COMMUTATIVE NON-ASSOCIATIVE ALGEBRAS AND IDENTITIES OF DEGREE FOUR

J. MARSHALL OSBORN

1. The main result of this paper is the following.

THEOREM 1. *Let A be a simple, commutative, finite-dimensional algebra containing an idempotent over a field of characteristic* 0, *and let the algebra A' obtained from A by adjoining a unity element satisfy an identity of degree $\leqq 4$ not implied by commutativity. Then either A is a Jordan algebra or A is two-dimensional over an appropriate field E. In the latter case, there exist elements e and g which form a basis of A over E, and which satisfy the relations $e^2 = e$, $eg = -g$, and $g^2 = \alpha e$, where α is some non-zero element of E.*

A first big step toward this result was taken in (**1**), where we proved the following.

THEOREM. *Let A be a commutative (non-associative) algebra with unity element over a field of characteristic not* 2 *or* 3, *and let A satisfy an identity of degree $\leqq 4$ not implied by the commutative law. Then A satisfies at least one of the following three identities*:

$$(x^2x)x = x^2x^2, \tag{1}$$

$$2(yx\cdot x)x + yx^3 = 3(yx^2)x, \tag{2}$$

$$2(y^2x)x + 2(x^2y)y + (yx)(yx) = 2(yx\cdot y)x + 2(yx\cdot x)y + y^2x^2. \tag{3}$$

It is well known that simple finite-dimensional algebras of characteristic 0 satisfying (1) are Jordan algebras (see, for example, **4**, p. 132). It has also been proved that simple finite-dimensional algebras of characteristic 0 satisfying (2) and containing an idempotent are Jordan algebras (**2**; **3**). Then, in order to prove Theorem 1, it suffices to show that the theorem holds with the added hypothesis that A satisfies (3). This paper is devoted to an investigation of algebras satisfying (3), culminating in a proof of Theorem 1 for such algebras.

In addition to the desire to prove Theorem 1, we were motivated to study algebras satisfying (3) because these algebras are noticeably less well behaved in several critical respects than those that have been studied in the literature, and we were interested in knowing what methods would work in such

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a situation. As an example of this behaviour, the linear transformation corresponding to multiplication by an idempotent in an algebra satisfying (3) may have any desired set of characteristic roots and multiplicities (see the first example in § 6). However, the Peirce decomposition still turns out to be a strong tool in the study of these algebras. A general Peirce decomposition for these algebras is studied in § 3, where we prove the following theorem.

THEOREM 2. *Let A be a simple commutative (possibly infinite-dimensional) algebra satisfying* (3) *over a field F of characteristic not* 2 *or* 3, *let A contain an idempotent e, let $B_\lambda = \{x \in A \mid x(R_e - \lambda I)^n = 0$, for some positive integer $n\}$ for each $\lambda \in F$, and suppose that for each $x \in A$ there exists a polynomial $f_x(R_e) \in F[R_e]$ annihilating x. Then either*

(i) $A = B_1 + B_{-1}$ *and* $B_1 = B_{-1}B_{-1}$, *or*

(ii) $A = B_1 + B_0$.

The first alternative of this theorem is pursued in § 4, where we prove that such an algebra (still possibly infinite-dimensional and of characteristic not 2, 3, or 5) is necessarily two-dimensional over an appropriate field and possesses a basis as described in Theorem 1. The second alternative is treated in § 5. For this case we assume finite dimensionality and characteristic 0 in order to be able to use a trace argument to conclude that A is associative. In the final section we give a few examples showing that some of our results cannot be sharpened.

The process of using a Peirce decomposition with multiple roots to prove results seems, inevitably, to involve an induction on the multiplicities of the roots involved. Luckily, it is possible to prove a general result which establishes conditions which are sufficient for such an inductive argument to work, and which are satisfied in almost all the cases that arise in this paper. This lemma on induction in algebras satisfying (3) is contained in § 2, together with some numerical results that are needed for this lemma and for Theorem 2.

2. We now begin a study of the Peirce decomposition for algebras satisfying (3). Let e be an idempotent in a commutative algebra A satisfying (3) over an algebraically closed field of characteristic not 2 or 3, and let $B_\lambda^{(n)} = \{x \in A \mid x(R_e - \lambda I)^n = 0\}$ for each $\lambda \in F$ and each positive integer n. Then

$$B_\lambda = \bigcup_{n=1}^{\infty} B_\lambda^{(n)},$$

and the sets B_λ are linearly independent. We shall assume in this section and most of the next that $A = \sum B_\lambda$. If $x \in B_\lambda$ and $x \neq 0$, we define the degree of x to be the smallest integer n such that $x \in B_\lambda^{(n)}$, and if $x = 0$ we let $\deg x = 0$. For $x \in B_\lambda$ we denote by $x', x'', \ldots, x^{(n)}$ the elements

$$x(R_e - \lambda I), x(R_e - \lambda I)^2, \ldots, x(R_e - \lambda I)^n,$$

respectively.

Next, for any $\lambda, \mu \in F$, let $\lambda \circ \mu$ and $\lambda \circ' \mu$ denote the two roots of the quadratic equation

$$(4) \qquad 2t^2 - (1 + \lambda + \mu)t + (\lambda + \mu + \lambda\mu - \lambda^2 - \mu^2) = 0.$$

Then $\lambda \circ \mu$ and $\lambda \circ' \mu$ are given explicitly as

$$(4)' \qquad \tfrac{1}{4}[1 + \lambda + \mu \pm \sqrt{(9\lambda^2 + 9\mu^2 - 6\lambda\mu - 6\lambda - 6\mu + 1)}].$$

In practice, this notation supplies a convenient shorthand. For example, $\lambda \circ \mu \neq \lambda \circ' \mu$ states that the roots of (4) are unequal, while $\lambda \circ \mu = 1$ states that one of the roots of (4) is equal to 1 without giving any information about the other root. If it is assumed in an argument that one of the roots of (4) satisfies a certain condition, $\lambda \circ \mu$ will be used to denote the root satisfying that condition and $\lambda \circ' \mu$ will be used to denote the other root (distinct from $\lambda \circ \mu$ exactly when the roots of (4) are distinct) throughout that argument.

LEMMA 1. *If* $\lambda \circ \mu \neq \lambda \circ' \mu$, *then* $B_\lambda^{(n)} B_\mu^{(m)} \subset B_{\lambda\circ\mu}^{(n+m-1)} + B_{\lambda\circ'\mu}^{(n+m-1)}$. *If* $\lambda \circ \mu = \lambda \circ' \mu$, *then* $B_\lambda^{(n)} B_\mu^{(m)} \subset B_{\lambda\circ\mu}^{(2n+2m-2)}$.

Proof. We begin by linearizing y in (3) to get

$$(5) \qquad \begin{aligned} 2(yz\cdot x)x + (x^2y)z + (x^2z)y + yx\cdot zx &= \\ (yx\cdot z)x + (zx\cdot y)x + (yx\cdot x)z &+ (zx\cdot x)y + yz\cdot x^2. \end{aligned}$$

Letting $x = e$, $y \in B_\lambda$, and $z \in B_\mu$ in (5), and using the relations $ye = \lambda y + y'$, $ye\cdot e = \lambda^2 y + 2\lambda y' + y''$, $ze = \mu z + z'$, and $ze\cdot e = \mu^2 z + 2\mu z' + z''$, we obtain

$$\begin{aligned} 2(yz)R_e^2 + (\lambda y + y')z + y(\mu z + z') + (\lambda y + y')(\mu z + z') &= \\ [(\lambda y + y')z]R_e + [y(\mu z + z')]R_e + (\lambda^2 y + 2\lambda y' + y'')z& \\ + y(\mu^2 z + 2\mu z' + z'') + (yz)R_e&, \end{aligned}$$

or

$$(6) \qquad \begin{aligned} &(yz)[2R_e^2 - (1 + \lambda + \mu)R_e + (\lambda + \mu + \lambda\mu - \lambda^2 - \mu^2)] \\ &\quad + (y'z)[-R_e + (1 + \mu - 2\lambda)] + (yz')[-R_e + (1 + \lambda - 2\mu)] \\ &\quad + y'z' - y''z - yz'' = 0. \end{aligned}$$

Defining the operator

$$\begin{aligned} T &= 2R_e^2 - (1 + \lambda + \mu)R_e + (\lambda + \mu + \lambda\mu - \lambda^2 - \mu^2)I \\ &= 2[R_e - (\lambda \circ \mu)I][R_e - (\lambda \circ' \mu)I], \end{aligned}$$

we show next that $(yz)T^{m+n-1} = 0$. If $m = n = 1$, then

$$y' = y'' = z' = z'' = 0,$$

and (6) reduces to the desired relation. If $m + n > 2$, we may assume by induction on $m + n$ that the products $y'z$, yz', $y'z'$, $y''z$, and yz'' are all annihilated by T^{m+n-2}. But then applying T^{m+n-2} to (6) gives the desired result.

We have proved that $[B_\lambda^{(n)} B_\mu^{(m)}]T^{n+m-1} = 0$ for all $\lambda, \mu \in F$. If

$$\lambda \circ \mu = \lambda \circ' \mu,$$

then $T = 2[R_e - (\lambda \circ \mu)I]^2$ and the second statement of Lemma 1 follows from the definition of $B_{\lambda\circ\mu}^{(2m+2n-2)}$. If $\lambda \circ \mu \neq \lambda \circ' \mu$ and if X is an indeterminate, then $[X - (\lambda \circ \mu)I]^{m+n-1}$ and $[X - (\lambda \circ' \mu)I]^{m+n-1}$ are relatively prime elements of the principal ideal domain $F[X]$, so that there exist $f(X)$, $g(X) \in F[X]$ satisfying

$$I = [X - (\lambda \circ' \mu)I]^{m+n-1}f(X) + [X - (\lambda \circ \mu)I]^{m+n-1}g(X).$$

Hence, $I = [R_e - (\lambda \circ' \mu)I]^{m+n-1}f(R_e) + [R_e - (\lambda \circ \mu)I]^{m+n-1}g(R_e)$ and

$$B_\lambda^{(n)} B_\mu^{(m)} = [B_\lambda^{(n)} B_\mu^{(m)}][R_e - (\lambda \circ' \mu)I]^{m+n-1}f(R_e) + [B_\lambda^{(n)} B_\mu^{(m)}][R_e - (\lambda \circ \mu)I]^{m+n-1}g(R_e).$$

Since $[B_\lambda^{(n)} B_\mu^{(m)}]T^{m+n-1} = 0$, the first term on the right side of the last equation is annihilated by $[R_e - (\lambda \circ \mu)]^{m+n-1}$ so it is in $B_{\lambda\circ\mu}^{(m+n-1)}$. Similarly, the second term is in $B_{\lambda\circ'\mu}^{(m+n-1)}$ and the first statement of Lemma 1 follows.

Next, we need to establish some relations between the elements λ, μ, $\lambda \circ \mu$, and $\lambda \circ' \mu$ of F.

LEMMA 2. *Let $\lambda, \mu, \nu \in F$. Then the following hold.*

(i) $\lambda \circ \mu = 1$ *if and only if* $\lambda^2 - \lambda\mu + \mu^2 = 1$.

(ii) *If* $\lambda \circ \mu = 1$ *and if neither* λ *nor* μ *is* 1, *then* $(\mu \circ \nu) \circ \lambda = \nu$ *if and only if* $\nu = 1$ *or* $\nu = \lambda$.

(iii) *If* $\lambda \circ \mu \neq \lambda \circ' \mu$, *then* $(\lambda \circ \mu) \circ \nu = (\lambda \circ' \mu) \circ \nu$ *implies that*

$$\nu(\lambda + \mu - 1) = \lambda\mu.$$

(iv) *If* $\lambda \circ \mu = 1$, *then* $\lambda \circ' \mu = \lambda \circ \mu$ *if and only if* $\lambda + \mu = 3$.

Proof. From the definition of $\lambda \circ \mu$, we see that $\lambda \circ \mu = 1$ if and only if $t = 1$ is a root of (4), or

$$0 = 2 - (1 + \lambda + \mu) + (\lambda + \mu + \lambda\mu - \lambda^2 - \mu^2) = 1 + \lambda\mu - \lambda^2 - \mu^2.$$

This establishes the first part of Lemma 2. Since $eB_\mu \subset B_\mu$ for all $\mu \in F$, it follows that $1 \circ \mu = \mu$ for all $\mu \in F$ (i.e., that μ is a root of (6) when $\lambda = 1$), from which it is easy to check that $\nu = 1$ and $\nu = \lambda$ are solutions of

$$(\mu \circ \nu) \circ \lambda = \nu$$

if $\lambda \circ \mu = 1$. Then (ii) will be established if we can prove that there are at most two distinct solutions of $(\mu \circ \nu) \circ \lambda = \nu$.

By definition, $(\mu \circ \nu) \circ \lambda = \nu$ implies that

$$2\nu^2 - (1 + \mu \circ \nu + \lambda)\nu + (\mu \circ \nu + \lambda + (\mu \circ \nu)\lambda - (\mu \circ \nu)^2 - \lambda^2) = 0.$$

Multiplying by -2 and rearranging terms, we get

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$$2(\mu \circ \nu)^2 - (2 + 2\lambda - 2\nu)(\mu \circ \nu) + (2\nu + 2\lambda\nu - 4\nu^2 + 2\lambda^2 - 2\lambda) = 0.$$

But, again by definition, $\mu \circ \nu$ satisfies

$$2(\mu \circ \nu)^2 - (1 + \mu + \nu)(\mu \circ \nu) + (\mu + \nu + \mu\nu - \mu^2 - \nu^2) = 0,$$

and subtracting this from the preceding equation gives

$$-(1 + 2\lambda - \mu - 3\nu)(\mu \circ \nu) + (1 + 2\lambda - \mu - 3\nu)\nu - \mu + \mu^2 + 2\lambda^2 - 2\lambda = 0.$$

Writing $\gamma = 1 + 2\lambda - \mu - 3\nu$ and $\delta = \mu - \mu^2 - 2\lambda^2 + 2\lambda$, and noting that $\mu \circ \nu = \frac{1}{4}[1 + \mu + \nu + \tau]$, where

$$\tau^2 = 9\mu^2 + 9\nu^2 - 6\mu\nu - 6\mu - 6\nu + 1,$$

we may put this equation in the form

$$\tfrac{1}{4}\gamma\tau = -\tfrac{1}{4}\gamma(1 + \mu + \nu) + \gamma\nu - \delta = -\tfrac{1}{4}[\gamma(1 + \mu - 3\nu) + 4\delta].$$

Thus

$$\begin{aligned}\gamma^2\tau^2 &= \gamma^2(9\mu^2 + 9\nu^2 - 6\mu\nu - 6\mu - 6\nu + 1)\\ &= \gamma^2(\mu^2 + 9\nu^2 - 6\mu\nu + 2\mu - 6\nu + 1) + 8\gamma\delta(1 + \mu - 3\nu) + 16\delta^2,\end{aligned}$$

or

$$0 = \gamma^2(-8\mu^2 + 8\mu) + 8\gamma\delta(1 + \mu - 3\nu) + 16\delta^2. \tag{7}$$

Since γ depends linearly on ν and since δ is independent of ν, this equation has degree $\leqq 2$ in ν. The coefficient of ν^2 in (7) is

$$\begin{aligned}9(-8\mu^2 + 8\mu) + 8 \times 9\delta &= 9 \times 16(\mu - \mu^2 - \lambda^2 + \lambda)\\ &= 9 \times 16[-(\lambda\mu - \lambda - \mu + 1) - (\lambda^2 - \lambda\mu + \mu^2 - 1)]\\ &= -9 \times 16(\lambda - 1)(\mu - 1),\end{aligned}$$

using the first part of the lemma. Since λ and μ are assumed to be different from 1, this coefficient does not vanish, so that ν satisfies a quadratic equation in λ and μ. Hence, there can be at most two distinct values of ν, and (ii) is established.

Suppose now that $\phi = (\lambda \circ \mu) \circ \nu = (\lambda \circ' \mu) \circ \nu$. Then ϕ satisfies the two equations

$$\begin{aligned}&2\phi^2 - (1 + \lambda \circ \mu + \nu)\phi\\ &\qquad + (\lambda \circ \mu + \nu + (\lambda \circ \mu)\nu - (\lambda \circ \mu)^2 - \nu^2) = 0,\\ &2\phi^2 - (1 + \lambda \circ' \mu + \nu)\phi\\ &\qquad + (\lambda \circ' \mu + \nu + (\lambda \circ' \mu)\nu - (\lambda \circ' \mu)^2 - \nu^2) = 0.\end{aligned} \tag{8}$$

Subtracting the second from the first gives

$$\begin{aligned}-(\lambda \circ \mu - \lambda \circ' \mu)\phi + (\lambda \circ \mu - \lambda \circ' \mu) + (\lambda \circ \mu - \lambda \circ' \mu)\nu&\\ - [(\lambda \circ \mu)^2 - (\lambda \circ' \mu)^2] = 0,&\end{aligned}$$

and since $\lambda \circ \mu - \lambda \circ' \mu \neq 0$, we may divide out this factor to get

$$\phi = 1 + \nu - (\lambda \circ \mu + \lambda \circ' \mu) = 1 + \nu - \tfrac{1}{2}(1 + \lambda + \mu) = \tfrac{1}{2}(1-\lambda-\mu+2\nu).$$

Multiplying the first equation of (8) by -2 and rearranging, we get

$$(8)' \; 2(\lambda \circ \mu)^2 - (2 + 2\nu - 2\phi)(\lambda \circ \mu) + (-2\nu + 2\nu^2 + 2\phi + 2\nu\phi - 4\phi^2) = 0.$$

Since $2\phi = 1 - \lambda - \mu + 2\nu$ implies $2 + 2\nu - \phi = 1 + \lambda + \mu$, subtracting

$$2(\lambda \circ \mu)^2 - (1 + \lambda + \mu)(\lambda \circ \mu) + (\lambda + \mu + \lambda\mu - \lambda^2 - \mu^2) = 0$$

from $(8)'$ yields

$$(-2\nu + 2\nu^2 + 2\phi + 2\nu\phi - 4\phi^2) - (\lambda + \mu + \lambda\mu - \lambda^2 - \mu^2) = 0.$$

Setting $2\phi = 1 - \lambda - \mu + 2\nu$ in this equation and cancelling terms, we easily obtain

$$0 = 3\lambda\nu + 3\mu\nu - 3\lambda\mu - 3\nu,$$

to complete the proof of (iii).

To prove that last part of Lemma 2, we observe that $\lambda \circ' \mu = \lambda \circ \mu$ is equivalent to

$$\begin{aligned} 0 &= 9\lambda^2 + 9\mu^2 - 6\lambda\mu - 6\lambda - 6\mu + 1 \\ &= (\lambda^2 + \mu^2 + 2\lambda\mu - 6\lambda - 6\mu + 9) + (8\lambda^2 + 8\mu^2 - 8\lambda\mu - 8) \\ &= (\lambda + \mu - 3)^2 + 8(\lambda^2 - \lambda\mu + \mu^2 - 1). \end{aligned}$$

Using (i), this is equivalent to $\lambda + \mu = 3$, as desired.

We turn now to the lemma on induction mentioned in the introduction. In order to be able to state this lemma, we need to develop some terminology. Let $\lambda_1, \ldots, \lambda_r$ be elements of F and let $x_1, \ldots, x_r$ be variables ranging over the sets $B_{\lambda_1}, \ldots, B_{\lambda_r}$, respectively. We shall call an equation E in $e, x_1, \ldots, x_r$ *standard* if it is homogeneous of degree 1 in each x_i, and if it is a linear combination of equations which arise by linearizing (3), substituting some product of one of more elements of the set $\{e, x_1, \ldots, x_r\}$ for each of the four variables, and then possibly multiplying by one or more products of this sort. By an *unsubscripted reduced term* of E, we mean any product of $x_1, \ldots, x_r$ which arises by taking a term of E and deleting all the e's that occur as well as the coefficient. If, in forming a reduced term T in $x_1, \ldots, x_r$, we keep only one of the two possible components that may arise at each stage using Lemma 1 (and indicate which component by a subscript on the pair of parentheses enclosing this part of the term), we shall say that T is a *fully subscripted reduced term*. For example, if $r = 4$, $((x_1x_2)x_3)x_4$ is an unsubscripted term; and if $\rho = \lambda_1 \circ \lambda_2$, $\sigma = \rho \circ \lambda_3$, and $\tau = \sigma \circ \lambda_4$, then $(((x_1x_2)_\rho x_3)_\sigma x_4)_\tau$ is a fully subscripted term. Clearly, each unsubscripted reduced term of E is a sum of a finite number of fully subscripted reduced terms.

The set of all pairs (λ, μ) of elements of F with the property that a product of the form $B_\lambda B_\mu$ arises in the formation of a fully subscripted term T are called the *root pairs* associated with T. For instance, the root pairs associated with the example just above are (λ_1, λ_2), (ρ, λ_3), and (σ, λ_4). A root pair is called *outer* if it corresponds to the last product made in the formation of T, and it is called *inner* if it corresponds to one of the other products. Thus, in our example, the pair (σ, λ_4) is outer and the other two are inner.

By a *fully subscripted term associated with* E, we shall mean any fully subscripted reduced term of E, or any term arising from a fully subscripted reduced term of E by distributing primes on the variables or on the parentheses in any fashion (recall that for $x \in B_\lambda$, $x' = x(R_e - \lambda I)$). Note that E may be expressed uniquely as a linear combination of fully subscripted terms associated with E. If E is expressed in this fashion and if we delete all the terms in this equation which contain one or more primes, the resulting equation will be called the *derived equation* of E and will be denoted by E^*. We are now finally ready to state Lemma 3.

LEMMA 3 (INDUCTION LEMMA). *Let $\nu, \lambda_1, \ldots, \lambda_r \in F$, let $C_1, \ldots, C_r$ be subspaces of $B_{\lambda_1}, \ldots, B_{\lambda_r}$, respectively, such that $eC_i \subset C_i$ for each i, and let D be a subspace of A such that $eD \subset D$. Let $E_1, \ldots, E_s$ be a set of standard equations in the variables $x_1, \ldots, x_r$ assumed to lie in $C_1, \ldots, C_r$, respectively, and suppose that at least one of the following conditions holds.*

(a) *$\lambda \circ \mu \neq \lambda \circ' \mu$ for each root pair (λ, μ) of each fully subscripted term of each E_i.*

(b) *$r = 3$ and, for each fully subscripted term $T = ((x_i x_j)_\rho x_k)_\nu$, where $\rho = \lambda_i \circ \lambda_j$ and $\nu = \rho \circ \lambda_k$ and where T is not in D by the definition of D, the relations $\lambda_i \circ \lambda_j = \lambda_i \circ' \lambda_j$ and $(\lambda_i \circ \lambda_j) \circ \lambda_k = 1 + \lambda_k - 2\lambda_i \circ \lambda_j$ do not both hold.*

Suppose, further, that the components in B_ν of the equations $E^*_1, \ldots, E^*_s$ imply by simultaneous linear solution that the fully subscripted reduced terms of $E_1, \ldots, E_s$ lying in B_ν also lie in D. Then the equations $E_1, \ldots, E_s$ themselves imply that the fully subscripted reduced terms of $E_1, \ldots, E_s$ lying in B_ν also lie in D for each way of choosing $x_1, \ldots, x_r$ in $C_1, \ldots, C_r$.

Proof. Let $G = G(x_1, \ldots, x_r)$ denote the set of all fully subscripted terms associated with $E_1, \ldots, E_s$, lying in B_ν and let G_j denote the set of elements of G that contain exactly j primes, for each non-negative integer j. We shall prove by induction on $d = \sum_{i=1}^{r} \deg x_i$ that the elements of G lie in D, given that the equations $E_1, \ldots, E_s$ hold, and given that the equations $E^*_1, \ldots, E^*_s$ imply that the elements of G_0 lie in D. If $d < r$, then at least one of the x_i's is zero and the result is trivially true. Thus, we may assume inductively that the result holds for smaller values of d. In particular, any element of G which has a prime on at least one of the x_i's (say on x_k) may be assumed to be already in D, since it may be regarded as a term of degree $d - 1$ by thinking of it as a term in the variables $x_1, \ldots, x'_k, \ldots, x_r$ rather than as a term in $x_1, \ldots, x_k, \ldots, x_r$.

Assume now that there exists a term $T \in G$, $T \notin G_0$, which does not lie in D. Among those terms with the largest number of primes not in D, we

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may find one, say T_0, with the property that, if any prime in T_0 is moved from a particular pair of parentheses to one of the factors within that pair of parentheses, or if this process is done twice, the modified term will lie in D. Such a T_0 exists since any term with a prime on one of the x_i's is in D as shown in the last paragraph. Consider now any pair of parentheses in T_0 which has at least one prime on it but no primes occurring inside it, and let the expression inside this pair of parentheses be yz, where y and z are each a fully subscripted product of one or more of the x_i's. Let $\lambda, \mu \in F$ be such that $y \in B_\lambda$ and $z \in B_\mu$.

Suppose first that condition (a) is satisfied. Then $\lambda \circ \mu \neq \lambda \circ' \mu$, and the component in $B_{\lambda\circ\mu}$ of the first term of (6) reduces to

$$\begin{aligned}[(yz)T]_{\lambda\circ\mu} &= 2(yz)_{\lambda\circ\mu}[R_e - (\lambda\circ\mu)I][R_e - (\lambda\circ'\mu)I] \\ &= 2(yz)'_{\lambda\circ\mu}[R_e - (\lambda\circ'\mu)I] \\ &= 2(yz)'_{\lambda\circ\mu}\{(\lambda\circ\mu - \lambda\circ'\mu)I + [R_e - (\lambda\circ\mu)I]\} \\ &= 2(\lambda\circ\mu - \lambda\circ'\mu)(yz)'_{\lambda\circ\mu} + 2(yz)''_{\lambda\circ\mu}.\end{aligned}$$

All other terms of the $B_{\lambda\circ\mu}$ component of (6) except those two just computed will have at least one prime on either y or z. We now multiply x_i's or products of x_i's and apply primes and subscripts in such an order on the $B_{\lambda\circ\mu}$ component of (6) so that the first term of the new equation becomes T_0. All the other terms of this new equation either have more primes than T_0 or have the same number with one or two primes shifted inwards, and hence they are all in D. But then $T_0 \in D$, contrary to hypothesis. This shows that all the terms of G, except possibly those in G_0, are in D. The components in B_ν of the equations $E_1, \ldots, E_s$ now reduce, modulo D, to the components in B_ν of the equations $E^*_1, \ldots, E^*_s$, and, by hypothesis, these equations may be solved linearly to show that the elements of G_0 also lie in D.

Suppose now that condition (b) holds, and suppose that

$$T_0 = ((x_ix_j)_\rho{}^{(m)}x_k)_\nu{}^{(n)}.$$

If $\lambda_i \circ \lambda_j \neq \lambda_i \circ' \lambda_j$, the argument of the preceding case shows that $m = 0$. On the other hand, if $\lambda_i \circ \lambda_j = \lambda_i \circ' \lambda_j$, then the coefficient of $(x_ix_j)'$ vanishes when x_i, x_j are substituted for y, z in (6), while the coefficient of $(x_ix_j)''$ does not vanish. If $m \geqq 2$, we may take the component in B_ρ of this equation, apply the operator $(R_e - \rho I)^{m-2}R_{x_k}(R_e - \nu I)^n$ to it, and conclude, as before, from the component of this in B_ν that T_0 lies in D, contrary to our assumption. Thus, $m = 0$ or 1. Similarly, substituting $(x_ix_j)_\rho{}^{(m)}$ and x_k for y and z in (6) and proceeding as above, shows that $n = 0$ if $\rho \circ \lambda_k \neq \rho \circ' \lambda_k$, and that $n = 0$ or 1 if $\rho \circ \lambda_k = \rho \circ' \lambda_k$.

Let us first consider the case when $m = n = 1$. Then the elements of G_i are in D for $i > 2$, so that, applying $(R_e - \nu I)^2$ to the components of $E_1, \ldots, E_s$ in B_ν, these equations reduce modulo D to the component in B_ν of the equations $E^*_1, \ldots, E^*_s$ with $(R_e - \nu I)^2$ applied. These equations may be solved by

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hypothesis to show that the elements of $G_0(R_e - \nu I)^2$ lie in D. But, setting $y = (x_i x_j)_\rho$ and $z = x_k$ in (6), taking the component in B_ν, and applying $(R_e - \nu I)$, we obtain an equation in which the only term not in D is T_0, which occurs with coefficient

$$-\nu + 1 + \lambda_3 - 2\rho = -(\lambda_1 \circ \lambda_2) \circ \lambda_3 + 1 + \lambda_3 - 2\lambda_1 \circ \lambda_2,$$

which is non-zero, by hypothesis. Thus, the case $m = n = 1$ cannot occur.

Suppose next that $m = 1$ and $n = 0$. Then the elements of G_i are in D for $i > 1$, and we may apply $(R_e - \nu I)$ to the components in B_ν of the equations $E_1, \ldots, E_s$ to get, modulo D, the components in B_ν of $E^*_1, \ldots, E^*_s$ with $(R_e - \nu I)$ applied. Thus, the elements of $G_0(R_e - \nu I)$ are in D. Setting $y = (x_i x_j)_\rho$ and $z = x_k$ in (6) and taking the component in B_ν, we again obtain an equation in which every term lies in D except for T_0, which occurs with the non-zero coefficient $-\nu + 1 + \lambda_3 - 2\rho$. Hence, this case cannot occur either.

If $m = 0$ and $n = 1$, then the elements of G_i are in D for $i > 1$, and we may show that the elements of $G_0(R_e - \nu I)$ lie in D as in the last case. But this contradicts the fact that $T_0 \in G_0(R_e - \nu I)$, showing that this case also cannot occur. Therefore, the elements of G_i are in D for $i > 0$, and the components in B_ν of $E_1, \ldots, E_s$ reduce directly modulo D to the components in B_ν of $E^*_1, \ldots, E^*_s$, showing that the elements of G_0 are in D also.

3. We are now ready to work more specifically toward the proof of Theorem 2. We retain the assumptions that F is algebraically closed of characteristic not 2 or 3, and that $A = \sum B_\lambda$. First, we linearize x in (5), set the new variable equal to e, and let $x \in B_\nu$, $y \in B_\lambda$, and $z \in B_\mu$, to obtain

$$\begin{aligned}2(yz\cdot x)e &+ 2(yz\cdot e)x + 2\nu yx\cdot z + 2yx'\cdot z + 2\nu zx\cdot y + 2zx'\cdot y + \lambda zx\cdot y \\ &+ zx\cdot y' + \mu yx\cdot z + yx\cdot z' = (yx\cdot z)e + \lambda(yz\cdot x) + y'z\cdot x + (zx\cdot y)e \\ &+ \mu zy\cdot x + z'y\cdot x + (yx\cdot e)z + \lambda yx\cdot z + y'x\cdot z + (zx\cdot e)y \\ &+ \mu zx\cdot y + z'x\cdot y + 2\nu yz\cdot x + 2yz\cdot x',\end{aligned}$$

or

$$\begin{aligned}(yz\cdot x)[2R_e - (\lambda + \mu + 2\nu)] + 2(yz\cdot e)x - y'z\cdot x - yz'\cdot x - 2yz\cdot x' &= \\ (yx\cdot z)[R_e + (\lambda - \mu - 2\nu)] + (yx\cdot e)z &+ y'x\cdot z \\ - 2yx'\cdot z - yx\cdot z' + (zx\cdot y)[R_e + (\mu - \lambda - 2\nu)&] \\ + (zx\cdot e)y + z'x\cdot y - 2zx'\cdot y &- zx\cdot y'.\end{aligned} \tag{9}$$

We prepare for Theorem 2 with three lemmas.

LEMMA 4. *Let $x, y \in B_\lambda$ and $z \in B_\mu$, where $\lambda \circ \mu = 1$ and where λ and μ are not* 1, 0, *or* -1. *Then* $[[yz]_1 \cdot x]_\lambda = 0$.

Proof. We shall prove first that $[yz\cdot x]_\lambda = [[yz]_1 x]_\lambda$ and $[yx\cdot z]_\lambda = 0$ under the hypotheses of Lemma 4. The first relation clearly holds unless

$$\lambda \circ \mu \neq \lambda \circ' \mu \quad \text{and} \quad (\lambda \circ \mu) \circ \lambda = (\lambda \circ' \mu) \circ \lambda.$$

But in this case, Lemma 2 gives $\lambda(\lambda + \mu - 1) = \lambda\mu$, or $\lambda(\lambda - 1) = 0$, which is ruled out here by hypothesis. The second relation holds unless

$$(\lambda \circ \lambda) \circ \mu = \lambda,$$

in which case Lemma 2 implies that either $\lambda = 1$ or $\lambda = \mu$. In the latter case, the first part of Lemma 2 shows that

$$1 = \lambda^2 - \lambda\mu + \mu^2 = \lambda^2 - \lambda^2 + \lambda^2 = \lambda^2,$$

so that $\lambda = \pm 1$. Thus the second relation is also valid under the given hypothesis.

The argument at the beginning of the last paragraph also gives the first step of the relation

$$[(yz\cdot e)x]_\lambda = [(yz\cdot e)_1 x]_\lambda = [(yz)_1 e\cdot x]_\lambda = [yz\cdot x]_\lambda + [[yz]_1'\cdot x]_\lambda.$$

Using this relation and the two shown in the last paragraph, we see that

$$(yz\cdot x)_\lambda[2(R_e - \lambda I) - (\lambda + \mu)] + 2[(yz\cdot e)x]_\lambda =$$
$$2(yz\cdot x)'_\lambda - (\lambda + \mu)(yz\cdot x) + 2(yz\cdot x)_\lambda - 2[(yz)'_1 x]_\lambda,$$
$$(zx\cdot y)_\lambda[(R_e - \lambda) + (\mu - 2\lambda)] + [(zx\cdot e)y]_\lambda =$$
$$(zx\cdot y)'_\lambda + (\mu - 2\lambda)(zx\cdot y)_\lambda + (zx\cdot y)_\lambda + [(zx)'_1 y]_\lambda,$$

so that the component in B_λ of (9) is reduced here to

$$[(2 - \lambda - \mu)yz\cdot x + 2(yz\cdot x)' + 2(yz)'\cdot x - y'z\cdot x - yz'\cdot x - 2yz\cdot x']_\lambda =$$
$$[(1 - 2\lambda + \mu)zx\cdot y + (zx\cdot y)' + (zx)'\cdot y + z'x\cdot y - 2zx'\cdot y - zx\cdot y']_\lambda.$$

We will show that this equation and its dual (obtained by switching x and y) imply that $[yz\cdot x]_\lambda = [zx\cdot y]_\lambda = 0$ for all $x, y \in B_\lambda$ and $z \in B_\mu$. Applying the Induction Lemma with $C_1 = C_2 = B_\lambda$, $C_3 = B_\mu$, $D = 0$, $\nu = \lambda$, $x_1 = x$, $x_2 = y$, and $x_3 = z$, it is sufficient to prove that condition (b) holds and that the derived equations

$$\begin{aligned}(2 - \lambda - \mu)[yz\cdot x]_\lambda &= (1 - 2\lambda + \mu)[zx\cdot y]_\lambda,\\ (1 - 2\lambda + \mu)[yz\cdot x]_\lambda &= (2 - \lambda - \mu)[zx\cdot y]_\lambda,\end{aligned} \tag{10}$$

imply that $[yz\cdot x]_\lambda = [zx\cdot y]_\lambda = 0$. But condition (b) does hold here, since the relation $(\lambda \circ \mu) \circ \lambda = 1 + \lambda - 2\lambda \circ \mu$ is equivalent to $\lambda = 1 + \lambda - 2$, which is not valid for any value of λ.

The two equations (10) can have a non-zero solution only if

$$2 - \lambda - \mu = 1 - 2\lambda + \mu$$

or if $2 - \lambda - \mu = -(1 - 2\lambda + \mu)$. But neither of these relations can hold under the hypotheses of Lemma 4, the former because it reduces to $1+\lambda = 2\mu$

which, together with the relation in (i) of Lemma 2, implies $\lambda = 1$, and the latter because it reduces to $\lambda = 1$ directly. Thus, $[yz \cdot x]_\lambda = [zx \cdot y]_\lambda = 0$, and the lemma is proved.

LEMMA 5. *Let $x \in B_\nu$, $y \in B_\lambda$, and $z \in B_\mu$, where $\lambda \circ \mu = 1$ and where $\lambda, \mu \neq 1$ and $\nu \neq 1, \lambda, \mu$. Then $[[yz]_1 x]_\nu = 0$.*

Proof. We shall again use the Induction Lemma, this time with $C_1 = B_\nu$, $C_2 = B_\lambda$, $C_3 = B_\mu$, $D = 0$, $x_1 = x$, $x_2 = y$, and $x_3 = z$. Let E_1 be (9) and let E_2 be the equation obtained from (9) by switching x and z (and hence ν and μ). Since $[yx \cdot z]_\nu = [zx \cdot y]_\nu = 0$ by the second part of Lemma 2, the only fully subscripted reduced terms in B_ν are $[[yz]_1 \cdot x]_\nu$ and $[[yz]_\rho \cdot x]_\nu$, where $\rho = \lambda \circ' \mu$. And since $(\lambda \circ \mu) \circ \nu = \nu \neq \nu - 1 = 1 + \nu - 2\lambda \circ \mu$, condition (b) is satisfied here. Thus, it is sufficient to solve the equations

$$(11) \qquad \begin{aligned} [(2 - \lambda - \mu)[yz]_1 \cdot x + (2\rho - \lambda - \mu)[yz]_\rho \cdot x]_\nu &= 0, \\ [(1 + \lambda - 2\mu)[yz]_1 \cdot x + (\rho + \lambda - 2\mu)[yz]_\rho \cdot x]_\nu &= 0, \end{aligned}$$

if $\rho \neq 1$, and to solve the equations

$$(12) \qquad (2 - \lambda - \mu)[[yz]_1 \cdot x]_\nu = 0, \qquad (1 + \lambda - 2\mu)[[yz]_1 \cdot x]_\nu = 0$$

if $\rho = 1$. Since $1 + \rho = \lambda \circ \mu + \lambda \circ' \mu = \frac{1}{2}(1 + \lambda + \mu)$, we have

$$\rho = \tfrac{1}{2}(\lambda + \mu - 1),$$

so that equations (11) may be reduced to

$$(13) \qquad \begin{aligned} [(2 - \lambda - \mu)[yz]_1 \cdot x - [yz]_\rho \cdot x]_\nu &= 0, \\ [(1 + \lambda - 2\mu)[yz]_1 \cdot x + \tfrac{1}{2}(3\lambda - 3\mu - 1)[yz]_\rho \cdot x]_\nu &= 0. \end{aligned}$$

Suppose first that $\lambda \neq \mu$. Then, subtracting the first of these from twice the second gives

$$[(3\lambda - 3\mu)[yz]_1 \cdot x + (3\lambda - 3\mu)[yz]_\rho \cdot x]_\nu = 0,$$

or

$$(14) \qquad [[yz]_1 \cdot x + [yz]_\rho \cdot x]_\nu = 0.$$

Adding (14) to the first equation of (13) now yields $(3 - \lambda - \mu)[[yz]_1 \cdot x]_\nu = 0$. When $\rho \neq 1$, we have $3 - \lambda - \mu \neq 0$ by Lemma 2, and hence

$$[[yz]_1 \cdot x]_\nu = [[yz]_\rho \cdot x]_\nu = 0.$$

If $\rho = 1$, then $\lambda + \mu = 3$, and the first equation of (12) implies that $[[yz]_1 \cdot x]_\nu = 0$.

On the other hand, if $\lambda = \mu$, then the relation $\lambda^2 - \lambda\mu + \mu^2 = 1$ of Lemma 2 (i) implies that $\lambda = \pm 1$. Since $\lambda \neq 1$ by hypothesis, we have $\lambda = -1 = \mu$, so that $\rho = \frac{1}{2}(\lambda + \mu - 1) = -3/2$ and the first equation of (13) reduces to $[4[yz]_1 \cdot x - [yz]_\rho \cdot x]_\nu = 0$. If $[[yz]_1 \cdot x]_\nu \neq 0$, then $[[yz]_\rho \cdot x]_\nu \neq 0$, in which case Lemma 2 (iii) implies that $\nu(-1 - 1 - 1) = (-1)^2$ or $\nu = -\frac{1}{3}$. Since

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$[[yz]_\rho \cdot x]_\nu \neq 0$, we must have $(-3/2) \circ (-\frac{1}{3}) = -\frac{1}{3}$. But substitution in (4) or (4)′ shows that this last relation is not true. Thus we must have $[[yz]_1 x]_\nu = 0$ in this case also.

LEMMA 6. *If*

$$B = \sum_{\lambda \neq 0,1} B_\lambda,$$

then the set $B + AB$ is an ideal of A.

Proof. It is clearly sufficient to prove that

$$([AB]_1 + [AB]_0)(B_1 + B_0) \subset B + AB.$$

Since A will still satisfy (3) after adjoining a unit element (**1**, Theorem 1), we need only prove that $[AB]_1(B_1 + B_0) \subset B + AB$, and the rest will follow by symmetry. If $\lambda \circ \mu = 1$, we may easily check from the relation $\lambda^2 - \lambda\mu + \mu^2 = 1$ that $\lambda = 1$ implies that $\mu = 0, 1$, and that $\lambda = 0$ implies that $\mu = \pm 1$. Thus, $[AB]_1 = [BB]_1 + [B_0 B_{-1}]_1$.

We prove first that $[AB]_1 B_1 \subset B + AB$ using the Induction Lemma. Let $x \in B_1 = C_1$, $y \in B_\lambda = C_2$, $z \in B_\mu = C_3$, $D = [B_\lambda B_\mu]_1 \subset B + AB$, and let $\lambda \circ \mu = 1$, where $\lambda \neq 1$ and $\mu \neq 0, 1$. This time, the hypothesis $eD \subset D$ is no longer trivial. Since $eB \subset B$, we must show that $e[B_\lambda B_\mu]_1 \subset [B_\lambda B_\mu]_1$, and then $e[B_\lambda B_\mu]_0 \subset [B_\lambda B_\mu]_0$ will follow by symmetry. We prove that

$$[yz]_1^{(n)} \subset [B_\lambda B_\mu]_1$$

by induction on $\deg y + \deg z$. Using the inductive hypothesis, (6) reduces to $[(3 - \lambda - \mu)(yz)' + 2(yz)'']_1 \equiv 0$ modulo terms of lower degree sum in $[B_\lambda B_\mu]_1$. If $[yz]'_1$ is not a sum of terms of lower degree sum in $[B_\lambda B_\mu]_1$, let k be the largest integer such that $[yz]_1^{(k)}$ is not such a sum and apply $(R_e - I)^{(k-1)}$ onto the congruence above to get $(3 - \lambda - \mu)[yz]_1^{(k)} \equiv 0$. If $\lambda + \mu \neq 3$, this gives a contradiction, showing that $e[B_\lambda B_\mu]_1 \subset [B_\lambda B_\mu]_1$.

Next we see from Lemma 2 (iii) that $(\lambda \circ 1) \circ \mu = (\lambda \circ' 1) \circ \mu$ only if $\mu\lambda = \lambda$, which can hold under the hypotheses $\mu \neq 1$ and $\lambda \circ \mu = 1$ only if $\lambda = 0$ and $\mu = -1$. But a direct calculation using (4)′ shows that

$$(0 \circ 1) \circ (-1) \neq (0 \circ' 1) \circ (-1).$$

Thus $[yx \cdot z]_1 = [[yx]_\lambda \cdot z]_1 \in D$, and, similarly, $[zx \cdot y]_1 = [[zx]_\mu \cdot y]_1 \in D$. From this and from $eD \subset D$ it is clear that the component in B_1 of the right side of (9) is contained in D. We also have that $[yz \cdot x]_1 = [[yz]_1 \cdot x]_1$ by Lemma 2, since $1(\lambda + \mu - 1) = \lambda\mu$ is equivalent to $(\lambda - 1)(\mu - 1) = 0$ which is ruled out by hypothesis. Hence, in order to show that $[[yz]_1 \cdot x]_1$ is in D, we need only show that (9) implies $[yz \cdot x]_1 \in D$. For the case $\lambda + \mu \neq 3$ (which implies condition (b) by Lemma 2 (iv)), it is sufficient by the Induction Lemma to prove that the derived equation $(2 - \lambda - \mu)[yz \cdot x]_1 \equiv 0$ implies that $[yz \cdot x]_1 \equiv 0$. But this follows since $\lambda + \mu = 2$ implies $\lambda^2 + 2\lambda\mu + \mu^2 = 4$, and subtracting this from $4\lambda^2 - 4\lambda\mu + 4\mu^2 = 4$ gives $3(\lambda^2 - 2\lambda\mu + \mu^2) = 0$, or $\lambda = \mu = 2 - \lambda$, which contradicts the hypothesis $\lambda \neq 1$. This proves that $[B_\lambda B_\mu]_1 B_1 \subset [B_\lambda B_\mu]_1$ for $\lambda + \mu \neq 3$.

For the case where $\lambda + \mu = 3$, we apparently have to prove

$$[B_\lambda B_\mu]_1^{(n)} \subset [B_\lambda B_\mu]_1 \quad \text{and} \quad [B_\lambda B_\mu]_1 B_1 \subset [B_\lambda B_\mu]_1$$

in the same joint induction. Thus, we cannot use the Induction Lemma itself, although we shall use pieces of the proof. Since

$$(\lambda \circ \mu) \circ 1 \neq 0 = 1 + 1 - 2\lambda \circ \mu,$$

condition (b) is satisfied here. Modulo $B + AB$, equation (9) reduces to

$$\begin{aligned}[(2 - \lambda - \mu)yz\cdot x + 2(yz\cdot x)' + 2(yz)'\cdot x - y'z\cdot x - yz'\cdot x - 2yz\cdot x']_1 &\equiv \\ [(yx\cdot z)' + (zx\cdot y)']_1.\end{aligned} \tag{15}$$

We shall prove that $[[yz]_1^{(m)}\cdot x]_1^{(n)}$, $[yz]^{(l)}$, $[yx\cdot z]_1^{(p)}$, $[zx\cdot y]_1^{(q)}$ are in $D = B + AB$ for all non-negative integers m, n, l, p, q by induction on $d = \deg x + \deg y + \deg z$. Suppose first that for some positive integer k, $[(yx)_\lambda^{(k)}\cdot z]'_1 \notin D$. Letting k be the largest such integer, we substitute x for z in (6), take the component in B_λ, apply $(R_e - \lambda I)^{k-1}R_z(R_e - I)$, and take the component in B_1 to get an equation in which the first term is $[[yx]_\lambda^k\cdot z]'_1$ with coefficient $3\lambda - 2$, and in which the other terms are known to be in D. Since $3\lambda = 2$ is incompatible with the conditions $\lambda + \mu = 3$ and

$$\lambda^2 - \lambda\mu + \mu^2 = 1,$$

this coefficient is non-zero and $[(yx)_\lambda^{(k)}\cdot z]'_1 \in D$. This contradiction shows that $[[yx]'_\lambda\cdot z]'_1 \in D$. We may now substitute $[yx]_\lambda$ for y in (6) and observe that the component in B_1 of each of the terms of this equation except $[[yx]_\lambda\cdot z]''_1$ is known to be in D, so that $[[yx]_\lambda\cdot z]''_1 \in D$ also. Setting $\eta = \lambda \circ' 1$ and noting that $\eta = \frac{1}{2}(1 + 1 + \lambda) - \lambda = 1 - \frac{1}{2}\lambda \neq \lambda$ since $3\lambda \neq 2$, we have $[[yx]_\eta\cdot z]''_1 \in D$ by the first part of the proof, yielding $[yx\cdot z]''_1 \in D$. By symmetry, $[zx\cdot y]''_1 \in D$.

We have shown that the right side of (15) is annihilated modulo D by $(R_e - I)$. The proof of the Induction Lemma using condition (b) now shows that $[[yz]'_1\cdot x]_1 \in D$, since the argument up to this point uses the original equations only after applying $(R_e - I)$ at least once. Since $\eta \neq \lambda$, we have $\deg yx < \deg y + \deg x$ by Lemma 1. Hence, the element $[[yx\cdot z]'_1\cdot e]_1$ is in D, since it has the same form as $[[yz]'_1\cdot x]_1$, and

$$\deg yx + \deg z + \deg e \leqq \deg y + \deg z + \deg x.$$

But then $[[yx\cdot z]'_1\cdot e]_1 = [yx\cdot z]'_1 + [yx\cdot z]''_1 \in D$, or $[yx\cdot z]'_1 \in D$, and by symmetry, $[zx\cdot y]'_1 \in D$. Having eliminated the right side of (15), the rest of the proof of the Induction Lemma may be applied and the derived equation solved as in the last case. Taking $x = e$ in $[yx\cdot z]^{(l)} \in D$ immediately gives $[yz]^{(l)} \in D$. This completes the proof of $[AB]_1B_1 \subset B + AB$ and $e(B + AB) \subset B + AB$.

Next we prove that $[BB]_1B_0 \subset B + AB$. Let $x \in B_0 = C_1$, $y \in B_\lambda = C_2$, $z \in B_\mu = C_3$, $D = B + AB$, where $\lambda \circ \mu = 1$ and $\lambda, \mu \neq 0, 1$. Then $[yx \cdot z]_0 = 0$ by the second part of Lemma 2, and $yz \cdot x \equiv [yz]_1 \cdot x$ modulo D by the third part of Lemma 2. Since $(\lambda \circ \mu) \circ 0 = 1 \circ 0 \neq -1 = 1 + 0 - 2\lambda \circ \mu$ for either the case $1 \circ 0 = 1$ or the case $1 \circ 0 = 0$, condition (b) is satisfied for either the component of $yz \cdot x$ in B_1 or in B_0. For the component in B_1, we take as our equations, equation (9) and the equation arising from (9) by switching x and z (and hence, ν and μ). This gives the derived equations

$$(4 - \lambda - \mu)[yz \cdot x]_1 \equiv 0, \qquad (2 + \lambda - 2\mu)[yz \cdot x]_1 \equiv 0,$$

which may always be solved to give $[yz \cdot x]_1 \equiv 0$, since $\lambda + \mu = 4$ and $-\lambda + 2\mu = 2$ imply $\lambda = \mu = 2$ which violates $\lambda^2 - \lambda\mu + \mu^2 = 1$. For the component in B_0, we just take equation (9) whose derived equation is $(2 - \lambda - \mu)[yz \cdot x]_0 \equiv 0$. This implies $[yz \cdot x]_0 \equiv 0$, since $\lambda + \mu = 2$ implies $\lambda = 1$ as in the first part of the proof.

It remains to prove that $[B_0B_{-1}]B_0 \subset B + AB$. Taking $x, z \in B_0$ and $y \in B_{-1}$, we have condition (b) holding, since $1 = 0 \circ (-1) \neq 0 \circ' (-1) = -1$. Both for the component in B_1 and for the component in B_0, we take as our equations, equation (9) and the equation arising from it by interchanging x and z. The two sets of derived equations are then

$$5[[yz]_1 \cdot x]_1 \equiv [[yx]_1 \cdot z]_1, \qquad [[yz]_1 \cdot x]_1 \equiv 5[[yx]_1 \cdot z]_1,$$

and

$$3[[yz]_1 \cdot x]_0 \equiv [[yx]_1 \cdot z]_0, \qquad [[yz]_1 \cdot x]_0 \equiv 3[[yx]_1 \cdot z]_0$$

modulo $D = B + AB$, giving

$$[[yz]_1 \cdot x]_1 \equiv [[yx]_1 \cdot z]_1 \equiv [[yz]_1 \cdot x]_0 \equiv [[yx]_1 \cdot z]_0 \equiv 0.$$

We are now finally ready to prove Theorem 2. Suppose first that F is algebraically closed (as we have been assuming up to this point). Then the hypothesis that, for each $x \in A$, there exists a polynomial $f_x(R_e) \in F[R_e]$ annihilating x, is just equivalent to the assumption $A = \sum B_\lambda$ that we have been using. Hence, any of the lemmas proved so far may be applied in this case. In particular, by Lemma 6, $B + AB$ is an ideal of A, so that either $B = 0$ or $A = B + AB$, since A is assumed to be simple. If $B = 0$, the second alternative of Theorem 2 holds, so that we may assume that $A = B + AB$.

We shall complete the proof for F algebraically closed by showing that $A = B + AB$ implies (without assuming simplicity) that $A = B_1 + B_{-1}$ and that $B_1 = B_{-1}B_{-1}$. If $A = B + AB$, then there must exist elements $y_1, \ldots, y_n$, $z_1, \ldots, z_n$ such that $e = \sum_{i=1}^{n} [y_i z_i]_1$, where $y_i \in B_{\lambda_i}$, $z_i \in B_{\mu_i}$, $\lambda_i \circ \mu_i = 1$, and $\mu_i \neq 0$ or 1 for $i = 1, \ldots, n$. Since $\lambda = 1$ and $\lambda \circ \mu = 1$ imply $\mu = 0, 1$ by Lemma 2 (i), it follows that $\lambda_i \neq 1$ for $i = 1, \ldots, n$. Then, choosing $x \in B_\nu$ for some $\nu \neq 1, 0, -1$, we see from Lemma 5 that $[[y_i z_i]_1 x]_\nu = 0$ unless $\nu = \lambda_i$ or μ_i, and we see from Lemma 4 that $[[y_i z_i]_1 x]_\nu = 0$ in these cases also unless $\mu_i = -1$ or $\lambda_i = 0, -1$. If either λ_i or μ_i is 0 or -1, it

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follows readily from the relation $\lambda_i^2 - \lambda_i\mu_i - \mu_i^2 = 1$ of Lemma 2 (i) that the other is 1, 0, or -1, contrary to the assumptions that $\nu = \lambda_i$ or μ_i and that $\nu \neq 1, 0, -1$. Thus, $ex = \sum_i[[y_iz_i]_1x]_\nu = 0$ and so $x \in B_0$. Since $x \in B_\nu$ for some $\nu \neq 0$ and since the B_λ's are linearly independent, $x = 0$. Hence $B_\nu = 0$ for $\nu \neq 1, 0, -1$ and we have proved that $A = B_1 + B_0 + B_{-1}$. But then $A = B_{-1} + AB_{-1} = B_{-1} + (B_0 + B_1 + B_{-1})B_{-1} \subset B_{-1} + B_1$. Hence, $B_0 = 0$ and $B_1 = B_{-1}B_{-1}$.

Suppose now that F is not algebraically closed, let $\bar{F}$ be the algebraic closure of F, and let $\bar{A}$ be the scalar extension of A over $\bar{F}$. We also let $B = \{x \in A \mid \exists f_x(R_e) \in F[R_e]$ such that $f_x(1) \neq 0$, $f_x(0) \neq 0$, and $f_x(R_e)$ annihilates $x\}$, and we let $\bar{B}$ be the subspace of $\bar{A}$ spanned by the elements of B. Then $\bar{B}$ is the subspace of $\bar{A}$ called B in Lemma 6, and hence $\bar{B} + \bar{A}\bar{B}$ is an ideal of $\bar{A}$ and $G = (\bar{B} + \bar{A}\bar{B}) \cap A$ is an ideal of A. Since $B + AB \subset G$, the relation $G = 0$ implies that $B = 0$ and hence that the second alternative of Theorem 2 holds. On the other hand, if $A = G$, then $\bar{A} = \bar{G} \subset \bar{B} + \bar{A}\bar{B}$, showing that $\bar{A} = \bar{B}_1 + \bar{B}_{-1}$ and $\bar{B}_1 = \bar{B}_{-1}\bar{B}_{-1}$ by the second paragraph of this proof. But then we must also have $A = B_1 + B_{-1}$ and $B_1 = B_{-1}B_{-1}$.

4. We now suppose that A is a simple algebra satisfying (3) over any field F of characteristic not 2, 3, or 5, and that $A = B_1 + B_{-1}$ and $B_1 = B_{-1}B_{-1}$. This section is devoted to showing that such an algebra is necessarily of the type described in Theorem 1. We first observe that $1 = 1 \circ 1 \neq 1 \circ' 1 = \frac{1}{2}$, $-1 = 1 \circ (-1) \neq 1 \circ' (-1) = 2$, $1 = (-1) \circ (-1) \neq (-1) \circ' (-1) = -3/2$ using (4), so that there are no multiple roots and the relations $B_1B_1 \subset B_1$, $B_1B_{-1} \subset B_{-1}$, and $B_{-1}B_{-1} = B_1$ are valid. The lack of multiple roots implies that condition (a) of the Induction Lemma always holds in this section.

We shall use the convention in this section that lower case letters at the beginning of the alphabet stand for element of B_1 and that lower case letters at the end of the alphabet stand for elements of B_{-1}. Since the product of two B_λ's here is contained in just one B_λ, it will never be necessary to add subscripts to products to indicate which component we are interested in.

We now state, for easy reference, the general equations derived from (3) which we will need in this section:

$$5(yx)' + 2(yx)'' + y'x - (y'x)' + yx' - (yx')' + y'x' - y''x - yx'' = 0, \tag{16}$$

$$-5(yb)' + 2(yb)'' + 5y'b - (y'b)' - yb' - (yb')' + y'b' - y''b - yb'' = 0, \tag{17}$$

$$(bc)' + 2(bc)'' - b'c - (b'c)' - bc' - (bc')' + b'c' - b''c - bc'' = 0, \tag{18}$$

$$2yz\cdot x + (yz\cdot x)' + (yz)'\cdot x - y'z\cdot x - yz\cdot x' = y'x\cdot z - yx'\cdot z + 2zx\cdot y + (zx\cdot y)' + (zx)'\cdot y - zx'\cdot y - zx\cdot y', \tag{19}$$

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$$(20)\quad \begin{aligned} &2yb\cdot x + (yb\cdot x)' + (yb)'\cdot x - y'b\cdot x - yb\cdot x' = \\ &\qquad y'x\cdot b - yx'\cdot b + 2bx\cdot y + (bx\cdot y)' + (bx)'\cdot y - bx'\cdot y - bx'\cdot y', \end{aligned}$$

$$(21)\quad \begin{aligned} &2yx\cdot b + (yx\cdot b)' + (yx)'\cdot b - yx'\cdot b - yx\cdot b' = \\ &\qquad (yb\cdot x)' + (yb)'\cdot x - yb'\cdot x - yb\cdot x' - 2bx\cdot y - b'x\cdot y + bx'\cdot y, \end{aligned}$$

$$(22)\quad \begin{aligned} &-4zb\cdot c + (zb\cdot c)' + (zb)'\cdot c - zb'\cdot c - zb\cdot c' = \\ &\qquad cb'\cdot z - c'b\cdot z - 4cz\cdot b + (cz\cdot b)' + (cz)'\cdot b - c'z\cdot b - cz\cdot b'. \end{aligned}$$

Equations (16)–(18) are just the three special cases of (6) obtained by letting λ and μ range over the values 1 and -1. For (19), we set $\lambda = \mu = \nu = -1$ in (9), subtract from this the equation obtained by interchanging x and y, and divide the result by 3. Each of the equations (20)–(22) arises by taking an appropriate linear combination of two special cases of (9) in a similar manner. It will also be convenient to have the derived equations of (19)–(22) written out explicitly for easy reference. We state these as Lemma 7.

LEMMA 7. *Let $b, c \in B_1$ and $x, y, z \in B_{-1}$. Then there exist standard equations implied by* (3) *whose derived equations are*

$$(23)\qquad yz\cdot x = y\cdot zx,$$

$$(24)\qquad yb\cdot x = y\cdot bx,$$

$$(25)\qquad yx\cdot b = -y\cdot xb,$$

$$(26)\qquad zb\cdot c = zc\cdot b.$$

Armed with this lemma, we can establish most of the results that we need in this section with a minimum of effort. We shall, in fact, simply make use of the equations given in Lemma 7 as if they held in A, with the understanding that each such use stands for an obvious application of the Induction Lemma. The obvious choices of the sets C_i and D will either be trivially invariant under R_e, or this property will follow easily by using (6), since the coefficient of $(yz)'$ will never be zero here. The one thing that we have to watch out for in using the equations of Lemma 7 is that the Induction Lemma requires that *all* reduced terms of our set of equations must be proven to be in D using the reduced equations, not just those that appear with non-zero coefficients in the derived equations. For example, if $zc\cdot b \in D$, it would only be valid to use (26) to conclude that $zb\cdot c \in D$ if we can also show that the remaining reduced term $z\cdot bc$ of the original equation is also in D.

We begin now to work toward proving that A is two-dimensional. Writing $A_\lambda = B_\lambda^{(1)} = \{x \in A \mid xe = \lambda x\}$ for convenience, we prove the following lemma.

LEMMA 8. *If $a \in A_{-1}A_{-1}$, then $aB_{-1} + aB_{-1}\cdot B_{-1}$ is an ideal of A.*

Proof. If $a \in A_{-1}A_{-1}$, there exist $x_1, \ldots, x_n, y_1, \ldots, y_n \in A_{-1}$ such that $a = \sum x_i y_i$. Letting $b \in B_1$ and $z \in B_{-1}$, and letting $D = aB_{-1}$, we may use the equations of Lemma 7 to obtain

$$0 \equiv a \cdot bz = \sum x_i y_i \cdot bz \equiv \sum x_i(y_i \cdot bz) \equiv \sum y_i(x_i \cdot bz) \equiv -\sum y_i(x_i z \cdot b)$$
$$\equiv \sum y_i(x_i b \cdot z) \equiv \sum y_i z \cdot x_i b \equiv \sum (y_i \cdot x_i b) z \equiv \sum (x_i \cdot y_i b) z \equiv -\sum (y_i x_i \cdot b) z$$
$$= -ab \cdot z$$

and

$$\sum x_i(y_i \cdot bz) \equiv -\sum x_i(y_i z \cdot b) \equiv \sum x_i(y_i b \cdot z) \equiv \sum x_i z \cdot y_i b \equiv \sum (x_i z \cdot y_i) b \equiv$$
$$\sum (x_i \cdot z y_i) b \equiv \sum (x_i y_i \cdot z) b = az \cdot b, \text{ modulo } D.$$

Since the equations of Lemma 7 imply that all of the reduced terms are zero, we may conclude that $az \cdot b \in aB_{-1}$, or that $aB_{-1} \cdot B_1 \subset aB_{-1}$.

Next, for $x, z \in B_{-1}$, $b \in B_1$, we have that

$$ax \cdot zb \in aB_{-1} \cdot B_{-1} \quad \text{and} \quad (ax \cdot b)z \in (aB_{-1} \cdot B_1)B_{-1} \subset aB_{-1} \cdot B_{-1}.$$

Hence, $(ax \cdot z)b \in aB_{-1} \cdot B_{-1}$ by (25), showing that $(aB_{-1} \cdot B_{-1})B_1 \subset aB_{-1} \cdot B_{-1}$. And, for $x, y, z \in B_{-1}$, we have $ax \cdot yz \in aB_{-1} \cdot B_1 \subset aB_{-1}$. Then (23) gives $(ax \cdot y)z \equiv ax \cdot yz \equiv 0$ and $(ax \cdot z)y \equiv ax \cdot zy \equiv 0$ modulo aB_{-1}, giving

$$(aB_{-1} \cdot B_{-1})B_{-1} \subset aB_{-1}.$$

Hence $aB_{-1} + aB_{-1}B_{-1}$ is an ideal.

LEMMA 9. *If t is a non-zero element of either A_1 or A_{-1}, then $tA_{-1} \neq 0$.*

Proof. Suppose that G is any subset of B_{-1} such that $eG \subset G$ and that $GA_{-1} = 0$. Then, letting $w \in G$, $x \in B_{-1}$, and $y \in A_{-1}$, we have

$$wx \cdot y = w \cdot xy = wy \cdot x = 0$$

by using (23), showing that $GB_{-1} \cdot A_{-1} = 0$. And if $b \in B_1$, we get

$$wb \cdot x = w \cdot bx = -wx \cdot b = 0$$

using (24) and (25), giving $GB_1 \cdot A_{-1} = 0$. Similarly, if H is a subset of B_1 such that $eH \subset H$ and that $HA_{-1} = 0$, and if $c \in H$, we obtain

$$cx \cdot y = -c \cdot xy = cy \cdot x = 0$$

from (25), or $HB_{-1} \cdot A_{-1} = 0$. Then the set HB_{-1} satisfies the hypotheses of the set G above, so that we may conclude from the relation $GB_{-1} \cdot A_{-1} = 0$ that $(HB_{-1} \cdot B_{-1})A_{-1} = 0$. Hence $(c \cdot xz)y = -(cx \cdot z)y = -(cz \cdot x)y = 0$ for $z \in B_{-1}$ by (25) and (24), giving $HB_1 \cdot A_{-1} = (H \cdot B_{-1}B_{-1})A_{-1} = 0$.

Using the relations that we have just shown for G and H, it follows easily that, if $tA_{-1} = 0$, then the ideal generated by t annihilates A_{-1}. Since $t \neq 0$ and since A is simple, the ideal generated by t must be all of A. Thus, $AA_{-1} = 0$, showing that A_{-1} is an ideal of A. But since $A_{-1} \neq A$, it follows that $A_{-1} = 0$, giving $B_{-1} = 0$ and $B_1 = B_{-1}B_{-1} = 0$. This contradiction proves that $tA_{-1} \neq 0$.

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LEMMA 10. *If $a \in A_{-1}A_{-1}$, then R_a is a one-to-one mapping of A onto itself, and $R_aR_e = R_eR_a$.*

Proof. If a is a non-zero element of $A_{-1}A_{-1}$, then $aB_{-1} \neq 0$ by the last lemma, and hence the ideal $aB_{-1} + aB_{-1}\cdot B_{-1}$ is all of A. Thus, $aB_{-1} = B_{-1}$ and $aB_{-1}\cdot B_{-1} = B_1$. If $x, y \in B_{-1}$, we have that $ax\cdot y \equiv ay\cdot x \equiv -a\cdot yx \equiv 0$ modulo aB_1 by using (24) and (25), giving $B_1 = aB_{-1}\cdot B_{-1} \subset aB_1$ or $aB_1 = B_1$. The two relations $aB_{-1} = B_{-1}$ and $aB_1 = B_1$ show that R_a maps A onto itself.

Next we show that $(ya)' = y'a$ by induction on the degree of $y \in B_{-1}$. Replacing b by a in (17) yields

$$-5(ya)' + 2(ya)'' + 5y'a - (y'a)' - y''a = 0,$$

and assuming that $(za)' = z'a$ for $z \in B_{-1}$ of degree less than the degree of y, this becomes

$$5[-(ya)' + y'a] - 2[-(ya)' + y'a]' = 0.$$

If $-(ya)' + y'a \neq 0$, let k be the largest integer such that $[-(ya)'+y'a]^{(k)} \neq 0$, and apply $(R_e + I)^k$ to this equation to get a contradiction. Thus, $(ya)' = y'a$, giving $yR_aR_e = -(ya) + (ya)' = (-y + y')a = yR_eR_a$. Similarly, $b = a$ in (18) gives

$$(ca)' + 2(ca)'' - c'a - (c'a)' - c''a = 0,$$

which leads, by the same argument, to $(ca)' = c'a$ and $cR_aR_e = cR_eR_a$. Hence, $R_aR_e = R_eR_a$ on all of A.

Let $K = \{t \in A \mid ta = 0\}$ and let $K_1 = K \cap B_1$ and $K_{-1} = K \cap B_{-1}$. Then $K = K_1 + K_{-1}$, since an element of A can only annihilate a if both its components do. Since $R_aR_e = R_eR_a$, it is clear that $eK_1 \subset K_1$ and $eK_{-1} \subset K_{-1}$. If $w \in K_{-1}$ and $y \in B_{-1}$, then (25) gives $w\cdot ay = -wy\cdot a = wa\cdot y = 0$, or $w\cdot B_{-1} = w\cdot aB_{-1} = 0$. But then $K_{-1} \cap A_{-1} = 0$ from Lemma 9, giving $K_{-1} = 0$. If $c \in K_1 \cap A_1$ and $z \in A_{-1}$, and if $a = \sum_{i=1}^{n} x_iy_i$, where $x_1, \ldots, x_n$, $y_1, \ldots, y_n \in A_{-1}$, then the equations of Lemma 7 give

$$cz\cdot a = \sum cz\cdot x_iy_i = \sum (cz\cdot x_i)y_i = \sum (cx_i\cdot z)y_i = \sum (cx_i\cdot y_i)z =$$
$$-\sum (c\cdot x_iy_i)z = -ca\cdot z = 0$$

(note that the Induction Lemma is not really being used this time, since everything has degree 1). Thus $cA_{-1}\cdot a = 0$ or $cA_{-1} \subset K_{-1} = 0$, and Lemma 9 gives $c = 0$ or $K_1 \cap A_1 = 0$. Hence, $K_1 = 0$ and R_a is one-to-one.

LEMMA 11. *$A_{-1}A_{-1}$ is a field.*

Proof. Let $a, b \in A_{-1}A_{-1}$, let $a = \sum_{i=1}^{n} x_iy_i$ for $x_1, \ldots, x_n, y_1, \ldots, y_n \in A_{-1}$, and let $z \in A_{-1}$. Then the relations of Lemma 7 give

$$ab = \sum x_i y_i \cdot b = -\sum x_i \cdot y_i b \in A_{-1}A_{-1}$$

and

$$z \cdot ab = -\sum z(x_i \cdot y_i b) = -\sum zx_i \cdot y_i b = -\sum (zx_i \cdot y_i)b = -\sum (z \cdot x_i y_i)b = -za \cdot b,$$

so that $A_{-1}A_{-1}$ is a subalgebra and $R_{ab} = -R_a R_b$ on A_{-1}. If a and b are nonzero, then R_a and R_b are one-to-one on A_{-1}, so that $R_{ab} = -R_a R_b$ is one-to-one on A_{-1}, and $ab \neq 0$. Thus, $A_{-1}A_{-1}$ has no zero divisors. If $c \in A_{-1}A_{-1}$, then $ca \cdot b = \sum (c \cdot x_i y_i)b = -\sum (cx_i \cdot y_i)b = -\sum cx_i \cdot y_i b = -\sum c(x_i \cdot y_i b) = \sum c(x_i y_i \cdot b) = c \cdot ab$, showing that $A_{-1}A_{-1}$ is associative.

Assume now that $a = \sum x_i y_i \neq 0$ (such elements exist by Lemma 9), and consider the element $f = -\sum x_i (y_i R_a^{-1})$. Since

$$(y_i R_a^{-1})e = (y_i e)R_a^{-1} = -y_i R_a^{-1},$$

we have $y_i R_a^{-1} \in A_{-1}$ and $f \in A_{-1}A_{-1}$. Then

$$fa = -[\sum x_i (y_i R_a^{-1})]a = \sum x_i (y_i R_a^{-1} \cdot a) = \sum x_i y_i = a,$$

giving $bfa = ba$, or $(bf - b)a = 0$, for any $b \in A_{-1}A_{-1}$. In view of the fact that there are no zero divisors in $A_{-1}A_{-1}$, this shows that f is the unity element of $A_{-1}A_{-1}$. Letting $a^* = \sum x_i (y_i R_a^{-2})$, we also have that

$$a^* a = [\sum x_i (y_i R_a^{-2})]a = -\sum x_i (y_i R_a^{-2} \cdot a) = -\sum x_i (y_i R_a^{-1}) = f,$$

showing that a has an inverse in $A_{-1}A_{-1}$. Hence $A_{-1}A_{-1}$ is a field.

LEMMA 12. *If $a \in A_{-1}A_{-1}$ and if $s, t \in A$, then*

$$(st)R_a R_e^{-1} = sR_a R_e^{-1} \cdot t = s \cdot tR_a R_e^{-1}.$$

Proof. This lemma is again proved by induction on the sum of the degrees of the elements involved. However, the Induction Lemma cannot be applied here, since it is set up to prove that appropriate products lie in a given set, whereas the present lemma involves establishing an identity. Here we have to keep account of the terms of lower degree and check that they cancel out using the inductive hypothesis. By linearity, it is sufficient to consider the case when both s and t are in B_{-1}, the case when one is in B_{-1} and the other in B_1, and the case when both are in B_1.

Writing $T = R_a R_e^{-1}$ for simplicity, we begin with the case where $s = y \in B_{-1}$ and $t = x \in B_{-1}$. We shall prove that $(yx)^{(k)}T = (yT \cdot x)^{(k)}$, for each nonnegative integer k, by induction on $\deg y + \deg x$. First, replacing y by yT in (16), and subtracting this equation from (16) with T applied to it, we obtain

$$5(yx)'T - 5(yT \cdot x)' + 2(yx)''T - 2(yT \cdot x)'' = 0 \tag{27}$$

after the lower terms have been cancelled using the induction hypothesis

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and using the fact that $R_e - I$ and $R_e + I$ both commute with T by Lemma 10. The latter fact also gives $(yx)''T = [(yx)'T]'$, and (27) becomes

$$(28) \qquad 5[(yx)'T - (yT\cdot x)'] + 2[(yx)'T - (yT\cdot x)']' = 0.$$

If $(yx)'T - (yT\cdot x)' \neq 0$, we let k be the greatest integer such that

$$[(yx)'T - (yT\cdot x)']^{(k)} \neq 0$$

and we apply $(R_e - I)^{(k)}$ to (28) to get a contradiction. Thus, $(yx)'T = (yT\cdot x)'$, and by applying $(R_e - I)^k$ to this equation we get $(yx)^{(k)}T = (yT\cdot x)^{(k)}$ for any positive integer k.

Next, setting $b = a$ in (21) recalling that $a' = 0$, and using the relations $ya = (ye)R_aR_e^{-1} = -yT + y'T$ and $ca = cT + c'T$ to express a in terms of the operator T, we obtain

$$\begin{aligned} 2(yx)T + 2(yx)'T + (yx)'T + (yx)''T + (yx)'T& \\ + (yx)''T - (yx')T - (yx')'T &= \\ -(yT\cdot x)' + (y'T\cdot x)' - y'T\cdot x + y''T\cdot x + yT\cdot x'& \\ - y'T\cdot x' + 2xT\cdot y - 2x'T\cdot y - x'T\cdot y + x''T\cdot y.& \end{aligned}$$

Pulling T to the outside in each term wherever possible and collecting terms, we get

$$\begin{aligned} 2(yx)T - 2xT\cdot y + 5(yx)'T + 2(yx)''T + (y'x)T - (y'x)'T + (yx')T& \\ -(yx')'T + (y'x')T - (y''x)T - (yx'')T = 0.& \end{aligned}$$

But, subtracting from this (16) with T applied, we obtain

$$2(yx)T - 2xT\cdot y = 0,$$

to establish the lemma for the case $s, t \in B_{-1}$.

We wish to show next that $(zb)T = zT\cdot b = z\cdot bT$ for $z \in B_{-1}$ and $b \in B_1$. Since this relation is linear in b and since $B_1 = B_{-1}B_{-1}$, it is sufficient to show that $(z\cdot yx)^{(k)}T = (zT\cdot yx)^{(k)} = (z\cdot (yx)T)^{(k)}$, for all non-negative integers k, using induction on $\deg z + \deg y + \deg x$. Replacing y by zT and b by yx in (17), and subtracting this from (17) with y and b replaced by z and yx and with T applied, we obtain

$$-5(z\cdot yx)'T + 5(zT\cdot yx)' + 2(z\cdot yx)''T - 2(zT\cdot yx)'' = 0$$

after cancelling the terms of lower degree sum using the inductive hypothesis. (Note that the relation $(yx)' \in \sum B_{-1}{}^{(i)} B_{-1}{}^{(j)}$, where the sum is over all i, j such that $i + j < \deg y + \deg x$, follows easily from (16); thus, the inductive hypothesis implies that $(z\cdot (yx)')T = zT\cdot (yx)'$.) As in the first part of this proof, we get $(z\cdot xy)'T = (zT\cdot xy)'$, which leads to

$$(z\cdot xy)^{(k)}T = (zT \cdot xy)^{(k)}$$

for all positive integers k. An identical argument gives

$$(z\cdot xy)^{(k)}T = (z\cdot (xy)T)^{(k)}$$

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for all positive integers k. Continuing in a manner similar to the first part of the proof, we set $b = a$ and $c = yx$ in (22) to yield

$$\begin{aligned}4zT\cdot yx &- 4z'T\cdot yx - (zT\cdot yx)' + (z'T\cdot yx)' - z'T\cdot yx \\ &+ z''T\cdot yx + zT\cdot(yx)' - z'T\cdot(yx)' = \\ &- (yx)'T\cdot z - (yx)''T\cdot z + 4(yx\cdot z)T - 4(yx\cdot z)'T - (yx\cdot z)'T \\ &\quad + (yx\cdot z)''T - (yx\cdot z')T + (yx\cdot z)''T + ((yx)'\cdot z)T - ((yx)'\cdot z)'T\end{aligned}$$

or

$$\begin{aligned}4(z\cdot yx)T - 4zT\cdot yx - 5(z\cdot yx)'T + 2(z\cdot yx)''T + 5(z'\cdot yx)T \\ - (z'\cdot yx)'T - (z\cdot(yx)')T - (z\cdot(yx)')'T + (z'\cdot(yx)')T \\ - (z''\cdot yx)T - (z\cdot(yx)'')T = 0.\end{aligned}$$

Replacing y by z and b by yx in (17), applying T, and subtracting from the equation just above, we obtain $(z\cdot yx)T = zT\cdot yx$.

Since we are inducting on the sum of the degrees of x, y, and z, anything that we prove for $z\cdot yx$ must also hold for $y\cdot zx$ and $x\cdot yz$. But then we get the same thing from any term on the left side of (19) either by replacing x by xT or by applying T to the whole term. Using the inductive hypothesis and the fact that $(zx)T = zT\cdot x$ has already been proved, we also get the same thing from any term on the right side with a prime in it, either by replacing x by xT or by applying T to the whole terms. Thus the same must be true of the remaining term, or $(z\cdot xT)\cdot y = (zx)T\cdot y = (zx\cdot y)T$. Interchanging y and z gives $(z\cdot yx)T = z\cdot(yx)T$, to complete this case.

It remains to prove that $cbT = cT\cdot b$. Again, it is sufficient to prove that $(yx\cdot b)^{(k)}T = ((yx)T\cdot b)^{(k)}$ for each non-negative integer k. As in the two preceding cases, we may replace b by $(yx)T$ and c by b in (18) and subtract this equation from (18) with b replaced by yx and c by b and with T applied, and the relation so obtained leads immediately to $(yx\cdot b)^{(k)}T = ((yx)T\cdot b)^{(k)}$ for k positive. We now observe that, using the previous two cases of this proof, we obtain the same thing from any term on the right side of (21) by either replacing x by xT or by applying T to the whole term. By the inductive hypothesis, the same is true of every term on the left side containing a prime, and we are left with $(y\cdot xT)\cdot b = (yx)T\cdot b = (yx\cdot b)T$, to complete the proof.

LEMMA 13. *Regarding A as an algebra over the appropriate field, A_1 and A_{-1} are both one-dimensional and $A_1 = A_{-1}A_{-1}$.*

Proof. Let $E = \{R_a R_e^{-1} \mid a \in A_{-1}A_{-1}\}$ and consider the mapping $\phi\colon A_{-1}A_{-1} \to E$ defined by $\phi(a) = R_a R_e^{-1}$. Using Lemma 12 and the fact that R_e is the identity operator on A_1, we have that

$$\begin{aligned}t\phi(a)\phi(b) &= (ta)R_e^{-1}\,R_bR_e^{-1} = [(ta)R_bR_e^{-1}]R_e^{-1} = \\ &\qquad [t(aR_bR_e^{-1})]R_e^{-1} = (t\cdot ab)R_e^{-1} = t\phi(ab)\end{aligned}$$

for $t \in A$ and $a, b \in A_{-1}A_{-1}$. And since $\phi(a + b) = \phi(a) + \phi(b)$ from the definition, it follows that ϕ is a homomorphism. From Lemma 10 we see that ϕ is an isomorphism and that the elements of E act as automorphisms on the additive structure of A. Lemma 11 shows that E is a field and Lemma 12 shows that the elements of E act as scalars on A. Then A is an algebra over E.

If the unity element of $A_{-1}A_{-1}$ is denoted by f, then the unity element of E is $\phi(f) = R_f R_e^{-1} = I$. Hence $0 = (R_f R_e^{-1} - I)R_e = R_f - R_e = R_{f-e}$, and the simplicity of A implies that $f - e = 0$, or $e \in A_{-1}A_{-1}$. We see from this that E contains F in a natural way. Also, if $b \in A_1$, and $e = \sum x_i z_i$ for $x_1, \ldots, x_n, z_1, \ldots, z_n \in A_{-1}$, then $b = eb = -\sum x_i(z_i b) \in A_{-1}A_{-1}$, showing that $A_1 = A_{-1}A_{-1}$ and that every element of A_1 is an E-multiple of e. Letting $x, z \in A_{-1}$ and $x \neq 0$, we have from Lemma 9 the existence of a $y \in A_{-1}$ such that $xy \neq 0$, and the result just established shows that $xy = \alpha e$, where α is a non-zero element of E. Then, $\alpha z = -\alpha e \cdot z = -(xy)z = -x(yz) = -x(\beta e) = \beta x$ for some $\beta \in E$, showing that x and z are E-dependent. Regarding A as an algebra over E, we have proved that A_1 and A_{-1} are one-dimensional.

We can now easily prove the main result of this section, namely, Theorem 3.

THEOREM 3. *Let A be a simple algebra satisfying* (3) *over a field F of characteristic not* 2, 3, *or* 5, *and let A contain an idempotent e such that $A = B_1 + B_{-1}$ and $B_1 = B_{-1}B_{-1}$ with respect to e. Then there exists an extension field E of F such that A is an algebra over E of dimension* 2. *If g is a non-zero element of B_{-1}, then e and g form a basis of A over E and multiplication in A is given by $e^2 = e$, $eg = -g$, and $g^2 = \alpha e$ for some non-zero $\alpha \in E$. Conversely, any two-dimensional algebra with this multiplication table is a simple algebra satisfying* (3).

Proof. Let E be as in Lemma 13 and let g be a non-zero element of A_{-1}. Then $g^2 = \alpha e \neq 0$ since $A_1 = A_{-1}A_{-1}$. If $B_{-1} \neq A_{-1}$, then there must exist an element $h \in B_{-1}$ such that $h' = g$. Setting $y = h$ and $x = g$ in (16) and noting that $(hg)'' = 0$ by Lemma 1, we obtain $5(hg)' + g^2 = 0$, or

$$(hg)' = -\tfrac{1}{5}g^2 = -\tfrac{1}{5}\alpha e;$$

and setting $y = h$ and $z = x = g$ in (19) and applying $(R_e + I)$, we get $2(hg \cdot g)' = (2g^2 \cdot h)' = 2\alpha(eh)' = 2\alpha eg = -2\alpha g$, or $(hg \cdot g)' = -\alpha g$. But then the substitution $b = hg$ and $y = g$ in (17) gives

$$0 = -5(hg \cdot g)' - (hg)' \cdot g = 5\alpha g + \tfrac{1}{5}\alpha eg = \alpha(5 - \tfrac{1}{5})g,$$

which is a contradiction since $\alpha \neq 0$ and $g \neq 0$. Thus, $B_{-1} = A_{-1}$, and also $B_1 = B_{-1}B_{-1} = A_{-1}A_{-1} = A_1$, proving all of the theorem except for the last sentence.

Suppose now that A is a commutative algebra with basis e, g over F such that $e^2 = e$, $eg = -g$ and $g^2 = \alpha e$ for $\alpha \in F$ and $\alpha \neq 0$. In order to prove that A satisfies (3), it is sufficient to linearize (3) and to check that all possible ways of setting these variables equal to e and g gives a valid equation. But since (3) is irreducible (**1**), any way of setting three or four of the variables equal in the linearization of (3) gives an equation which vanishes identically. It is also easy to check that any way of setting two of the variables equal in the linearization and setting the other two equal also just gives us (3) back again. Thus it is sufficient just to check that setting $y = e$ and $x = g$ in (3) gives a valid relation. But this gives $-2\alpha e + 2\alpha e + \alpha e = 2\alpha e - 2\alpha e + \alpha e$, which is valid. It is trivial to check that A is simple when $\alpha \neq 0$.

5. We turn now to the study of simple algebras of the form $A = B_1 + B_0$. The methods that we shall employ here are quite different from the methods we have employed so far. Our approach in this section is more elegant and yields results much more rapidly, but requires finite dimensionality and characteristic zero to work here, and does not work at all except in this case. Conversely, the approach used up to now does not seem strong enough to handle this case. We begin with a more general result:

LEMMA 14. *Let A be an algebra satisfying* (3) *over a field of characteristic not* 2 *or* 3, *and let $G(A)$ be the subspace of A spanned by the elements of the form (x, y, y) for all $x, y \in A$. Then $G(A)$ is an ideal of A.*

Proof. Consider the identity

$$x(y, z, w) + (x, y, z)w = (xy, z, w) - (x, yz, w) + (x, y, zw), \tag{29}$$

which holds in any ring. Setting $z = x$ and $w = y$ in this identity and noting that $(y, x, y) = 0$ in a commutative algebra, we get

$$0 = (xy, x, y) - (x, yx, y) + (x, y, xy). \tag{30}$$

But, modulo $G(A)$, we have $(x, y, xy) \equiv -(x, xy, y) \equiv (xy, x, y)$, and using this in (30) gives $0 \equiv 3(xy, x, y) \equiv -3(x, yx, y)$, or

$$(yx \cdot x)y \equiv (yx)(yx) \equiv (yx \cdot y)x.$$

Noting also that $(x^2y)y \equiv x^2y^2 \equiv (y^2x)x$, we find that (3) reduces to

$$3(y^2x)x \equiv 3(yx \cdot y)x, \quad \text{or} \quad (y, y, x)x \equiv 0, \quad \text{modulo } G(A).$$

Linearizing x in this relation and using $(y, y, x) = -(x, y, y)$, we obtain

$$x(y, y, w) - (x, y, y)w \equiv 0. \tag{31}$$

Equation (29) also gives

$$\begin{aligned} x(y, y, w) + (x, y, y)w &= (xy, y, w) - (x, y^2, w) + (x, y, yw) \\ &\equiv -(y, xy, w) + (y^2, x, w) - (y, x, yw) \\ &\equiv (y^2, x, w) - (y, yx, w) + (y, y, xw) - (y, x, wy) \\ &= y(y, x, w) + (y, y, x)w - (y, x, wy) \end{aligned}$$

or

$$x(y, y, w) + 2(x, y, y)w \equiv y(y, x, w) - (y, x, wy).$$

Interchanging left and right in this equation and switching x and w gives

$$2x(y, y, w) + (x, y, y)w \equiv (x, w, y)y - (yx, w, y),$$

and adding the last two equations yields

$$3x(y, y, w) + 3(x, y, y)w \equiv y(x, w, y) + (y, x, w)y - (yx, w, y) - (y, x, wy).$$

But the right side of this equation is equal to $-(y, xw, y) = 0$ by (29), showing that

$$x(y, y, w) + (x, y, y)w \equiv 0.$$

Subtracting (31) from this, we have that $2(x, y, y)w \equiv 0$, or $G(A) \cdot A \subset G(A)$, as desired. With this preparation, we can now prove the following theorem.

THEOREM 4. *Let A be a simple finite-dimensional algebra satisfying* (3) *over a field of characteristic* 0, *and let $A = B_1 + B_0$. Then A is a field.*

Proof. $G(A)$ is an ideal of A by Lemma 14, so that either $G(A) = 0$ or $G(A) = A$. If the latter possibility holds, then A has a basis of elements of the form (y, x, x). Now we may write (5) in operator form as

$$R_{(y,x,x)} = [R_x, R_{yx}] + [R_{x^2}, R_y] + [R_y, R_x{}^2] + [R_yR_x, R_x],$$

and the trace of the right side is zero, implying that the trace of right multiplication by any basis element is zero. By linearity, the trace of any right multiplication is zero and, in particular, the trace of R_e is zero.

On the other hand let, us select a basis for A by choosing first a basis for $A_1 = B_1{}^{(1)}$, augmenting this to a basis of $B_1{}^{(2)}$, augmenting this to a basis of $B_1{}^{(3)}$, and so on until we have a basis of B_1. We then build up a basis of B_0 in the same fashion and add it on to get a basis of A. In this basis, R_e is a triangular matrix having 1 on the diagonal for each basis vector in B_1, and 0 on the diagonal for each basis vector in B_0. Hence, the trace of R_e is the dimension of B_1.

The incompatibility of these two conclusions about the trace of R_e shows that $G(A) \neq A$, or that $G(A) = 0$. Then $(y, x, x) = 0$ for all $x, y \in A$, and A is a commutative alternative algebra. But a simple commutative alternative algebra of characteristic not 3 is known to be associative, and the theorem follows from associative theory.

6. We end with four examples illustrating a few of the ways in which algebras satisfying (3) can behave differently from the classes of algebras that have been studied before.

Example 1. Let V be a vector space (possibly infinite-dimensional) over any field F, and let T be any linear transformation from V into V. Consider the vector space direct sum $A = eF + V$ made into an algebra by defining $e^2 = e$, $ve = vT$ and $uv = 0$ for all $u, v \in V$. We claim that A satisfies (3). Since V is an ideal of A whose square is zero, any way of picking more than one of the variables in the linearization of (3) to be in V will make every term vanish, so that we need only consider those substitutions into the linearization of (3) in which at least three of the variables are set equal to e. But any equation arising from the linearization of (3) by setting three variables equal must vanish identically since (3) is irreducible, so that A satisfies (3) automatically. This example shows that R_e can be completely arbitrarily given, aside from the obvious condition that it must send e into itself.

Example 2. Let A have a basis e, y, z, w over F and let $e^2 = e$, $ey = \lambda y$, $ez = \mu z$, $ew = (\lambda \circ \mu)w$, $yz = w$, and $y^2 = z^2 = yw = zw = w^2 = 0$, where λ, μ, $(\lambda \circ \mu) \in F$ and where $\lambda \circ \mu$ is either of the roots of (4). As we have remarked before, any way of setting more than two of the variables equal to e gives an equation which vanishes identically. On the other hand, any product of either three of four elements chosen from the set $\{y, z, w\}$ will give zero, and the same is true of the product of any two of them unless those two are y and z. Thus, the only cases that need to be checked are when two of the variables in the linearization of (3) are set equal to e, and the other two are set equal to y and z, respectively. It can be easily verified that any way of setting two variables equal in the linearization of (3) always yields (5). Hence, it is sufficient to check that (6) is satisfied with e, y, z as in this example. But this follows trivially from the way that our example is constructed.

This example shows that the relation $B_\lambda B_\mu \subset B_{\lambda\circ\mu} + B_{\lambda\circ'\mu}$ cannot be strengthened in general. By modifying this example, it is easy to show that yz can have non-zero components in both $B_{\lambda\circ\mu}$ and $B_{\lambda\circ'\mu}$. This example can also be built up to show that the information given in Lemma 1 relating the degrees of y, z, and yz is best possible.

Example 3. Let A have a basis s, t, where $s^2 = s + t$, $st = \frac{1}{2}t$, and $t^2 = 0$. This example again satisfies (3), since any way of setting the variables in the linearization of (3) equal to s and t either has three variables equal, or has two t's and every term vanishes. If the element $\alpha s + \beta t$ is idempotent, then $(\alpha s + \beta t)^2 = \alpha^2 s + \alpha^2 t + \alpha\beta t = \alpha s + \beta t$, giving the relations $\alpha^2 = \alpha$ and $\alpha^2 + \alpha\beta = \beta$, which have only the solution $\alpha = \beta = 0$. Thus, A contains no idempotents. However, the homomorphic image of A whose kernel is the ideal spanned by t does have an idempotent. This shows that one cannot expect to lift idempotents in algebras satisfying (3). If a unity element is adjoined to A and if the new algebra is called A^*, we observe that A^* has a unity element which cannot be expressed as a sum of two idempotents, but that the unity element can be so decomposed in a homomorphic image. The examples A and A^* also satisfy (2) and are mentioned in (**3**).

Example 4. Let A have a basis e_1, e_2, f, g, where $e_1^2 = e_1$, $e_2^2 = e_2$, $e_1e_2 = e_1g = g^2 = 0$, $e_1f = g$, $e_2f = f - g$, $e_2g = g$, $f^2 = f$, and $fg = \frac{1}{2}g$. Denoting the subspace spanned by e_1, f, g by B, we observe that B is a subalgebra of A and that A is the result of adjoining the unity element $e_1 + e_2$ to the algebra B. Then A will satisfy (3) if we can show that (3) is satisfied by B (**1**, Theorem 1). As in the previous examples, the linearization of (3) vanishes if three or more of the variables are set equal, and since the multiples of g form an ideal that squares to zero, this equation will also vanish if two variables are set equal to g.

If two of the variables in the linearization of (3) are set equal to e, then the equation reduces to (6). Since $f, g \in A_0(e_1)$, setting $y = z = f$ in (6) yields $-g + \frac{1}{2}g + \frac{1}{2}g = 0$, and setting $y = f$ and $z = g$ in (6) yields $0 = 0$.

The only remaining case to be considered is when two of the variables in the linearization of (3) are set equal to f, and the remaining two variables are replaced by e and g, respectively. But taking $x = f$, $y = g$, and $z = e$ in (5), we may check that all the terms vanish identically. Therefore, A satisfies (3).

Decomposing A with respect to e_1, we see that R_{e_1} has characteristic roots 0 and 1, and that 0 is a multiple root. Furthermore, $B_0(e_1)$ contains an idempotent f which is not orthogonal to e_1. It is easy to check that e_2 is not the sum of two orthogonal idempotents, despite the fact that $B_1(e_2) = B_0(e_1)$ contains idempotents not equal to e_2. Writing the unity element of A as the sum of the two primitive idempotents f and $h = e_1 + e_2 - f$, we see that R_f and R_h have the characteristic roots 0, $\frac{1}{2}$, and 1, and that these are all simple roots. Since $fg = \frac{1}{2}g$ and $f(e_1 - 2g) = 0$, we have

$$e_1 = (e_1 - 2g) + 2g \in A_0(f) + A_{1/2}(f),$$

although $e_1 - 2g$ is not idempotent itself.

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University of Wisconsin,
Madison, Wisconsin;
Yale University,
New Haven, Connecticut