Ergod. Th. & Dynam. Sys., (2023), **43**, 873–886
doi:10.1017/etds.2021.174

A non-singular version of the Oseledeč ergodic theorem

ANTHONY H. DOOLEY and JIE JIN

School of Mathematical and Physical Sciences,
University of Technology Sydney, Ultimo 2007, Australia
(e-mail: anthony.dooley@uts.edu.au, Jie.Jin-3@student.uts.edu.au)

(*Received* 6 *October* 2020 *and accepted in revised form* 21 *December* 2021)

Abstract. Kingman's subadditive ergodic theorem is traditionally proved in the setting of a measure-preserving invertible transformation T of a measure space (X, μ). We use a theorem of Silva and Thieullen to extend the theorem to the setting of a not necessarily invertible transformation, which is non-singular under the assumption that μ and $\mu \circ T$ have the same null sets. Using this, we are able to produce versions of the Furstenberg–Kesten theorem and the Oseledeč ergodic theorem for products of random matrices without the assumption that the transformation is either invertible or measure-preserving.

Key words: non-singular ergodic theory, random ergodic theorem, critical dimension
2020 Mathematics Subject Classification: 37A40, 37A30 (Primary); 37A50 (Secondary)

1. *Introduction*

The study of ergodic theorems is an important bridge between functional analysis and probability theory. Originally proved by Birkhoff [4] in 1931, the Birkhoff ergodic theorem has become the fundamental theorem in the study of measure-preserving transformations of a measure space. The subadditive ergodic theorem, obtained by Kingman [10] in 1968, is an important extension of this fundamental result, which has found many applications. One important application is the Furstenberg–Kesten theorem [6], on the structure of multiplicative cocycles from a measure-preserving transformation T of a measure space (X, μ), with values in $GL(d, \mathbb{R})$. The Furstenberg–Kesten theorem has been extended and refined by the well-known Oseledeč ergodic theorem on the products of randomly chosen matrices [11].

A statement of Kingman's theorem is as follows.

THEOREM 1. (The sub-additive ergodic theorem) *Let $(X, \mathcal{B}, \mu)$ be a probability space, and $T : X \to X$ an invertible and measure-preserving transformation. Let $f_n \in L^1$ be a* *sequence of functions satisfying the subadditivity condition:* $f_{m+n}(x) \leq f_n(x) + f_m(T^n x)$ *for almost all* $x \in X$. *Then*

$$\lim_{n\to\infty} \frac{f_n(x)}{n} = f(x) < \infty$$

exists μ*-almost everywhere (a.e). Furthermore,* $f(x)$ *is a* T*-invariant measurable function over* $(X, \mathcal{B}, \mu)$.

There have been several proofs of this theorem since Kingman's original version. See [1] for a survey of these. Most of them have made the assumption that the measure μ is invariant under the transformation T. However, [14, Theorem 3.4] is a version of the subadditive ergodic theorem under the assumption that T is a Markovian transformation of $(X, \mathcal{B}, \mu)$.

Note that the theorem generalizes the following result in elementary analysis, which we recover in the case where the f_n are all constant functions.

LEMMA 1. *If* (f_n) *is a subadditive sequence then*

$$\lim_n \frac{f_n}{n} = \inf_n \frac{f_n}{n} < \infty.$$

The aim of this paper is to use [14, Theorem 3.4] to extend the Furstenberg–Kesten theorem and the Ocеledeč theorem to the setting of non-singular transformations. The key idea is to define subadditive sequences by

$$f_{m+n}(x) \leq f_n(x) + \omega_n(x) f_m(T^n x)$$

where $\omega_n(x) = (d\mu \circ T^n)/d\mu$ is the Radon–Nikodým derivative.

The non-singular version theorem of [14] allows us to conclude that

$$\lim_{n\to\infty} \frac{f_n(x)}{\sum_{i=0}^{n-1} \omega_i(x)} = f_*(x) < \infty$$

exists μ-a.e.

After some preliminary remarks and definitions in §2, we review [14, Theorem 3.4] in §3. In §4 we state and prove non-singular versions of the Furstenberg–Kesten theorem (Theorem 6) and the Oseledeč ergodic theorem (Theorem 7).

We expect these results to lead to new applications of these theorems in the non-singular setting. One key application of the Oseledeč theorem in the measure-preserving case is the calculation of Lyapunov exponents for random dynamical systems; see [2]. In future work, we will extend this construction to non-singular random dynamical systems.

2. *Preliminaries*

The dynamical system $(X, \mathcal{B}, \mu, T)$ is said to be *non-singular* if the map $T : X \to X$ is a non-singular transformation of (X, μ), that is, for any $N \in \mathcal{B}$, $\mu(TN) = 0$ if and only if $\mu(N) = 0$. Recall that the system is measure-preserving if $\mu(A) = \mu(TA)$ for all $A \in \mathcal{B}$. By the Poincaré recurrence lemma, measure-preserving transformations are conservative.

The structure of non-singular transformations is given by the Hopf decomposition theorem, a proof of which can be found in [1].

https://doi.org/10.1017/etds.2021.174 Published online by Cambridge University Press

THEOREM 2. (Hopf decomposition) *Let T be a non-singular transformation. There exist disjoint invariant sets $C, D \in \mathcal{B}$ such that $X = C \sqcup D$, T restricted to C is conservative, and $D = \bigsqcup_{n=-\infty}^{\infty} T^n W$, where W is a wandering set. If $f \in L^1(X, \mu)$, $f > 0$, then $C = \{x : \sum_{i=1}^{n-1} f(T^i x)\omega_i(x) = \infty$ a.e.$\}$ and $D = \{x : \sum_{i=1}^{n-1} f(T^i x)\omega_i(x) < \infty$ a.e.$\}$*

The set C is called the *conservative part* of T. If the non-singular transformation is invertible, so that T and its inverse T^{-1} are measurable, then we have both $\mu \circ T^{-1} \sim \mu$ and $\mu \circ T \sim \mu$. However, we do not assume further the transformation is invertible.

We will denote the Radon–Nikodým derivative $d(\mu \circ T^i)/d\mu$ by ω_i. Note that the Radon–Nikodým derivatives must satisfy the cocycle identity

$$\omega_{i+j}(x) = \omega_i(x)\omega_j(T^i x)$$

for almost every x and for every $i, j \in \mathbb{Z}$. Clearly, T is measure-preserving if and only if $\omega_i(x) = 1$ for almost every x for all i.

It follows that for every $f \in L^1(X, \mu)$

$$\int_X f(x)\, d\mu(x) = \int_X f(Tx)\omega_1(x)\, d\mu(x) = \int_X f(T^n x)\omega_n(x)\, d\mu(x).$$

If $f_n = \sum_{i=0}^{n-1} f(T^i x)\omega_i(x), n \geq 1$, where $\omega_0(x) = 1$, it is easy to show that $f_{m+n}(x) = f_n(x) + \omega_n(x) f_m(T^n x)$. The Hurewicz ergodic theorem [7] is a generalization of the Birkhoff ergodic theorem to the setting of non-singular conservative transformations.

THEOREM 3. (Hurewicz ergodic theorem) *Let $(X, \mathcal{B}, \mu)$ be a probability space, and $T : X \to X$ a non-singular conservative transformation. If $f \in L^1(\mu)$, then*

$$\lim_{n\to\infty} \frac{\sum_{i=0}^{n-1} f(T^i x)\omega_i(x)}{\sum_{i=0}^{n-1} \omega_i(x)} = f_*(x)$$

exists μ-a.e. Furthermore, $f_*(x)$ is T- invariant and

$$\int_X f(x)\, d\mu(x) = \int_X f_*(x)\, d\mu(x).$$

Note that if T is measure-preserving, the left-hand side becomes

$$\lim_{n\to\infty} (1/n) \sum_{i=0}^{n-1} f(T^i x),$$

and so we recover the Birkhoff theorem.

3. *Non-singular Kingman theorem*

Thus, let T be a conservative non-singular transformation of a measure space $(X, \mathcal{B}, \mu)$, and denote by ω_i the Radon–Nikodým derivative $(d\mu \circ T^i)/d\mu$.

Definition 1. We say that $\{f_n\}$ in $L^1(X, \mu)$ is a *subadditive sequence* for T if, for all integers m and n,

$$f_{m+n}(x) \leq f_n(x) + \omega_n(x) f_m(T^n x).$$

https://doi.org/10.1017/etds.2021.174 Published online by Cambridge University Press

It is easy to see that if f is integrable, then

$$f_n(x) = \sum_{i=0}^{n-1} f(T^i x)\omega_i(x).$$

is subadditive.

Similarly, we say that $\{f_n\}$ in $L^1(X, \mu)$ is *superadditive* for T if, for all integers m and n,

$$f_{m+n}(x) \geq f_n(x) + \omega_n(x) f_m(T^n x).$$

Observe that f_n is a superadditive sequence if and only if $-f_n$ is a subadditive sequence.

We now state the non-singular Kingman theorem,

THEOREM 4. (Non-singular Kingman ergodic theorem) *Let $(X, \mathcal{B}, \mu)$ be a probability space, and $T : X \to X$ a non-singular conservative transformation. Let $f_n \in L^1$ be a sequence of functions satisfying the subadditivity relation $f_{m+n}(x) \leq f_n(x) + \omega_n(x) f_m(T^n x)$ for almost all $x \in X$. Then*

$$\lim_{n\to\infty} \frac{f_n(x)}{\sum_{i=0}^{n-1} \omega_i(x)} = f_*(x) < \infty$$

exists μ-a.e. Furthermore, $f_*(x)$ is T-invariant and

$$\int_X f(x)\, d\mu(x) = \int_X f_*(x)\, d\mu(x).$$

This theorem follows easily from the following fact, which follows from the maximal ergodic theorem:

$$\int_X \limsup_{n\to\infty} \frac{f_n}{\sum_{i=0}^{n-1} \omega_i(x)}\, d\mu \leq \int L\, d\mu \leq \int_X \liminf_{n\to\infty} \frac{f_n}{\sum_{i=0}^{n-1} \omega_i(x)}\, d\mu$$

where L is

$$\lim_{n\to\infty} \frac{f_n}{\sum_{i=0}^{n-1} \omega_i(x)}.$$

Proof. Theorem 3.4 of [14] states that if $\{f_n\}$ is subadditive and $\{g_n\}$ is superadditive, the limit

$$\lim_{n\to\infty} \frac{f_n}{g_n} = \frac{\lim_{n\to\infty}(1/n)\mathbb{E}_{h\mu}[f_n/h|\mathcal{I}]}{\lim_{n\to\infty}(1/n)\mathbb{E}_{h\mu}[g_n/h|\mathcal{I}]}$$

for any positive μ-integrable function h (where $\mathcal{I}$ is the invariant σ-algebra). In our case, we may take $h = 1$, since X is a probability space, and noting that $g_n(x) = \sum_{i=0}^{n-1} \omega_i(x)$ is a superadditive sequence of non-negative functions, the result follows. □

We would like to thank the referee for pointing out this elegant proof of the theorem. Silva and Thieullen's proof of their Theorem 3.4 uses a maximal function argument: their Lemma 2.4 extends the maximal ergodic theorem to subadditive sequences. Using it, we obtain the following maximal function estimate for non-singular actions.

THEOREM 5. *Suppose that (f_n) is a subadditive sequence of functions satisfying the integrability condition $f_1^+ \in L^1$, and that $p \geq 1$ an integer. Define*

$$A = \{x \in X : f_k(x) \geq 0 \text{ for all } 1 \leq k \leq p\}$$

and

$$B = \{x \in X : f_k(x) > 0 \text{ for some } 1 \leq k \leq p\}.$$

Suppose further that for all integers $n \geq p$ and for almost every $x \in X$ we have

$$f_n \leq \sum_{i=0}^{n-1} f_1 \circ T^i \chi_A \circ T^i \omega_i + \sum_{i=n-p}^{n-1} \| f_1 \circ T^i \| \omega_i$$

and

$$f_n^+ \leq \sum_{i=0}^{n-1} f_1 \circ T^i \chi_B \circ T^i \omega_i + \sum_{i=n-p}^{n-1} \| f_1 \circ T^i \| \omega_i .$$

Then

$$\inf_{n\geq 1} \left(\frac{1}{n}\right) \int f_n \, d\mu = \lim_{n\to\infty} \left(\frac{1}{n}\right) \int f_n \, d\mu \leq \int_A f_1 \, d\mu,$$

$$0 \leq \lim_{n\to\infty} \left(\frac{1}{n}\right) \int f_n^+ \, d\mu \leq \int_B f_1 \, d\mu.$$

Many proofs of the Kingman theorem deduce it from a maximal inequality; see [**8**, **9**]. There are other types of proofs [**3**, **15**] which do not rely on a maximal inequality. One can also follow these approaches, replacing the quantity $1/n$ with $1/\Omega_n(x)$, where $\Omega_n(x) = \sum_{i=0}^{n-1} \omega_i(x)$, to find an alternative proof of Theorem 4.

Note that in the measure-preserving case, we have $\Omega_n(x) = n$, which gives us back the standard Kingman theorem, Theorem 1: our theorem shows how to replace the quantity $1/n$ with $1/\Omega_n(x)$, which is the key to proving the multiplicative ergodic theorem below.

In the case where the measure μ is non-singular and has critical dimension $\alpha \in [0, 1]$ (see [**5**]), we have that $\Omega_n(x)/n^\alpha$ is non-zero a.e., and the conclusion of Theorem 6 is equivalent to $\lim_{n\to\infty}(f_n(x)/n^\alpha) = f(x) < \infty$.

4. *The multiplicative ergodic theorem*

We now introduce the notion of cocycles with values in $GL(d)$ of a non-singular transformation T of $(X, \mathcal{B}, \mu)$; see [**13**]. A *cocycle* with respect to the action of T is a function $\Phi : \mathbb{N} \times X \to GL(d)$ satisfying $\Phi(n + m, x) = \Phi(n, x)\Phi(m, T^{n-1}x)$.

Cocycles can be generated by choosing a (random) $d \times d$ matrix, $A(x)$, for each $x \in X$, and defining

$$\Phi(n, x) = A(x) \times A(Tx) \times A(T^2x) \cdots \times A(T^{n-1}x).$$

It is easy to see that this formula defines a cocycle. We will say that $A(x)$ is the *generator* of Φ.

The operator norm of a square matrix A of dimension d is defined as

$$\|A\| = \sup\left\{\frac{\|Av\|}{\|v\|} : v \in \mathbb{R}^d\backslash\{0\}\right\}.$$

It follows directly from the definition that the norm of the product of two matrices is less than or equal to the product of the norms of those matrices. Thus

$$\|\Phi(n, x)\| \leq \|A(x)\|\|A(Tx)\| \cdots \|A(T^{n-1}x)\|.$$

If T is measure-preserving, the Furstenberg–Kesten theorem is an application of the Kingman subadditive ergodic theorem, applied to the subadditive sequence:

$$\log \|\Phi(n+m, x)\| \leq \log \|\Phi(n, x)\| + \log \|\Phi(m, T^m x)\|.$$

In the non-singular case, we define a (non-singular) subadditive sequence by

$$\log \|\Phi(n, x)\| \leq \sum_{i=0}^{n-1} \omega_i \log \|A(T^i x)\|.$$

We define singular values and exterior powers before we introduce the theorem.

Definition 2. (Exterior power) Let V be a vector space with dimension r. For $1 < k < r$, the k-fold exterior power of V is $\wedge^k V$, which is the vector space of alternating k-linear forms on the dual space. The k-fold exterior power of a matrix A is $\wedge^k A$, which has the following properties:

(i) $(AB)^{\wedge k} = A^{\wedge k} B^{\wedge k}$;

(ii) $(A^{\wedge k})^{-1} = (A^{-1})^{\wedge k}$;

(iii) $(cA)^{\wedge k} = c^{\wedge k} A^{\wedge k}$, where $c \in \mathbb{R}$.

The singular valued decomposition of exterior powers is $\wedge^k A = \wedge^k V \wedge^k D \wedge^k U$, where $\wedge^k D$ is a diagonal matrix with entries $\{\delta_{i_1}\delta_{i_2} \cdots \delta_{i_k}, 1 \leq i_1 \leq \cdots \leq i_k \leq r\}$. The largest singular value is $\delta_{r-k+1} \cdots \delta_r$, and the smallest value is $\delta_1 \cdots \delta_k$. The norm of $\wedge^k A$ is the largest singular value.

THEOREM 6. (Non-singular Furstenberg–Kesten theorem) *Let Φ be a linear cocycle with one side in discrete time over the non-singular dynamical system $(\Omega, \mathcal{F}, \mu, T)$. Assume that the generator $A : X \to Gl(d, \mathbb{R})$ of Φ satisfies*

$$\log^+ \|A\| \in L^1$$

Then the following statements hold.

(1) *For each $k = 1, \ldots, d$, the sequence*

$$f_n^k(x) = \log\|\wedge^k \Phi(n, x)\|, \ n \in \mathbb{N},$$

is subadditive and $f_1^{k+} \in L^1(X, \mathcal{F}, \mu)$. That is,

$$f_{n+m}^k(x) \leq f_m^k(x) + f_n^k(T^m x)\omega_m.$$

https://doi.org/10.1017/etds.2021.174 Published online by Cambridge University Press

(2) *There is an invariant set* $\bar{\Omega}$ *of full measure and measurable functions* $\gamma^k : X \to \mathbb{R}$ *with* $\gamma^{k+} \in L^1(X, \mathcal{F}, \mu)$*:*

$$\gamma^k(x) = \lim_{n\to\infty} \frac{\log\|\wedge^k \Phi(n, x)\|}{\sum_{i=0}^{n-1} \omega_i(x)}$$

and

$$\gamma^k(Tx) = \gamma^k(x), \quad \gamma^{k+l}(x) \le \gamma^k(x) + \gamma^l(x).$$

Let Λ_k *be the function defined by* $\Lambda_k = \gamma^{k+1} - \gamma^k$*, and let* δ_k *be the corresponding singular value of* $\Phi(n, x)$*. Then*

$$\Lambda_k = \lim_{n\to\infty} \frac{\log \delta_k(\Phi(n, x))}{\sum_{i=0}^{n-1} \omega_i(x)}.$$

Proof. Note that $A(x) = \Phi(1, x)$. For all k,

$$f_n^k(x) = \log\|\wedge^k \Phi(n, x)\|$$

is a subadditive sequence, and

$$f_{n+1}^k(x) = \log\|\wedge^k \Phi(n+1, x)\| \le f_n^k(Tx)\omega_1(x) + \log\|\wedge^k A(x)\|.$$

Hence subadditivity of $f_n^k(x)$ follows. By Theorem 4, we have

$$\gamma^k(x) = \lim_{n\to\infty} \frac{\log\|\wedge^k \Phi(n, x)\|}{\sum_{i=0}^{n-1} \omega_i(x)}.$$

Since $\|\wedge^{k+l}\Phi(n, x)\| \le \|\wedge^k \Phi(n, x)\| \|\wedge^l \Phi(n, x)\|$, γ^k is a subadditive sequence. For $k = 1, \ldots, d$,

$$\log\|\wedge^k \Phi(n, x)\| = \sum_{i=1}^{k} \log \delta_i(\Phi(n, x))$$

where δ_i is the corresponding singular value of $\Phi(n, x)$. □

We now consider the behaviour of $\|\Phi(n, x)v\|$ for $v \in \mathbb{R}^d$ as $n \to \infty$. If $A \in M_d(\mathbb{R})$ with transpose A^*, both A^*A and AA^* are symmetric and positive semidefinite. Any positive semidefinite and symmetric matrix S may be written in the form

$$S = C^{-1}DC$$

where D is diagonal with non-negative entries in non-deceasing order and C is orthogonal.

The polar decomposition of a matrix A is

$$A = C(AA^*)^{1/2}C' = C''(A^*A)^{1/2}$$

where C', C'' are orthogonal matrices. Applying the polar decomposition to $\Phi(n, x)$ in the theorem, we obtain

$$\Phi(n, x) \approx C_n'' A^n(x)$$

https://doi.org/10.1017/etds.2021.174 Published online by Cambridge University Press

for some orthogonal matrix C''_n. Since orthogonal matrices are isometries, we have $\|C''_n v\| = \|v\|$. Thus

$$\|\Phi(n, x)v\| = \|A^n(x)v\|.$$

Returning to the symmetric matrix $\Phi(n, x)^*\Phi(n, x)$, we know that $\Phi(n, x)^*\Phi(n, x) = C_n^* D_n C_n$, and $\Phi(n, x) = L_n(D_n)^{1/2}C_n$, and hence

$$\lim_{n\to\infty} (\Phi(n, x)^*\Phi(n, x))^{1/2n} = \lim_{n\to\infty} C_n^*(D_n)^{1/2n}C_n.$$

If the limit exists, then there are an orthogonal matrix $C = \lim C_n$ and a diagonal matrix $D = \lim D_n^{1/2n}$. By Theorem 5, we see that $\lim_{n\to\infty}(\log\|\wedge^k\Phi(n, x)\|/\sum_{i=0}^{n-1}\omega_i(x))$ converges for all k to $-\infty$ or a finite limit. Hence $\lim_{n\to\infty}(\log \delta_i(\Phi(n, x))/\sum_{i=0}^{n-1}\omega_i(x))$ converges to a finite limit for every i. Now we can assume that $D_n^{1/2n}$ converges as $n \to \infty$.

By the monotonicity of Λ^i,

$$\Lambda^r(x) \geq \cdots \geq \Lambda^1(x).$$

There is a unique partition I, given by

$$I = \{1 = i_2 < i_1 < \cdots < i_p < i_{p+1} = r + 1\},$$

such that $\Lambda^{i_q} = \Lambda^{i_{q+1}-1} < \Lambda^{i_{q+1}}$. This partition splits $\{1, 2, \ldots, r\}$ into finitely many intervals $[i_q, i_{q+1} - 1]$. If $\Lambda^i = \Lambda^j$, then they belong to same interval. Let $\Sigma(I, q)(x)$ be a vector subspace of $\mathbb{R}^r$; it is a union of the zero vector 0 and the set of all eigenvectors corresponding to eigenvalues smaller than or equal to $\Lambda^{i_{q+1}-1}$. It is easily seen that $\Sigma(I, 0)(x)$ is $\{0\}$ and $\Sigma(I, p)(x)$ is $\mathbb{R}^r$. We see that $C_n^{-1}e_i$ is an eigenvector of $(\Phi(n, x)^*\Phi(n, x))^{1/2}$ with eigenvalue δ_i^2. We know that $D_n^{1/2n}$ converges, but the question is whether the vector space spanned by $C_n^{-1}e_i$ converges. We will formulate a one-sided multiplicative ergodic theorem which is based on Raghunathan's version [12].

THEOREM 7. (Non-singular Oseledeč theorem) *Let (X, S, m) be a probability space. Suppose that T is a non-singular transformation and $u : Z \times X \to M(r, \mathbb{R})$ is a measurable cocycle over T such that* $\log^+ \|\Phi(1, \dot{)}\| \in L^1(X, S, m)$. *We set*

$$B = \left\{(x, v) \in X \times \mathbb{R}^r : \frac{\log \|\Phi(n, x)v\|}{\sum_{i=0}^{n-1}\omega_i(x)} \text{ tends to a finite limit or } -\infty\right\}$$

and

$$X' = \{x \in X | (x, v) \in B \text{ for all } v \in \mathbb{R}^r\}.$$

Then there is a subset Y of X' which has full measure and a sequence of functions $\Lambda^1(x) \leq \cdots \leq \Lambda^r(x)$ (taking values in $\mathbb{R} \cup -\infty$) such that the following statements hold.

(i) *Let $I = \{1 = i_1 < i_2 < \cdots < i_p < i_{p+1} = r + 1\}$ be $n + 1$ tuples of integers. Define $Y(I) = \{x \in X' | \Lambda^i(x) = \Lambda^j(x)$ for $i_q \leq i, j < i_{q+1}$ and $\Lambda^{i_q}(x) < \Lambda^{i_{q+1}}(x)$ for all q with $1 < q < p\}$. Then for all $x \in Y(I)$, $1 < q < p$,*

$$\Sigma(I, q)(x) = \left\{v \in \mathbb{R}^r | \lim_{n\to\infty} \frac{\log \|\Phi(n, x)v\|}{\sum_{i=0}^{n-1}\omega_i(x)} \leq \Lambda^{i_q}(x)\right\}$$

is a vector subspace of $\mathbb{R}^r$ with dimension $i^{q+1} - 1$.

(ii) *If* $\Sigma(I, 0) = \{0\}$ *then*

$$\lim_{n\to\infty} \frac{\log \|\Phi(n, x)v\|}{\sum_{i=0}^{n-1} \omega_i(x)} = \Lambda^{i_q}(x)$$

for any vector $v \in \Sigma(I, q)(x) - \Sigma(I, q-1)(x)$.

(iii) *For* $x \in Y$, *the sequence*

$$A(n, x) = (\Phi(n, x)^* \Phi(n, x))^{1/2n}$$

converges to a matrix $A(x) \in M(r, \mathbb{R})$. *The eigenspace of* $A(x)$ *is the orthogonal complement of* $\Sigma(I, q)(x)$ *in* $\Sigma(I, q+1)(x)$ *corresponding to the eigenvalue* $\exp \Lambda^{i_{q+1}}$.

LEMMA 2. *Suppose that* $\log^+ \|\Phi(1, \dot{})\|$ *is a measurable function and* T *a non-singular transformation. There is a set* $Y \subset X$ *of full measure such that for every* $x \in Y$, *the following statements hold.*

(i) *The sequence* $S_n = \sum_{0\le q<n} \log^+ \|\Phi(1, T^q(x))\| / \sum_{i=0}^{n-1} \omega_i(x)$ *converges to a limit a.e.*

(ii) $\lim_{n\to\infty} \|\Phi(1, T^n(x))\| = 0$.

Proof. (i) This follows directly from the Hurewicz ergodic theorem.

(ii) By (i), the sequence S_n converges to a limit,

$$S_n = \frac{\sum_{0\le q<n} \log^+ \|\Phi(1, T^q(x))\|}{\sum_{i=0}^{n-1} \omega_i(x)},$$

$$S_{n-1} = \frac{\sum_{0\le q<n-1} \log^+ \|\Phi(1, T^q(x))\|}{\sum_{i=0}^{n-2} \omega_i(x)},$$

$$S_n = \frac{\sum_{i=0}^{n-2} \omega_i(x)}{\sum_{i=0}^{n-1} \omega_i(x) \sum_{i=0}^{n-2} \omega_i(x)} \sum_{0\le q<n-1} \log^+ \|\Phi(1, T^q(x))\| + \frac{\log^+ \|\Phi(1, T^n(x))\|}{\sum_{i=0}^{n-1} \omega_i(x)}.$$

Since $n \to \infty$ and $|S_n - S_{n-1}| \to 0$, we can conclude that

$$\frac{\log^+ \|\Phi(1, T^n(x))\|}{\sum_{i=0}^{n-1} \omega_i(x)} \to 0$$

for all $x \in Y$. □

Now, given $\varepsilon > 0$, there exists $N(\varepsilon, x)$ such that for all $n > N$,

$$\|\Phi(1, T^n(x))\| < \exp \sum_{i=0}^{n-1} \omega_i(x)\varepsilon.$$

$\|\Phi(n, x)\|$ satisfies the cocycle identity: $\Phi(n+1, x) = \Phi(1, T^n(x))\Phi(n, x)$.

https://doi.org/10.1017/etds.2021.174 Published online by Cambridge University Press

For a unit vector $v \in \Sigma(I, q, n)(x)$,

$$\|\Phi(n+1, x)v\| \leq \|\Phi(1, T^n(x))\| \|\Phi(n, x)v\|,$$

$$\|\Phi(n+1, x)v\| \leq \exp\bigg(\sum_{i=0}^{n-1} \omega_i(x)(\Lambda^{i_q} + 2\varepsilon)\bigg).$$

Choose a unit vector $v \in \Sigma(I, q, n)(x)$, and let $v' \in \Sigma(I, q, n+1)(x)$ be the orthogonal projection of v on $\Sigma(I, q, n+1)(x)$. The orthogonal complement of v' in $\Sigma(I, q, n+1)(x)$ has the form $C_{n+1}^{-1} \sum_{i \geq_{q+1} - 1} b_i e_i$.

LEMMA 3. *Given $\varepsilon > 0$, there exists $N(\varepsilon, x)$, $x \in Y$, with the following property. There is a unit vector $v \in \Sigma(I, q, n)(x)$, for some number $b_i \in \mathbb{R}$ and $v' \in \Sigma(I, q, n+1)(x)$, such that*

$$v = v' + C_{n+1}^{-1} \sum_{i \geq i_{q+1}} b_i e_i.$$

Then $|b_i| < \exp\{-\sum_{i=0}^{n-1} \omega_i(x)(\Lambda^i - \Lambda^{i_q} - \varepsilon)\}$ for $n \geq N$.

Proof. We have $\|\Phi(n+1, x)v\| \leq \exp \sum_{i=0}^{n-1} \omega_i(x)(\Lambda^{i_q} + 2\varepsilon)$ by Lemma 2. Notice that

$$\|\Phi(n+1, x)v\| \geq \bigg\| \sum_{i \geq i_{q+1}} |b_i| \Phi(n+1, x) C_{n+1}^{-1} e_i \bigg\|.$$

Now $\|b_i \Phi(n+1, x) C_{n+1}^{-1} e_i\| \leq \| \sum_{i \geq i_{q+1}} b_i \Phi(n+1, x) C_{n+1}^{-1} e_i\|$ as Λ^i is non-decreasing. Hence

$$\|\Phi(n+1, x)v\| \geq \|b_i \Phi(n+1, x) C_{n+1}^{-1} e_i\|.$$

Now $\|\Phi(n+1, x) C_{n+1}^{-1} e_i\|$ is the ith eigenvalue, since

$$\|b_i \Phi(n+1, x) C_{n+1}^{-1} e_i\| = |b_i| \|L_{n+1}(D_{n+1})^{1/2} C_{n+1} C_{n+1}^{-1} e_i\| = |b_i| \delta_i(\Phi_{n+1}(x)).$$

Let Λ^i be the limit of $\log \delta_i(\Phi_{n+1}(x)) / \sum_{i=0}^{n} \omega_i(x)$. It follows from the above that

$$\|\Phi(n+1, x)v\| \geq |b_i| \exp\bigg(\bigg(\sum_{i=0}^{n} \omega_i(x)\bigg)(\Lambda^i - \varepsilon)\bigg).$$

Thus Λ^i is in a bounded interval which is greater than Λ^{i_q}, $0 < \varepsilon < 1$. For large n, we may assume

$$\exp\bigg(\bigg(\sum_{i=0}^{n} \omega_i(x)\bigg)(\Lambda^i - \varepsilon)\bigg) \geq \exp\bigg(\bigg(\sum_{i=0}^{n-1} \omega_i(x)\bigg)(\Lambda^i - 2\varepsilon)\bigg).$$

Thus

$$|b_i| \leq \exp -\bigg(\sum_{i=0}^{n-1} \omega_i(x)\bigg)(\Lambda^i - 4\varepsilon - \Lambda^{i_q})$$

which completes the proof of the lemma. □

https://doi.org/10.1017/etds.2021.174 Published online by Cambridge University Press

Lemma 3 shows that a vector in $\Sigma(I, q, n)(x)$ can be combined with the projection on $\Sigma(I, q, n+1)(x)$ and the orthogonal complement of $\Sigma(I, q, n+1)(x)$, that is,

$$C_n^{-1} \sum_{i=1}^{i_{q+1}-1} K e_i = C_{n+1}^{-1} \sum_{j=1}^{r} b_j e_j \quad \text{and} \quad v_n = v'_{m+1} + w_{m+1}.$$

Now v'_{m+1} is the orthogonal projection of v_n onto $\Sigma(I, q, n+1)(x)$ and the norm of v'_{m+1} is given by

$$\|v'_{m+1}\| = \bigg\| v_n - C_{n+1}^{-1} \sum_{i \geq i_{q+1}} b_i e_i \bigg\|.$$

Since v_n is a unit vector, we have the following upper and lower bounds for $\|v'_{m+1}\|$:

$$1 - \bigg\| C_{n+1}^{-1} \sum_{i \geq i_{q+1}} b_i e_i \bigg\| \leq \|v'_{m+1}\| \leq 1 + \bigg\| C_{n+1}^{-1} \sum_{i \geq i_{q+1}} b_i e_i \bigg\|.$$

We want to show that $\|v_{n+1} - v_n\| \leq 2r \exp\{-(\sum_{i=0}^{n-1})\omega_i(x)(\Lambda^{i_{q+1}} - \Lambda^{i_q} - \varepsilon)\}$.

LEMMA 4. *If $v_{n+1} \in \Sigma(I, q, n+1)(x)$ is a unit vector and $v_n \in \Sigma(I, q, n)(x)$ satisfies the conditions of Lemma 3, then*

$$\|v_{n+1} - v_n\| \leq 2r \exp\bigg\{ - \bigg(\sum_{i=0}^{n-1} \omega_i(x) \bigg) (\Lambda^{i_{q+1}} - \Lambda^{i_q} - \varepsilon) \bigg\}.$$

Proof.

$$\|v_{n+1} - v_n\| = \|v_n - v_{n+1}\| = \|v_n - v'_{n+1} + v'_{n+1} - v_{n+1}\|$$

and

$$\|v_n - v'_{n+1} + v'_{n+1} - v_{n+1}\| \leq \|v_n - v'_{n+1}\| + \|v'_{n+1} - v_{n+1}\|.$$

It follows that

$$\|v_n - v'_{n+1}\| = \bigg\| C_{n+1}^{-1} \sum_{i \geq i_{q+1}} b_i e_i \bigg\|.$$

On the other hand, let $v'_{n+1} = a v_{n+1}$, and we have $1 - \|C_{n+1}^{-1} \sum_{i \geq i_{q+1}} b_i e_i\| \leq \|v'_{m+1}\| \leq 1 + \|C_{n+1}^{-1} \sum_{i \geq i_{q+1}} b_i e_i\|$. Thus

$$1 - \bigg\| C_{n+1}^{-1} \sum_{i \geq i_{q+1}} b_i e_i \bigg\| \leq a \leq 1 + \bigg\| C_{n+1}^{-1} \sum_{i \geq i_{q+1}} b_i e_i \bigg\|.$$

Now $\|v'_{n+1} - v_{n+1}\| = \|(a-1)v_{n+1}\|$, which is smaller than $\|C_{n+1}^{-1} \sum_{i \geq i_{q+1}} b_i e_i\|$. We thus have

$$\|v_{n+1} - v_n\| \leq 2 \bigg\| C_{n+1}^{-1} \sum_{i \geq i_{q+1}} b_i e_i \bigg\|.$$

https://doi.org/10.1017/etds.2021.174 Published online by Cambridge University Press

It follows that

$$\left\| C_{n+1}^{-1} \sum_{i \geq i_{q+1}} b_i e_i \right\| \leq \sum_{i \geq i_{q+1}} \|b_i e_i\| \leq r |b_{i_{q+1}}|$$
$$\leq r \exp\left\{ -\left(\sum_{i=0}^{n-1} \omega_i(x) \right) (\Lambda^{i_{q+1}} - \Lambda^{i_q} - \varepsilon) \right\}.$$

This completes the proof. □

We will show that $\|v_{n+k} - v_{n+l}\|$ is a Cauchy sequence. In fact, the sequence $\sum_{i=l}^{\infty} r \exp\{-(n+i)(\Lambda^{i_{q+i}} - \Lambda^{i_q} - \varepsilon)\}$ is the sum of a geometric series. Thus

$$\|v_{n+k} - v_{n+l}\|$$
$$\leq \sum_{i=l}^{\infty} \|v_{n+k} - v_{n+l}\|$$
$$= 2r \frac{1}{1 - \exp\{-(\Lambda^{i_{q+1}} - \Lambda^{i_q} - \varepsilon)} \exp\left\{ -\left(\sum_{i=0}^{n-1} \omega_i(x) + l \right) (\Lambda^{i_{q+1}} - \Lambda^{i_q} - \varepsilon) \right\}$$
$$= C \exp\left\{ -\left(\sum_{i=0}^{n-1} \omega_i(x) + l \right) (\Lambda^{i_{q+1}} - \Lambda^{i_q} - \varepsilon) \right\}.$$

This shows that v_n is a Cauchy sequence, as claimed.

LEMMA 5. *Let* $\{v_n^1, v_n^2, \ldots, v_n^i\}$ *be a collection of vectors which is a basis for* $\Sigma(I, q, n)(x)$, *where* $0 \leq i < i_{q+1}$. *The sequence* $\{\Sigma(I, q, n)(x)\}$ *has limit* $\Sigma(I, q)(x)$. *Furthermore,*

$$\|v_{n+k} - v_{n+l}\| \leq \sum_{i=l}^{k} \|v_{n+k} - v_{n+l}\| \leq \sum_{i=l}^{\infty} \|v_{n+k} - v_{n+l}\|.$$

Proof. By Lemma 4, we see easily that

$$\|v_{n+k}^i - v_{n+l}^i\| \leq C \exp\left\{ -\left(\sum_{i=0}^{n-1} \omega_i(x) + \max(k, l) \right) (\Lambda^{i_{q+1}} - \Lambda^{i_q} - \varepsilon) \right\}$$

for $i_q \leq i < i_{q+1}$. Moreover,

$$\|v_n^i - v^i\| \leq C \exp\left\{ -\left(\sum_{i=0}^{n-1} \omega_i(x) \right) (\Lambda^{i_{q+1}} - \Lambda^{i_q} - \varepsilon) \right\}$$

where v^i is the limit of v_n^i. The sequence $\{v_n^i\}$ converges to a linearly independent set of vectors as $n \to \infty$. $\Sigma(I, q, n)(x)$ is the space spanned by $\{v_n^1, v_n^2, \ldots, v_n^i\}$. Thus $\Sigma(I, q, n)(x) \to \Sigma(I, q)(x)$ as $n \to \infty$. □

Writing the matrix $B = C_n C_{n+k}^{-1}$, a vector in the space $\Sigma(I, q, n)(x)$ has the form $C_n^{-1} e_i$ which we can split into an orthogonal projection in $\Sigma(I, q, n+1)(x)$ and its

https://doi.org/10.1017/etds.2021.174 Published online by Cambridge University Press

orthogonal complement in $\Sigma(I, q, n+1)(x)^{\perp}$. By Lemma 3, $C_n^{-1}e_i = C_{n+1}^{-1}\sum_{j=1}^{r} b_j e_j$. Hence $|C_{n+1}C_n^{-1}| = |b_j|$ as e_i is the standard basis. The inverse of $C_{n+1}C_n^{-1}$ is $C_nC_{n+1}^{-1}$. Thus we have a similar decomposition of a vector in $\Sigma(I, q, n+1)(x)$, namely,

$$v = v' + C_n^{-1}\sum_{i\geq_{q+1}} b_{i*}e_i,$$

where $v' \in \Sigma(I, q, n)(x)$ and $C_n^{-1}\sum_{i\geq_{q+1}} b_{i*}e_i \in \Sigma(I, q, n)(x)^{\perp}$. We set $a_i = \exp(\Lambda^i - i\varepsilon)$, and note that $|C_nC_{n+1}^{-1}| = |b_{i*}|$. It then follows that

$$|b_{i*}| \leq C \exp -\bigg(\sum_{i=0}^{n-1}\omega_i(x)\bigg)(\Lambda^i - \Lambda^{i_q} - r\varepsilon),$$

since we have a cycle of length r.

LEMMA 6. *If* $v^i \in \Sigma(I, q)(x)$, *then* $\log\|\Phi(n, x)(v_n^i)\|/\Omega(x) \leq \lambda^{i_q}(x)$.

Proof. Firstly, we can see that

$$\limsup(\log\|\Phi(n, x)(v_n^i)\|/\Omega(x)) = \lim_{n\to\infty}(\log\delta_i(\Phi(n, x))/\Omega(x)) = \Lambda^i.$$

If $i < i_{q+1}$, we have $\limsup(\log\|\Phi(n, x)(v_n^i)\|/\Omega(x)) \leq \Lambda^{i_q}$.

On the other hand, $v^i - v_n^i = w + C_n^{-1}\sum_{i\geq_{q+1}} b_{i*}e_i$. For $w \in \Sigma(I, q, n)(x)$, $\limsup(\log\|\Phi(n, x)(w)\|/\Omega(x)) \leq \Lambda^{i_q}$. Hence

$$\begin{aligned}
&\limsup \frac{\log\|\Phi(n, x)C_n^{-1}\sum_{i\geq i_{q+1}} b_{i*}e_i\|}{\sum_{i=0}^{n-1}\omega_i(x)}\\
&\quad\leq \limsup \frac{\log\{C\exp -(\Omega(x))(\Lambda^i - \Lambda^{i_q} - r\epsilon)\times\delta_i(\Phi(n, x))\}}{\Omega(x)}\\
&\quad= -\Lambda^i + \Lambda^{i_q} + r\epsilon + \Lambda^i = \Lambda^{i_q} + r\epsilon
\end{aligned}$$

for $i \geq i_{q+1}$.

The triangle inequality implies that

$$\|\Phi(n, x)(v^i)\| \leq \|\Phi(n, x)(v_n^i)\| + \|\Phi(n, x)(v^i - v_n^i)\| \leq 2\exp\bigg(\sum_{i=0}^{n-1}\omega_i(x)\bigg)(\Lambda^{i_q} + \varepsilon),$$

and thus $\limsup(\log\|\Phi(n, x)(v^i)\|/\sum_{i=0}^{n-1}\omega_i(x)) \leq \Lambda^{i_q}$. □

LEMMA 7. *If the vector* v^i *is not in* $\Sigma(I, q-1)(x)$, *for large n, the projection* $v^{i\prime}$ *belongs to* $\Sigma(I, q-1, n)(x)$ *with* $\|v^{i\prime}\| > c > 0$. *Then*

$$\liminf \frac{\log\|\Phi(n, x)(v^i)\|}{\sum_{i=0}^{n-1}\omega_i(x)} \geq \Lambda^{i_q}(x).$$

Proof. This proof is quite straightforward. We take a unit vector v^i which is not in $\Sigma(I, q-1)(x)$. There is a $\delta \in V$ such that $v^i + \delta \in \Sigma(I, q-1)(x)$. When n is large enough, the vector v_n^i has projection $v^{i\prime}$ in $\Sigma(I, q-1, n)(x)$ and orthogonal complement $v^{i\prime\prime} \in \Sigma(I, q-1, n)(x)^{\perp}$. We take the difference $\|v^i - v^{i\prime}\| \geq \delta/2$, obtaining

$$\begin{aligned}\|\Phi(n, x)(v^i)\| &\approx \|\Phi(n, x)v_n^i\| \\ &\geq \|\Phi(n, x)(v^i - v^{i\prime})\| \\ &\geq \frac{\delta}{2} \exp\bigg(\sum_{i=0}^{n-1} \omega_i(x)\bigg)(\Lambda^{i_q}(x) - \varepsilon).\end{aligned}$$ □

Combining Lemmas 6 and 7, we can conclude that

$$\lim(\log \|\Phi(n, x)(v^i)\| / \sum_{i=0}^{n-1} \omega_i(x)) = \Lambda^{i_q}(x),$$

for $v_i \in \Sigma(I, q)(x) \setminus \Sigma(I, q-1)(x)$. We now see that the eigenspace is $C_n^{-1}e_i \to C^{-1}e_i$ and the eigenvalue is $\Lambda^k(x) = \lim_{n\to\infty}(\log \delta_k(\Phi(n, x))/\sum_{i=0}^{n-1} \omega_i(x))$, so that the limit matrix $A(x) = \lim_{n\to\infty}(\Phi(n, x)^*\Phi(n, x))^{1/2n}$ exists.

Acknowledgements. We have been influenced by Raghunathan's elegant proof of the Oseledeč theorem [12]. We would like to thank Anthony Quas for drawing our attention to this paper, and for useful discussions. We would also like to thank the referee for several useful remarks.

REFERENCES

[1] J. Aaronson. *An Introduction to Infinite Ergodic Theory*. American Mathematical Society, Providence, RI, 1997.
[2] L. Arnold. *Random Dynamical Systems*. Springer, Berlin, 2003.
[3] A. Avila and J. Bochi. On the subadditive ergodic theorem. *Preprint*, 2009, http://www.mat.uc.cl/~jairo.bochi/docs/kingbirk.pdf.
[4] G. D. Birkhoff. Proof of the ergodic theorem. *Proc. Natl. Acad. Sci. USA* **17** (1931), 656–660.
[5] A. H. Dooley and G. Mortiss. On the critical dimension of product odometers. *Ergod. Th. & Dynam. Sys.* **29** (2008), 475–485.
[6] H. Furstenberg and H. Kesten. Products of random matrices. *Ann. Math. Stat.* **31** (1960), 457–469.
[7] W. Hurewicz. Ergodic theorem without invariant measure. *Ann. of Math. (2)* **45** (1944), 192–206.
[8] R. L. Jones. New proofs for the maximal ergodic theorem and the Hardy–Littlewood maximal theorem. *Proc. Amer. Math. Soc.* **87** (1983), 681–684.
[9] Y. Katznelson and B. Weiss. A simple proof of some ergodic theorems. *Israel J. Math.* **42** (1982), 291–296.
[10] J. F. C. Kingman. The ergodic theory of subadditive stochastic processes. *J. R. Stat. Soc. Ser. B.* **30** (1968), 499–510.
[11] I. Oseledets. A multiplicative ergodic theorem: Lyapunov characteristic numbers for dynamical systems. *Trans. Moscow Math. Soc.* **19** (1968), 197–231.
[12] M. S. Raghunathan. A proof of Oseledec's multiplicative ergodic theorem. *Israel J. Math.* **32** (1979), 356–362.
[13] K. Schmidt. *Algebraic Ideas in Ergodic Theory (CBMS Regional Conference Series in Mathematics, 76)*. American Mathematical Society, Providence, RI, 1990.
[14] C. E. Silva and P. Thieullen. The subadditive ergodic theorem and recurrence properties of Markovian transformations. *J. Math. Anal. Appl.* **154** (1991), 83–99.
[15] J. M. Steel. Kingman's subadditive ergodic theorem. *Ann. Inst. Henri Poincaré Probab. Stat.* **25** (1989), 93–98.