$$\int_{r=0}^{r=a}\int_{\phi=0}^{\phi=\pi}\int_{z=0}^{z=2r\sin\phi} r\,dz\,d\phi\,dr = \int_{r=0}^{r=a}\int_{\phi=0}^{\phi=\pi}\left[rz\right]_0^{2r\sin\phi} d\phi\,dr$$
$$= \int_{r=0}^{r=a}\int_{\phi=0}^{\phi=\pi} 2r^2 \sin\phi\,d\phi\,dr$$
$$= \int_{r=0}^{r=a}\left[-2r^2\cos\phi\right]_0^{\pi} dr$$
$$= \int_{r=0}^{r=a} 4r^2\,dr$$
$$= \left[\frac{4}{3}r^3\right]_0^{\pi}$$
$$= \frac{4}{3}a^3.$$

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107.04 A geometric telescope revisited

In 2002, Thomas Walker [1] introduced his nice and interesting result with following words,

"The two most basic series whose sums can be computed explicitly (geometric series, telescoping series) combine forces to demonstrate the assuming fact that

$$\sum_{m=2}^{\infty}\left(\zeta(m) - 1\right) = 1 \qquad (1)$$

where $\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$ is the Riemann zeta function."

It seems to be an interesting problem to investigate and extend the result where m runs over the terms of any AP:

$$a, a + d, a + 2d, \dots$$

In this direction we state and prove the main result of this short Note.

Theorem: For $k = 0, 1, 2, \dots$ and any two real numbers a, d (with $a + kd > 1$) the following holds,

$$\sum_{k=0}^{\infty} \left(\zeta(a + kd) - 1\right) = \sum_{n=2}^{\infty} \frac{1}{n^{a-d}} \cdot \left(\frac{1}{n^d - 1}\right) \tag{2}$$

Proof: To establish the above result, we directly follow the table used in [2] with slight modification as follows. Here we consider arrangement of numbers of the form $\dfrac{1}{n^{a+kd}}$, where $k = 0, 1, 2, \dots, n \geqslant 2$.

	$\zeta(a) - 1$	$\zeta(a + d) - 1$	$\zeta(a + 2d) - 1$	$\dots$		
2	$\frac{1}{2^a}$	$\frac{1}{2^{a+d}}$	$\frac{1}{2^{a+2d}}$	$\dots$	$= \frac{\frac{1}{2^a}}{1 - \frac{1}{2^d}}$	$= \frac{1}{2^{a-d}}\left(\frac{1}{2^d - 1}\right)$
3	$\frac{1}{3^a}$	$\frac{1}{3^{a+d}}$	$\frac{1}{3^{a+2d}}$	$\dots$	$= \frac{\frac{1}{3^a}}{1 - \frac{1}{3^d}}$	$= \frac{1}{3^{a-d}}\left(\frac{1}{3^d - 1}\right)$
4	$\frac{1}{4^a}$	$\frac{1}{4^{a+d}}$	$\frac{1}{4^{a+2d}}$	$\dots$	$= \frac{\frac{1}{4^a}}{1 - \frac{1}{4^d}}$	$= \frac{1}{4^{a-d}}\left(\frac{1}{4^d - 1}\right)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$= \dots \dots$	$= \dots \dots \dots \dots$

$$\vdots$$

$$\sum_{k=0}^{\infty} \left(\zeta(a + kd) - 1\right) = \sum_{n=2}^{\infty} \frac{1}{n^{a-d}} \left(\frac{1}{n^d - 1}\right)$$

Example: For $a = 3$, $d = 2$, we get from (2)

$$\sum_{k=0}^{\infty} \left(\zeta(3 + 2k) - 1\right) = \sum_{n=2}^{\infty} \frac{1}{n} \left(\frac{1}{n^2 - 1}\right) = \frac{1}{4}$$

which is precisely the result for the odd numbers.

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107.05 The final solution of a quasi-palindromic

Introduction

We consider the eighth roots of unity. They are obtained by solving the octic $x^8 - 1 = 0$. Also the primitive eighth roots of unity are obtained by solving the eighth cyclotomic $x^4 + 1 = 0$. Let us solve this cyclotomic. From $x^4 = -1$, we have $x^2 = \pm i$. Hence the four roots of the cyclotomic are $x = \pm\sqrt{i}, \pm\sqrt{-i}$. However, the radicals in these roots include the imaginary unit. In general, roots with the imaginary unit included in radicals would be inferior to ones without it. As the latter types of roots of the cyclotomic, $x = \frac{1 \pm i}{\sqrt{2}}, \frac{-1 \pm i}{\sqrt{2}}$ are well known. Comparing these two types of roots, we notice that the latter types are in the form of $u + vi$ where u, v are real. We thus introduce the concept of *final roots*. These are roots which are in the form of $u + vi$ where u, v are real. The process of finding these final roots is called a *final solution*.

The main result of this Note is the following. The final roots of the quartic

$$x^4 + Bx^3 + Cx^2 + Dx + E = 0$$

where B, C, D, E are real with $D^2 = B^2E$, $B \neq 0$ and $E \neq 0$ (the quartic with $D^2 = B^2E$ and $E \neq 0$ is called a quasi-palindromic) are as follows:

(i) If $B^2 - 4C + \frac{8D}{B} \geqslant 0$, then

$$x = \frac{1}{2}\left(-p_k \pm \sqrt{p_k^2 - \frac{4D}{B}}\right) \qquad (k = 1,\ 2)$$

where $p_1 = \frac{1}{2}\left(B + \sqrt{B^2 - 4C + \frac{8D}{B}}\right)$, $p_2 = \frac{1}{2}\left(B - \sqrt{B^2 - 4C + \frac{8D}{B}}\right)$.