f_{ij}; \Lambda\right)$$

is uniquely determined by its minimal vectors (the repeated vectors in effect being used only once). We thus have: If each

$$(f_{ij}; \Lambda) \qquad (1 \leqslant i < j \leqslant k)$$

is perfect, then $(f; \Lambda)$ is perfect.

(ii) Since the forms ϕ_{r_t} $(1 \leqslant t \leqslant k)$ are disjoint, the reciprocal form F of f is given by

$$F = \sum_{t=1}^{k} \Phi_{r_t},$$

where Φ_{r_t} is the reciprocal form of ϕ_{r_t} $(1 \leqslant t \leqslant k)$.

Using a similar result for the reciprocal form F_{ij} of f_{ij} $(1 \leqslant i < j \leqslant k)$, we obtain

(2.4)
$$(k-1)F = \sum_{i<j} F_{ij}.$$

If now each F_{ij} can be expressed as a positive linear combination of the squares of its minimal forms, then from (2.4) this is also true for F. This yields: If each $(f_{ij}; \Lambda)$ is eutactic, then $(f; \Lambda)$ is eutactic.

Using Voronoi's criterion, this completes the proof of the theorem.

3. The forms B_n, $L_n{}^r$, R_n, and P_n. Following Coxeter **(4)** we define the form A_r by

$$(3.1)\qquad A_r = x_1{}^2 - x_1 x_2 + x_2{}^2 - x_2 x_3 + \ldots - x_{r-1} x_r + x_r{}^2.$$

This form is known to be perfect and extreme for all r. The $\frac{1}{2}r(r+1)$ minimal vectors are given by

$$(3.2)\quad (1, 0, \ldots, 0)_r,\ (1, 1, 0, \ldots, 0)_{r-1},\ \ldots,\ (1, 1, \ldots, 1, 0)_2,\ (1, 1, \ldots, 1)_1,$$

where $(x_1, \ldots, x_r)_k$ represents the k minimal vectors obtained by permuting the variables cyclically $0, 1, \ldots, k-1$ times.

It is easily verified that no vector of (3.2) lies on the lattice

$$\sum_1^r x_i \equiv 0 \pmod{(r+1)}.$$

We therefore construct our new form $(f; \Lambda)$ by setting

$$(3.3)\qquad f(\mathbf{x}) = \sum_{t=1}^{k} A_{r_t}(\mathbf{x}^{(t)}),$$

with lattice the sublattice of the integral lattice

$$(3.4)\qquad \Lambda : \sum_1^n x_i \equiv 0 \pmod{(r_1+1)},$$

where

$$r_1 \geqslant r_2 \geqslant \ldots \geqslant r_k \geqslant 1, \qquad \sum_1^k r_t = n,$$

$$\mathbf{x} = (\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(k)}),$$

and A_r as defined by (3.1).

If now $r_1 = r_2 = \ldots = r_n = 1$, $(f; \Lambda)$ is the form

$$x^{(1)2} + x^{(2)2} + \ldots + x^{(n)2},$$

where

$$\sum_{i=1}^{n} x^{(i)} \equiv 0 \pmod{2},$$

i.e. the form B_n of **(2**, I), known to be perfect and extreme for $n \geqslant 3$.

If $r_1 = r_2 = \ldots = r_r = 2$, $r_{r+1} = \ldots = r_k = 1$, we obtain the forms $L_n{}^r$ considered in **(2**, I). These forms are shown to be perfect for $r \geqslant 3$, or $r = 2$, $n \geqslant 5$, and extreme if and only if $n = 2r$ or $2r + 1$.

The remaining cases are dealt with in **(9)**, where the forms, denoted by $R_n(r_1, \ldots, r_k)$, are shown to be perfect, provided certain restrictions are placed on the parameters $r_1, \ldots, r_k$.

In particular, if R_n contains just one term A_n, we obtain

$$f(\mathbf{x}) = A_n(\mathbf{x}) = x_1^2 - x_1 x_2 + x_2^2 - \ldots - x_{n-1} x_n + x_n^2$$

with lattice the sublattice of the integral lattice

$$\sum_1^n x_i \equiv 0 \pmod{(n+1)}.$$

Applying the integral unimodular transformation

$$\mathbf{x} = T\mathbf{y} = \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ . & 1 & 1 & \ldots & 1 \\ . & . & 1 & \ldots & 1 \\ . & . & . & & . \\ . & . & . & \ldots & 1 \end{bmatrix} \mathbf{y},$$

we obtain the equivalent form

$$2f(\mathbf{x}) = \sum_1^n y_i^2 + \left(\sum_1^n y_i\right)^2 \tag{3.5}$$

with lattice

$$\sum_{i=1}^n i y_i \equiv 0 \pmod{(n+1)}.$$

This is the form P_n of (**2**, I), known to be perfect and extreme for $n \geqslant 6$.

Sometimes we shall find it convenient to use (3.5) as an alternative definition of the form A_n.

4. The forms B_4, E_8 ($\sim B_8^2$), J_{12} and their analogues. The form B_4 is defined by

$$f(\mathbf{x}) = f(x_1, \ldots, x_4) = \sum_1^4 x_i^2 \tag{4.1}$$

with lattice the sublattice of the integral lattice

$$\sum_1^4 x_i \equiv 0 \pmod{2}. \tag{4.2}$$

For this form, $\Delta = 4$, $M = 2$, and the minimal vectors are given by

$$\mathbf{e}_i \pm \mathbf{e}_j \qquad (1 \leqslant i < j \leqslant 4),$$

where $\mathbf{e}_i$ is the ith unit vector.

We now consider the additional lattice given by

$$x_1 \equiv x_2 \equiv x_3 \equiv x_4 \pmod{2}. \tag{4.3}$$

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Clearly no minimal vector of B_4 satisfies (4.3).

We now define $(f;\Lambda)$ to be the $4k$-dimensional form

$$(4.4)\qquad f(\mathbf{x}) = f(x_1, \ldots, x_{4k}) = \sum_{t=1}^{k} B_4(x_1^{(t)}, x_2^{(t)}, x_3^{(t)}, x_4^{(t)}) = \sum_{t=1}^{k} B_4(\mathbf{x}^{(t)}),$$

with lattice

$$(4.5)\qquad \Lambda: \sum_1^k x_1^{(t)} \equiv \sum_1^k x_2^{(t)} \equiv \sum_1^k x_3^{(t)} \left(\equiv \sum_1^k x_4^{(t)}\right) \pmod 2,$$

where $B_4(\mathbf{x}^{(t)})$ is defined by (4.1) and (4.2) and $\mathbf{x} = (\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(k)})$. For $\mathbf{x}$ belonging to the lattice Λ, $B_4(\mathbf{x}) \geqslant 4$ as required, and in view of relations of the type (4.2) we have $d(\Lambda) = 2^2$. Hence using (2.3) we have

$$(4.6)\qquad \Delta(f;\Lambda) = \frac{1}{2^{4k}}\cdot 2^4 \cdot 4^k = 2^{4-2k} \qquad (k \geqslant 1).$$

Let $\mathbf{e}_i^{(t)}$ denote the unit vector in n-space corresponding to the co-ordinate $x_i^{(t)}$.

The minimal vectors of $(f;\Lambda)$ are given by

$$(4.7)\qquad 2e_i^{(t)} \qquad (1 \leqslant i \leqslant 4,\ 1 \leqslant t \leqslant k),$$

$$(4.8)\qquad e_1^{(t)} \pm e_2^{(t)} \pm e_3^{(t)} \pm e_4^{(t)} \qquad (1 \leqslant t \leqslant k),$$

$$(4.9)\qquad (e_i^{(t_1)} \pm e_j^{(t_2)}) \pm (e_i^{(t_2)} \pm e_j^{(t_2)}) \qquad (1 \leqslant i < j \leqslant 4,\ 1 \leqslant t_1 < t_2 \leqslant k),$$

$$(4.10)\qquad (\mathbf{e}_i^{(t_1)} \pm \mathbf{e}_j^{(t_1)}) \pm (\mathbf{e}_k^{(t_2)} \pm \mathbf{e}_l^{(t_2)})$$

$$((i, j, k, l) \text{ some permutation of } (1, 2, 3, 4),\ 1 \leqslant t_1 < t_2 \leqslant k).$$

The number of minimal vectors is easily found to be

$$s = 12k(4k-3) = 3n(n-3).$$

We now prove that all the forms $(f;\Lambda)$ defined by (4.4) and (4.5) are extreme. By Theorem 2.1, it is enough to show that the form with $k = 2$ is extreme. Changing the notation, this is the form

$$(4.11)\qquad f(\mathbf{x}) = \sum_{i=1}^{8} x_i^2$$

with lattice the sublattice of the integral lattice

$$\sum_1^4 x_i \equiv \sum_5^8 x_i \equiv 0 \pmod 2,$$

$$x_1 + x_5 \equiv x_2 + x_6 \equiv x_3 + x_7 \equiv x_4 + x_8 \pmod 2.$$

We could show directly that this form is extreme, but applying the linear transformation

$$(4.12)\quad \mathbf{x} = T\mathbf{y} = \frac{1}{6}\begin{bmatrix} 6 & \cdot & \cdot & \cdot & -6 & \cdot & \cdot & \cdot \\ -2 & 4 & 4 & 4 & -2 & -2 & -2 & -2 \\ 4 & 4 & -2 & -2 & 4 & -2 & -2 & -2 \\ \cdot & \cdot & 6 & -6 & \cdot & \cdot & \cdot & \cdot \\ -2 & 4 & -2 & -2 & -2 & 4 & -2 & -2 \\ 2 & -4 & 2 & 2 & 2 & 2 & -4 & -4 \\ \cdot & \cdot & \cdot & \cdot & \cdot & -6 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 6 & -6 \end{bmatrix}\mathbf{y},$$

we obtain the equivalent form $8g(\mathbf{y})$, where

$$g(\mathbf{y}) = 9\sum_1^8 y_i^2 - \left(\sum_1^8 y_i\right)^2,$$

with lattice

$$\sum_{i=1}^8 y_i \equiv 0 \pmod 3.$$

This form is shown in **(1)** to be the form E_8 of **(4)**, known to be perfect and extreme.

This completes the proof.

The forms for $n \leqslant 16$ are listed in Table I in §6.

5. The forms E_5 $(\sim B_5)$, E_6, E_7 $(\sim A_7^2)$, B_9^2, Φ_{10}, and their analogues. We next consider the possibility of combining the forms B_4 and A_r. From the vast number of possible cases we select those forms comprising $h = (k-1)$ B_4's and a single A_r $(1 \leqslant r \leqslant 3)$. For our purpose it is convenient to use a definition of the form A_n equivalent to that given in (3.5).

We now have that

(5.1) $A_3(x_1, x_2, x_3, x_4)$ is the *section* of $B_4(x_1, x_2, x_3, x_4)$ by

$$\sum_1^4 x_i = 0,$$

i.e. $A_3(x_1, \ldots, x_4)$ is obtained from $B_4(x_1, \ldots, x_4)$ by setting

$$\sum_1^4 x_i = 0;$$

(5.2) $A_2(x_1, x_2, x_3)$ is the section of $A_3(x_1, \ldots, x_4)$ by $x_4 = 0$;

(5.3) $A_1(x_1, x_2)$ is the section of $A_2(x_1, x_2, x_3)$ by $x_3 = 0$.

We retain the extra variable in each case for symmetry considerations. Thus, for example, $A_3(x_1, \ldots, x_4)$ is written as

$$\sum_1^4 x_i^2, \qquad \text{subject to } \sum_1^4 x_i = 0.$$

It is easily seen that the values assumed by a section of a form are a subset of the values assumed by that form, and hence the forms A_r $(1 \leqslant r \leqslant 3)$ satisfy the condition on the minimum required in §2. $(f; \Lambda)$ is now defined to be the $(4h + r)$-dimensional form

$$f(\mathbf{x}) = \sum_{t=1}^{h} B_4(\mathbf{x}^{(t)}) + A_r(x_1^{(k)}, \ldots, x_{r+1}^{(k)}) \qquad (1 \leqslant r \leqslant 3) \tag{5.4}$$

with lattice the sublattice of the integral lattice

$$\Lambda: \sum_{t=1}^{k} x_1^{(t)} \equiv \sum_{t=1}^{k} x_2^{(t)} \equiv \sum_{t=1}^{k} x_3^{(t)} \pmod 2; \tag{5.5}$$

compare (4.4) and (4.5).

By (2.3) we have

$$\Delta(f; \Lambda) = \frac{1}{2^{4h+r}} \cdot 2^4 \cdot 4^h \cdot (r + 1) = (r + 1) \cdot 2^{4-2h-r}.$$

The minimal vectors of $(f; \Lambda)$ are just those vectors (4.7)–(4.10) that lie on the lattice (5.5). The number of minimal vectors is found to be

$$s = 12h(4h - 3) + 4hr(r + 1) + \tfrac{1}{2}r(r - 1)(r - 2).$$

We shall show that all the forms $(f; \Lambda)$ defined by (5.4) and (5.5) are extreme. By Theorem 2.1, and using the results of §4, it suffices to show that the forms with $h = 1$ are extreme.

We take successively the sections $x_5 + x_6 + x_7 + x_8 = 0$, $x_8 = 0$, $x_7 = 0$ of the form (4.11), corresponding to (5.1), (5.2), (5.3) respectively. Under the transformation (4.12) these appear in y-co-ordinates as $y_8 = 0$, $y_7 = 0$, and $y_6 = 0$.

Immediately we obtain the forms E_7, E_6, and E_5, known to be perfect and extreme. Thus all the forms $(f; \Lambda)$ are extreme.

In Table I are listed the forms of §§4 and 5 for $n \leqslant 16$. The columns give respectively the value of n, the values of $(h, r) = (k - 1, r)$, the quantity $\Delta = (2/M)^n D$, the number s of pairs of minimal vectors, and the known symbol for those (equivalent) forms that have been found previously.

6. The forms K_{11}, K_{12}. The form K_{12} of **(5)** appears very simply by this method as a combination of two E_6's.

Barnes **(1)** showed that E_n $(5 \leqslant n \leqslant 8)$ can be defined as

$$f(\mathbf{x}) = f(x_1, \ldots, x_n) = 9\sum_1^n x_i^2 - \left(\sum_1^n x_i\right)^2 \tag{6.1}$$

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TABLE I

n	(h, r)	Δ	s	Symbol
4	(0, 0)	4	12	B_4
5	(1, 1)	4	20	$E_5(\sim B_5)$
6	(1, 2)	3	36	E_6
7	(1, 3)	2	63	$E_7(\sim A_7{}^2)$
8	(1, 0)	1	120	$E_8(\sim B_8{}^2)$
9	(2, 1)	1	136	$B_9{}^2)$
10	(2, 2)	3/4	168	Φ_{10}
11	(2, 3)	1/2	219	—
12	(2, 0)	1/4	324	J_{12}
13	(3, 1)	1/4	348	—
14	(3, 2)	3/16	396	—
15	(3, 3)	1/8	471	—
16	(3, 0)	1/16	624	—

with lattice the sublattice of the integral lattice

$$\sum_1^n x_i \equiv 0 \pmod{3}. \tag{6.2}$$

We have $M(E_n) = 18$, $\Delta(E_n) = 9 - n$, and the minimal vectors are given by

$$(1, -1, 0, \ldots, 0)', \quad (1, 1, 1, 0, \ldots, 0)', \quad (1, 1, 1, 1, 1, 1, 0, \ldots, 0)', \tag{6.3}$$

$$(1, 1, 1, 1, 1, 1, 1, 2)', \tag{6.4}$$

where the prime denotes all permutations of the co-ordinates, and the sets (6.4) exist only for $n = 8$.

Now the minimal vectors of E_6 are all eliminated by the lattice

$$3\sum_1^6 ix_i - 2\sum_1^6 x_i \equiv 0 \pmod{9},$$
$$x_1 + x_2 + x_3 \equiv 0 \pmod{3}.$$

We therefore define $(f; \Lambda)$ to be the form

$$f(\mathbf{x}) = f(x_1, \ldots, x_{6k}) = \sum_{t=1}^k E_6(x_1{}^{(t)}, \ldots, x_6{}^{(t)}), \tag{6.5}$$

with lattice the sublattice of the integral lattice

$$\sum_{t=1}^k \left\{3\sum_{i=1}^6 ix_i{}^{(t)} - 2\sum_{i=1}^6 x_i{}^{(t)}\right\} \equiv 0 \pmod{9}, \tag{6.6}$$

Λ:

$$\sum_{t=1}^k (x_1{}^{[t)} + x_2{}^{(t)} + x_3{}^{(t)}) \equiv 0 \pmod{3}. \tag{6.7}$$

In view of relations of the type (6.2), it is easily verified that $d(\Lambda) = 3^2$; also it is not difficult to show that $E_6(x_1, \ldots, x_6)$ takes no value between 18 and 36 for integral $x_1, \ldots, x_6$.

From (2.3) we have

$$\Delta(f;\Lambda) = \frac{1}{2^{6k}}\cdot 3^4\cdot 3^k = \frac{3^{4+k}}{2^{6k}}. \tag{6.8}$$

For $k = 2$, carrying out the transformation

$$\mathbf{x}^{(1)} = A\{y_1, y_2, y_3, y_7, y_8, y_9\},$$

$$\mathbf{x}^{(2)} = A\{y_{10}, y_{11}, y_{12}, y_4, y_5, y_6\},$$

where

$$A = \frac{1}{3}\begin{bmatrix} 2 & \cdot & -2 & 1 & \cdot & -1 \\ 2 & \cdot & 1 & 1 & \cdot & 2 \\ 2 & \cdot & 1 & 1 & \cdot & -1 \\ 1 & -1 & \cdot & 2 & -2 & \cdot \\ 1 & 2 & \cdot & 2 & 1 & \cdot \\ 1 & -1 & \cdot & 2 & 1 & \cdot \end{bmatrix},$$

we obtain the equivalent form

$$f(\mathbf{y}) = \sum_{i=1}^{6} (y_i^2 + y_i y_{i+6} + y_{i+6}^2) \tag{6.9}$$

with integral $\mathbf{y}$ satisfying

$$y_1 - y_7 \equiv y_2 - y_8 \equiv \ldots \equiv y_6 - y_{12} \pmod 3,$$

$$\sum_{1}^{12} y_i \equiv 0 \pmod 3. \tag{6.10}$$

This is the form K_{12} **(5)**, known to be perfect and extreme. Hence by Theorem 2.1, the forms $(f;\Lambda)$ are perfect and extreme for all $k \geqslant 2$. In fact, for $k = 1$, $(f;\Lambda)$ is again perfect and extreme and may be identified with the form L_6^3 of **(2**, I).

Taking $k = 2$, and $x_6^{(2)} = 0$ (corresponding to $f = E_6 + E_5$), we obtain the form K_{11} with $\Delta = 3^5/2^9$, $s = 216$. In y-co-ordinates this becomes the section of the form defined by (6.9), (6.10) obtained by setting

$$y_4 - y_{10} = y_5 - y_{11}.$$

7. Bounds for Δ_n ($13 \leqslant n \leqslant 16$). The minimal vectors (6.3), (6.4) of the form E_8 are eliminated by the lattice

$$3\sum_{1}^{8} ix_i - 2\sum_{1}^{8} x_i \equiv 0 \pmod 6,$$

$$x_1 + x_2 \equiv x_3 + x_4 \equiv \ldots \equiv x_7 + x_8 \pmod 2.$$

We therefore define $(f;\Lambda)$ to be the form

$$f(\mathbf{x}) = f(x_1, \ldots, x_{8k}) = \sum_{t=1}^{k} E_8(x_1^{(t)}, \ldots, x_8^{(t)}),$$

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with lattice the sublattice of the integral lattice

$$\Lambda: \quad \begin{aligned} &\sum_{t=1}^{k} \left\{ 3\sum_{i=1}^{8} i x_i^{(t)} - 2\sum_{i=1}^{8} x_i^{(t)} \right\} \equiv 0 \pmod 6, \\ &\sum_{t=1}^{k} (x_1^{(t)} + x_2^{(t)}) \equiv \ldots \equiv \sum_{t=1}^{k} (x_7^{(t)} + x_8^{(t)}) \pmod 2. \end{aligned}$$

Because of congruences of the type (6.2) it follows that $d(\Lambda) = 2^4$. Also, E_8 satisfies the required condition on the minimum, and so

$$\Delta(f;\Lambda) = \frac{1}{2^{8k}} \cdot 2^8 \cdot 1 = 2^{8-8k}.$$

For $k = 2$ ($n = 16$), we have $\Delta(f;\Lambda) = 1/2^8$; this form is obtained in **(3)**.

Taking sections of this 16-variable form, we obtain the following forms:

(i) Setting $x_8^{(2)} = 0$, a 15-variable form f' with

$$\Delta(f') = \frac{1}{2^{15}} \cdot 2^8 \cdot 2 = 2^{-6}; \qquad \text{see } \mathbf{(3)}.$$

(ii) Setting $x_7^{(2)} = 0 = x_8^{(2)}$, a 14-variable form f'' with

$$\Delta(f'') = \frac{1}{2^{14}} \cdot 2^8 \cdot 3 = 3 \cdot 2^{-6}.$$

(iii) Setting $x_6^{(2)} = 0 = x_7^{(2)} = x_8^{(2)}$, a 13-variable form f''' with

$$\Delta(f''') = \frac{1}{2^{13}} \cdot 2^8 \cdot 4 = \frac{1}{8}.$$

Hence

$$\Delta_{13} \leqslant 1/8, \qquad \Delta_{14} \leqslant 3/64, \qquad \Delta_{15} \leqslant 1/64, \qquad \Delta_{16} \leqslant 1/256.$$

We notice that Mordell's inequality **(8)**

$$\Delta_n \geqslant (\tfrac{1}{2}\Delta_{n-1})^{n/(n-2)}$$

would hold with equality for $n = 16$, if the bounds for Δ_n, Δ_{n-1} were precise.

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