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RELATIONS BETWEEN BOUNDARY VALUE FUNCTIONS FOR A NONLINEAR DIFFERENTIAL EQUATION AND ITS VARIATIONAL EQUATIONS*

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We consider here the nonlinear differential equation

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}) \tag{1.1}$$

where $x \in I = [a, \infty)$. We will make the following assumptions:
(A) f is continuous on $[a, \infty) \times R^n$,
(B) solutions of initial value problems (I.V.P.'s) are unique and extend throughout $[a, \infty)$.

$$(C) \qquad f_k(x, y, y', \ldots, y^{(n-1)}) = \frac{\partial f(x, y, y', \ldots, y^{(n-1)})}{\partial y^{(k)}}$$

$k=0, 1, 2, \ldots, n-1$ is continuous on $[a, \infty) \times R^n$.

We could take $I=[a, b)$ where $b<\infty$, but for simplicity of statements, we take $I=[a, \infty)$. Before we define the boundary value functions that we will be interested in we make the preliminary definition.

DEFINITION 1.1. We say that $y \in C^n[a, \infty)$ has an $(i_1, \ldots, i_k)$-distribution of zeros, $0 \le i_m \le n$, $\sum_{m=1}^{k} i_m = n$, on $[c, d] \subseteq [a, \infty)$ provided there are points $c \le t_1 < t_2 < \cdots < t_k \le d$ such that $y(x)$ has a zero of order at least i_m at t_m, $m=1, 2, \ldots, k$.

DEFINITION 1.2. Let $R = \{r > t$: there are distinct solutions $u(x)$ and $v(x)$ of (1.1) such that $u(x)-v(x)$ has an $(i_1, \ldots, i_k)$-distribution of zeros, $0 \le i_m \le n$, $\sum_{m=1}^{k} i_m = n$, on $[t, r]\}$. If $R \ne \varnothing$, then set $r_{i_1 \cdots i_m}(t) = \inf R$, and if $R = \varnothing$, set $r_{i_1 \cdots i_k}(t) = \infty$.

These boundary value functions were studied by the author in [7]. Note that if $t \le t_1 < t_2 < \cdots < t_m < r_{i_1 \cdots i_m}(t) \le \infty$, then solutions of (1.1) satisfying the boundary conditions

$$y^{(m_j)}(t_j) = A_{m_j, j} \tag{1.2}$$

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where $A_{m_j,j}$ is a constant, $j=1, 2, \ldots, m$; $m_j=0, 1, \ldots, i_j-1$, when they exist are unique. This is unlike the linear case where one also gets the existence of solutions of (1.1), (1.2). Also we will use the following definition.

DEFINITION 1.3. The first conjugate point $\eta_1(t)$ for the nonlinear equation (1.1) is defined by,

$$\eta_1(t) = \min r_{i_1 \cdots i_m}(t),$$

where the minimum is taken over all boundary value functions $r_{i_1 \cdots i_m}(t)$ where $\sum_{j=1}^m i_j = n$.

DEFINITION 1.4. If $y_0(x)$ is a solution of (1.1), then the linear differential equation

$$z^{(n)} = \sum_{k=0}^{n-1} f_k(x; y_0(x), \ldots, y_0^{(n-1)}(x)) z^{(k)} \tag{1.3}$$

is called the equation of variation of (1.1) along $y_0(x)$.

DEFINITION 1.5. We will let $r_{i_1 i_2 \cdots i_k}(t; y_0(x))$ denote the boundary value function defined in Definition 1.2 for equation (1.3). Similarly $\eta_1(t; y_0(x))$ is the first conjugate point of $x=t$ for the linear differential equation (1.3).

Our main interest is in discovering relations which exist between boundary value functions for equation (1.1) and the various variational equations (1.3). We also discuss the relationship of the first conjugate point function $\eta_1(t)$ for (1.1) and the first conjugate point function $\eta_1(t; y_0(x))$ for the various variational equations (1.3).

DEFINITION 1.6. By $y(x; x_0, y_0) = y(x; x_0, \overset{1}{y_0}, \overset{2}{y_0}, \ldots, \overset{n}{y_0})$ we mean the solution of equation (1.1) satisfying the initial conditions

$$y^{(i)}(x_0) = \overset{i+1}{y_0}, \qquad i = 0, 1, \ldots, n-1.$$

DEFINITION 1.7. Let $\{u_j(x; x_0, y_0(x))\}_{j=0}^{n-1}$ be the fundamental set of solutions of equation (1.3) which satisfy the initial conditions at $x=x_0$,

$$u_j^{(i)}(x_0; x_0, y_0(x)) = \delta_{ij},$$

$i, j=0, 1, \ldots, n-1$, and δ_{ij} is the Kronecker delta. Sometimes it will be convenient to write $u_j(x; x_0, y_0)$ or $u_j(x; x_0, \overset{1}{y_0}, \ldots, \overset{n}{y_0})$, $j=0, 1, \ldots, n-1$, to denote the solution $u_j(x; x_0, y_0(x))$ of (1.3) where $y_0(x)$ is the solution of (1.1) with $y_0^{(i-1)}(x_0) = \overset{i}{y_0}$, $i=1, 2, \ldots, n$.

We will frequently make use of the next lemma which follows from the proof of Theorem 5.3.1 of [2].

LEMMMA 1.8. *The solution $y(x; x_0, y_0)$ has continuous partial derivatives with respect to its $n+2$ variables and in particular*

$$\frac{\partial y(x; x_0, y_0)}{\partial y_0^{(k)}} = u_{k-1}(x; x_0, y_0),$$

where

$$u_{k-1}(x; x_0, y_0) = \lim_{h\to 0} \frac{y(x; x_0, y_0^1, \ldots, y_0^k+h, \ldots, y_0^n)-y(x; x_0, y_0)}{h},$$

$k=1, \ldots, n$, *converges uniformly on compact subintervals of* $[a, \infty)$. *Also* $u_k^{(i)}(x; x_0, y_0)$, $k=1, \ldots, n$, *is a continuous function of its* $n+2$ *variables for* $i=0, 1, \ldots, n-1$.

Next we state and prove a theorem proved by Erbe [1] although the proof we give is a somewhat shorter proof than Erbe's.

THEOREM 1.9. *If* $t\geq a$,

$$r_{n-1,1}(t) \geq \inf_{y_0(x)} r_{n-1,1}(t; y_0(x))$$

and

$$r_{1,n-1}(t) \geq \inf_{y_0(x)} r_{1,n-1}(t; y_0(x))$$

where $y_0(x)$ *is a solution of* (1.1).

Proof. We will prove only the first statement, the proof of the other statement being similar. Suppose that $r_{n-1,1}(t)<\inf_{y_0(x)} r_{n-1,1}(t; y_0(x))=c$. Then there exist two distinct solutions $y_1(x)$ and $y_2(x)$ of (1.1) having an $(n-1, 1)$-distribution of zeros in $[t, c)$ at the points $t\leq x_1<x_2<c$. Suppose $y_1^{(i-1)}(x_1)=y_1^i$, $1\leq i\leq n$, and let $p=y_1^{(n-1)}(x_1)-y_2^{(n-1)}(x_1)$, then

$$y_1(x) = y(x; x_1, y_1^1, \ldots, y_1^n)$$

and

$$y_2(x) = y(x; x_1, y_1^1, \ldots, y_1^{n-1}, y_1^n+p)$$

and since both solutions are equal at x_2 we have

$$\begin{aligned} O &= y(x_2; x_1, y_1^1, \ldots, y_1^n+p)-y(x_2; x_1, y_1^1, \ldots, y_1^n) \\ &= p\frac{\partial y(x_2; x_1, y_1^1, \ldots, y_1^n+\lambda p)}{\partial y_1^{(n-1)}} \end{aligned}$$

where $0<\lambda<1$. Hence by Lemma 1.8, $u_{n-1}(x_2; x_1, y_\lambda(x))=0$, where $y_\lambda(x)$ is the solution of (1.1) satisfying the initial conditions $y_\lambda^{(i-1)}(x_1)=y_1^i$, $1\leq i\leq n-1$, $y_\lambda^{(n-1)}(x_1)=\lambda p$. This contradicts $r_{n-1,1}(t; y_\lambda(x))\geq c$.

EXAMPLE 1.10. Consider

$$y^{(n)} = f(x, y) \tag{1.4}$$

where f satisfies (A) and (B), $f_y(x, y)$ is continuous on $[a, \infty)\times R$, and $f_y(x, y)\geq 0$. It is well known that for the differential equation

$$z^{(n)}-f_y(x, y_0(x))z = 0$$

that $r_{n-1,1}(a; y_0(x)) = +\infty$ for any solution $y_0(x)$ of (1.1). Hence by Theorem 1.9 we have for equation (1.4) that $r_{n-1,1}(a) = \infty$.

In analogy to a definition made by Peterson [5] we make the following definition.

DEFINITION 1.11. If $r_{ij}(t) = \infty$ for $i = k, k+1, \ldots, n-1$, then set $\rho_k(t) = \infty$, $k = 1, \ldots, n-1$. Otherwise set $\rho_k(t) = \min\{r_{n-1,1}(t), r_{n-2,2}(t), \ldots, r_{k,n-k}(t)\}$, $k = 1, 2, \ldots, n-1$. Similarly $\rho_k(t; y_0(x))$ is defined for equation (1.3).

For linear equations it is well known that $\eta_1(t) = r_{1\ldots1}(t) = \rho_1(t)$. For the nonlinear case (1.1) it is known that $\eta_1(t) = \rho_1(t)$ for $n = 2, 3$ [4], and with an additional condition for $n = 4$ [6]. Also for the nonlinear case (1.1) it is known [3] that $\eta_1(t) = r_{1\ldots1}(t)$. Peterson [5] has shown that $\rho_k(t) = r_{k11\ldots1}(t)$, $k = 1, 2, \ldots, n-2$, in the linear case if $\rho_k(t) < r_{k,n-k}(t)$.

THEOREM 1.12. *If* $\rho_k(t; y_0(x)) < r_{k,n-k}(t; y_0(x))$, *then*

$$r_{k1\cdots1}(t) \le \rho_k(t; y_0(x)),$$

where $y_0(x)$ *is a solution of* (1.1).

Proof. Let $\varepsilon > 0$ be given. Since $\rho_k(t; y_0(x)) < r_{k,n-k}(t; y_0(x))$ by Theorem 4, [5], there is an $x_0 \in [t, \rho_k(t; y_0(x))$ and a solution $w(x; y_0(x))$ of the variational equation (1.3) corresponding to the solution $y_0(x)$ of (1.1) having a zero of order exactly k at x_0, $n-k-1$ simple zeros in $(x_0, \rho_k(t; y_0(x)))$ and an odd order zero in $[\rho_k(t; y_0(x)), \rho_k(t; y_0(x)) + \varepsilon)$. Now $w(x; y_0(x))$ can be written as

$$w(x; y_0 x)) = A_k u_k(x; x_0, y_0(x)) + \cdots + A_{n-1} u_{n-1}(x; x_0, y_0(x))$$

and we know that the quotient

$$\begin{aligned}
&\frac{y(x; x_0, y_0^1, \ldots, y_0^{k+1} + hA_k, \ldots, y_0^n + hA_{n-1}) - y(x; x_0, y_0^1, \ldots, y_0^n)}{h} \\
&= \frac{y(x; x_0, y_0^1, \ldots, y_0^{k+1} + hA_k, \ldots, y_0^n + hA_{n-1})}{h} \\
&\quad - \frac{y(x; x_0, y_0^1, \ldots, y_0^{k+1}, y_0^{k+2} + hA_{k+1}, \ldots, y_0^n + hA_{n-1})}{h} \\
&\quad + \cdots + \frac{y(x; x_0, y_0^1, \ldots, y_0^{n-1}, y_0^n + hA_{n-1}) - y(x; x_0, y_0^1, \ldots, y_0^n)}{h} \\
&= A_k u_k(x; x_0, y_0^1, \ldots, y_0^{k+1} + \xi_k, y_0^{k+2} + hA_{k+1}, \ldots, y_0^n + hA_{n-1}) \\
&\quad + \cdots + A_{n-1} u_{n-1}(x; x_0, y_0^1, \ldots, y_0^{n-1}, y_0^n + \xi_{n-1})
\end{aligned}$$

where ξ_j is between 0 and $A_j h$, $j = k, \ldots, n-1$. Hence

$$\frac{y(x; x_0, y_0^1, \ldots, y_0^{k+1} + hA_k, \ldots, y_0^n + hA_{n-1}) - y(x; x_0, y_0^1, \ldots, y_0^n)}{h}$$

converges uniformly to $w(x; y_0(x))$ as $h \to 0$ on $[t, \rho_k(t; y_0(x))+\varepsilon]$. Thus for h sufficiently small the difference

$$y(x; x_0, y_0^1, \ldots, y_0^{k+1}+hA_k, \ldots, y_0^n+hA_{n-1})-y(x; x_0, y_0^1, \ldots, y_0^n)$$

has a $(k, 1, 1, \ldots, 1)$-distribution of zeros on $[t, \rho_k(t; y_0(x))+\varepsilon)$. Hence $r_{k1\ldots 1}(t)<\rho_k(t; y_0(x))+\varepsilon$. Since $\varepsilon>0$ is arbitrary $r_{k11\ldots 1}(t)\le\rho_k(t; y_0(x))$.

REMARK 1.13. Note that we could have stated Theorem 1.12 differently, namely, if $\rho_{k+1}(t, y_0(x))<r_{k,n-k}(t; y_0(x))$, then $r_{k11\ldots 1}(t)\le\rho_{k+1}(t; y_0(x))$. The proof of Theorem 1.12 is valid for this last statement for $k=0$.

COROLLARY 1.14. *$\eta_1(t)\le\eta_1(t; y_0(x))$ where $y_0(x)$ is any solution of* (1.1).

Proof. The inequality is trivially satisfied if $\eta_1(t; y_0(x))=\infty$. Hence we can assume $\eta_1(t; y_0(x))<\infty$. From Remark 1.13 since $\eta_1(t; y_0(x))=\rho_1(t; y_0(x))< r_{on}(t)=\infty$, we have that $r_{1\ldots 1}(t)\le\rho_1(t; y_0(x))$. Since $\eta_1(t)=r_{1\ldots 1}(t)$ the proof is complete.

COROLLARY 1.15. *For $n=2$ and 3,*

$$\eta_1(t) = \inf_{y_0(x)} \eta_1(t; y_0(x)).$$

Proof. By Theorem 1.9

$$\eta_1(t) = r_{11}(t) \ge \inf_{y_0(x)} r_{11}(t; y_0(x)) = \inf_{y_0(x)} \eta_1(t; y_0(x)),$$

and by Corollary 1.14, $\eta_1(t)\le\inf_{y_0(x)} \eta_1(t; y_0(x))$ and so $\eta_1(t)=\inf_{y_0(x)} \eta_1(t; y_0(x))$ for $n=2$.

In the case $n=3$, Jackson [4] showed that

$$\eta_1(t) = \min\{r_{12}(t), r_{21}(t)\},$$

and so by Theorem 1.9,

$$\begin{aligned}\eta_1(t) &\ge \min\left\{\inf_{y_0(x)} r_{12}(t; y_0(x)), \inf_{y_0(x)} r_{21}(t; y_0(x))\right\}\\ &\ge \inf_{y_0(x)} \eta_1(t; y_0(x)).\end{aligned}$$

Hence again by Corollary 1.14 we have equality.

The first result in the next theorem follows from the work of Erbe [1]. The second result in the next theorem with t replaced by $t+$ follows from work of Erbe [1] also. It is easy to see how to extend Erbe's result to the case given here.

THEOREM 1.16. *If $r_{n-1,1}(t; y_0(x))<r_{n-2,2}(t; y_0(x))$ for each $y_0(x)$ a solution of* (1.1), *then*

$$r_{n-1,1}(t) = \inf_{y_0(x)} r_{n-1,1}(t; y_0(x)).$$

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Also if $r_{1,n-1}(t; y_0(x)) < r_{2,n-2}(t; y_0(x))$ *for each* $y_0(x)$ *a solution of* (1.1), *then*

$$r_{1,n-1}(t) = \inf_{y_0(x)} r_{1,n-1}(t; y_0(x)).$$

Proof. We will prove only the second result in this theorem. We do this by contradiction, so suppose there is a solution $y_0(x)$ of (1.1) such that $r_{1,n-1}(t; y_0(x)) < r_{1,n-1}(t)$. Since we are assuming that $r_{1,n-1}(t; y_0(x)) < r_{2,n-2}(t; y_0(x))$ it follows [5] that $r_{1,n-1}(t; y_0(x))$ is a strictly increasing continuous real valued function in a right hand neighborhood of t. Hence there is an $x_0 > t$, close to t, such that if we let $\tau = r_{1,n-1}(x_0; y_0(x))$, then $\tau \in (r_{1,n-1}(t; y_0(x)), r_{2,n-2}(t; y_0(x)))$ and $\tau < r_{1,n-1}(t)$. It follows that $u_{n-1}(x_0; \tau, y_0(x)) = 0$. Since $r_{2,n-2}(t; y_0(x)) > \tau$ this zero is a simple zero.

Now

$$\frac{y(x; \tau, y_0^1, \ldots, y_0^{n-1}, y_0^n + h) - y(x; \tau, y_0^1, \ldots, y_0^n)}{h}$$

converges uniformly on $[t, r_{1,n-1}(t)]$ to $u_{n-1}(x; \tau, y_0(x))$ as $h \to 0$, and so for $h > 0$ sufficiently small we have that $y(x; \tau, y_0^1, \ldots, y_0^{n-1}, y_0^n + h) - y(x; \tau, y_0^1, \ldots, y_0^n)$ has a zero in (t, τ) as well as a zero at τ of order $n-1$. But this is a contradiction and the proof is complete.

EXAMPLE 1.17. Consider the nonlinear differential equation

$$y^{(n)} = f(x, y)$$

where f satisfies (A), (B), (C), and $f_y(x, y) \le 0$. It is well known that for the differential equation

$$z^{(n)} - f_y(x; y_0(x))z = 0$$

that $r_{n-2,2}(a; y_0(x)) = \infty$. Hence if $r_{n-1,1}(t; y_0(x)) < \infty$, $t \ge a$, for all solutions $y_0(x)$ of (1.1), then we have by Theorem 1.16 that

$$r_{n-1,1}(t) = \inf_{y_0(x)} r_{n-1,1}(t; y_0(x)).$$

Also it is well known that if n is even, then $r_{2,n-2}(t; y_0(x)) = \infty$. Hence if $r_{1,n-1}(t; y_0(x)) < \infty$, $t \ge a$, for all solutions $y_0(x)$ of (1.1), then

$$r_{1,n-1}(t) = \inf_{y_0(x)} r_{1,n-1}(t; y_0(x)).$$

The example given by Erbe [1] to show in the statement that if $r_{1,n-1}(t+; y_0(x)) < r_{2,n-2}(t+; y_0(x))$, then $r_{1,n-1}(t+) = \inf_{y_0(x)} r_{1,n-1}(t+; y_0(x))$ you cannot remove the hypothesis, gives us the corresponding example for the second result in Theorem 1.16.

EXAMPLE 1.18. For the differential equations for which solutions extend to $[a, \infty)$ of the form

$$y''' + 4y' + p(x)y^s = 0$$

where $s>1$ is the quotient of odd integers, $p \in C[a, \infty)$, $p(x)>0$ on $[a, \infty)$, $a>0$, one can show that

$$\begin{aligned}\infty = r_{12}(a) &> \inf_{y_0(x)} r_{12}(a; y_0(x)) = r_{12}(a; 0)\\ &= r_{21}(a; 0) = \inf_{y_0(x)} r_{21}(a; y_0(x)) = r_{21}(a)\\ &= \eta_1(a) = a+\pi.\end{aligned}$$

The following theorem generalizes Lemma 2.1, [1].

THEOREM 1.19. (*Local Uniqueness Theorem*). *If* $\rho_k(t; y_0(x))>b$, *where* $y_0(x)$ *is a solution of* (1.1), $x_0 \in [t, b)$, *and* $y_0^{(i-1)}(x_0)=y_0^i$, $i=1, 2, \ldots, n$, *then there exist positive numbers* p_{i0}, $i=1, \ldots, n-k$, *such that if* $0 \le |p_i| \le p_{i0}$, $i=1, 2, \ldots, n-k$, *with* $\sum_{m=1}^{n-k} |p_i| \ne 0$, *the difference*

$$y(x; x_0, y_0^1, \ldots, y_0^k, y_0^{k+1}+p_1, \ldots, y_0^n+p_{n-k})-y_0(x) \tag{1.5}$$

does not have n zeros on $[x_0, b]$.

Proof. Assume the conclusion is false, then there is a sequence $\{(p_{1j}, \ldots, p_{n-k,j})\}_{j=1}^{\infty}$, $\sum_{m=1}^{n-k} p_{mj}^2 \ne 0$, of points in $n-k$ space such that $\lim_{j\to\infty} p_{mj}=0$, $m=1, \ldots, n-k$, and $y(x; x_0, y_0^1, \ldots, y_0^k, y_0^{k+1}+p_{1j}, \ldots, y_0^n+p_{n-k,j})-y_0(x)$ has at least n zeros on $[x_0, b]$ for each $j \ge 1$. Let $t_j=(\sum_{m=1}^{n-k} p_{mj}^2)^{1/2}$, then $\{(p_{1j}/t_j, \ldots, p_{n-k,j}/t_j)\}_{j=1}^{\infty}$ are points on the unit sphere in $n-k$ space and so there is a subsequence, which for convenience we label the same as above, which converges to a point $(A_1, \ldots, A_{n-k})$ which is on the unit sphere, so that not all the A_i's are zero. Rewrite (1.5) as

$$\begin{aligned}&[y(x; x_0, y_0^1, \ldots, y_0^k, y_0^{k+1}+p_{1j}, \ldots, y_0^n+p_{n-k,j})\\ &\qquad -y(x; x_0, y_0^1, \ldots, y_0^{k+1}, y_0^{k+2}+p_{2j}, \ldots, y_0^n+p_{n-k,j})]\\ &\qquad +\cdots+ [y(x; x_0, y_0^1, \ldots, y_0^{n-1}, y_0^n+p_{n-k,j})-y(x; x_0, y_0^1, \ldots, y_0^n)]\\ &\quad = p_{1j}u_k(x; x_0, y_k(x))+\cdots+p_{n-k,j}u_{n-1}(x; x_0, y_{n-1}(x)) \text{ where } y_m(x),\end{aligned}$$

$m=k, \ldots, n-1$ is the solution of (1.1) with $y_m^{(i-1)}(x_0)=y_0^i$, $i=1, \ldots, m$; $y_m^{(m)}(x_0)=y_0^{m+1}+\bar{p}_{m-k+1,j}$; $y_m^{(i-1)}(x_0)=y_0^i+p_{i-k+1,j}$; $i=m+2, \ldots, n$, where $\bar{p}_{m-k+1,j}$ is between 0 and $p_{m-k+1,j}$; $m=k, \ldots, n-1$. Hence it follows that if we divide (1.5) by t_j, then this goes to

$$A_1u_k(x; x_0, y_0(x))+\cdots+A_{n-k}u_{n-1}(x; x_0, y_0(x)) \tag{1.6}$$

uniformly on $[x_0, b]$ as $j\to\infty$. Therefore equation (1.6) gives us a nontrivial solution of the variational equation (1.3) with n zeros on $[x_0, b]$. Since (Theorem 1, [5]), $r_{i_1\cdots i_m}(t; y_0(x)) \ge \rho_k(t; y_0(x))$ for $i_1 \ge k$ this is a contradiction.

It is easy to see from the proof of Theorem 1.19 that the following more general result is true.

THEOREM 1.20. *If* $\rho_k(t; y_0(x)) > b$ *where* $y_0(x)$ *is a solution of* (1.1), $x_0 \in [t, b)$, *and* $y_0^{(i-1)}(x_0) = y_0^i$, $i = 1, 2, \ldots, n$, *then there exist positive numbers* p_{i0}, $i = 1, 2, \ldots, n-k$, *such that if* $0 \le |p_i - q_i| \le p_{i0}$, $i = 1, 2, \ldots, n-k$, *with* $\sum_{i=1}^{n-k} |p_i - q_i| \ne 0$ *the difference*

$$y(x; x_0, y_0^1, \ldots, y_0^k, y_0^{k+1} + p_1, \ldots, y_0^n + p_{n-k}) - y(x; x_0, y_0^1, \ldots, y_0^k, y_0^{k+1} + q_1, \ldots, y_0^n + q_{n-k})$$

does not have n zeros on $[x_0, b]$.

Questions that remain open involve comparison theorems for boundary value functions of (1.1) and its variational equation (1.3). In particular is the following true.

CONJECTURE 1.21. *If* $r_{k-1\ n-k+1}(t) = \infty$ *then*

$$r_{i_1 \cdots i_m}(t) \ge \inf_{y_0(x)} \rho_k(t; y_0(x)) \tag{1.7}$$

for $i_1 \ge k$, $k = 1, 2, \ldots, n-1$.

Probably Theorem 1.19 could be used to prove this result.

REMARK 1.22. If this conjecture were true, then by Theorem 1.12 if $\rho_k(t; y_0(x)) < r_{k,n-k}(t; y_0(x))$ for all $y_0(x)$ a solution of (1.1), then

$$r_{k1\cdots 1}(t) = \inf_{y_0(x)} r_{k1\cdots 1}(t; y_0(x)) \tag{1.8}$$

$k = 1, 2, \ldots, n-1$. In particular it would follow from (1.7) with $k = 1$ and Corollary 1.14 that in general

$$\eta_1(t) = \inf_{y_0(x)} \eta_1(t; y_0(x)).$$

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