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MARKOV PROCESSES ASSOCIATED WITH POSITIVITY PRESERVING COERCIVE FORMS

ZHI-MING MA AND MICHAEL RÖCKNER

ABSTRACT. Coercive closed forms on L^2-spaces are studied whose associated L^2-semigroups are positivity preserving. Earlier work by other authors is extended by further developing the potential theory of such forms and completed by proving an analytic characterization of those of these forms which have a probabilistic counterpart, *i.e.*, are associated with (special standard) Markov processes. Examples with finite and infinite dimensional state spaces are discussed.

0. **Introduction.** In previous work (*cf. e.g.* [MR 92], [AMR 93a]) we studied Dirichlet forms $(\mathcal{E}, D(\mathcal{E}))$ on measure spaces, *i.e.*, coercive closed (positive definite bilinear) forms on $L^2(E;m)$ whose associated L^2-semigroups $(T_t)_{t>0}$ as well as its adjoint $(\hat{T}_t)_{t>0}$ are sub-Markovian (*i.e.*, $0 \leq u \leq 1 \Rightarrow 0 \leq \overset{(\wedge)}{T_t} u \leq 1$, $t > 0$). Here E is equipped with a σ-algebra $\mathcal{B}$ and m is a (σ-finite) positive measure on $(E, \mathcal{B})$. In case E is a Hausdorff topological space, extending fundamental results of M. Fukushima, M. L. Silverstein, S. Carrillo Menendez, and Y. Le Jan (*cf.* [F71, 80], [Si 74], [Ca-Me 75], [Le 77]) we proved an analytic characterization of all Dirichlet forms $(\mathcal{E}, D(\mathcal{E}))$ which are associated with a pair of nice (*i.e.*, special standard) Markov processes $(\mathbf{M}, \hat{\mathbf{M}})$, *i.e.*, the transition probabilities of $\mathbf{M}$, $\hat{\mathbf{M}}$ are given as ("quasi-continuous") m-versions of $(T_t)_{t>0}$ resp. $(\hat{T}_t)_{t>0}$. We called such Dirichlet forms *quasi-regular*.

The present paper is devoted to coercive closed forms $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E;m)$ whose associated L^2-semigroups $(T_t)_{t>0}$ (and hence $(\hat{T}_t)_{t>0}$) are merely positivity preserving (*i.e.*, $u \geq 0 \Rightarrow T_t u \geq 0$, $t > 0$). We call such forms also *positivity preserving*. They have been studied intensively already many years ago by several authors as J. Bliedtner [Bl 71], A. Ancona [An72a, b, 75], C. Preston [Pr 74] and even earlier by A. Beurling and J. Deny [BeDe 58] (see also [ReS 78, Theorem XIII 50]). The intention of this paper is to extend and complete the work of these authors by developing the corresponding probabilistic counterpart. To our knowledge this has not been done so far for coercive closed forms which are merely positivity preserving and not necessarily Dirichlet forms. Our main result (Theorem 5.2) states that also in this more general case (a slight modification of) quasi-regularity characterizes all positivity preserving coercive forms which are associated (in a certain sense) with a nice (*i.e*, special standard) Markov process on E,

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provided E is a Hausdorff topological space. In particular, this yields a probabilistic interpretation and representation for both corresponding L^2-semigroups $(T_t)_{t>0}$ and $(\hat{T}_t)_{t>0}$. The technique we use to achieve this is well-known in potential theory, the so-called *h-transformation*, by which we can reduce this situation to the case of a *semi-Dirichlet form* for which the said characterization has been proved recently in [MOR 93] (which extends the "Dirichlet form case" in [MR 92], [AMR 93a]). On the way we have to further develop the potential theory of positivity preserving coercive forms (beyond what has been done in [BeDe 58], [Bl 71], [An 72a, b, 75], [Pr 74]). However, much of the potential theory for (semi-) Dirichlet forms in [MR 92] and [MOR 93] extends almost word for word to this more general case. We only describe the necessary modifications. We also would like to draw the reader's attention to Theorem 1.7 which is an extension of the first Beurling-Deny criterion to the non-symmetric case without even requiring the (weak) sector condition (*cf.* [MR 92, I. (2.5)]).

As in [MR 92], [AMR 93a], [MOR 93] also in this paper we strictly distinguish the parts of the theory that can be developed in a purely measure theoretic framework from those for which a topology on the state space E is needed. In the latter case again we keep our topological assumptions as minimal as possible. We only assume E to be a Hausdorff topological space and (for simplicity) that its Borel σ-algebra $\mathcal{B} := \mathcal{B}(E)$ is generated by the continuous functions on E. In particular, we do not need local compactness.

The idea of using the h-transformation as indicated above is, of course, quite straightforward. Its implementation, however, in this general framework in order to obtain our main result, Theorem 5.2, bears some surprising complications. (One has to identify the precise role which the constant function 1 plays for semi-Dirichlet forms.)

Another central point of this work is to present further examples which are now covered by this more general case. The results can roughly be summarized as follows: any perturbation of a Dirichlet form which is still coercive and closable is a positivity preserving coercive form. This holds for example both with finite and infinite dimensional state spaces E. In case E is an open subset of $\mathbb{R}^d$ we can even allow the (symmetric) diffusion part, to be degenerate (*cf.* Subsection 2.1 below). The quasi-regularity is trivial here. In infinite dimensions we consider the cases where E is a separable Banach space and where $E := \mathcal{M}_1(S)$ is the space of probability measures on a Polish space S. One generic difference of these two cases is that in the latter case the "gradient-type" coercive forms are defined in terms of a "tangent bundle" (in a loose sense) over the "manifold" $E = \mathcal{M}_1(S)$ where the tangent space changes from point to point (*cf.* Subsection 2.3 below for details). In contrast to this in the first case the "tangent space" to the Banach space E at each point z is given by a fixed Hilbert space where only its inner product depends on $z \in E$ (*cf.* Subsection 2.2 below). Nevertheless, arguments similar to the finite dimensional case show that they are positivity preserving coercive forms. We also show that they are quasi-regular (*cf.* Example 5.4), hence by Theorem 5.2 that they give rise to associated infinite dimensional Markov processes (which are in fact diffusions, *cf.* [MR 92, V. Section 1] and [AMR 93b]). For more details about the resulting measure-valued diffusions in the last example in Subsection 2.3 we refer to [ORS 93].

We also would like to mention recent work by B. Schmuland [S 93] where the local property of positivity preserving forms is studied. As in the case of (semi-) Dirichlet forms this property is related to the continuity of the sample paths of the associated special standard Markov process.

Finally, we give a brief overview of the organization of this paper. In Section 1 we describe our framework and give the necessary definitions. Section 2 is devoted to the above mentioned examples. Section 3 is on the h-transformation and the potential theory of positivity preserving coercive forms. In Section 4 we introduce and study the notion of quasi-regularity. In Section 5 we prove our main characterization theorem and describe applications.

1. Positivity preserving coercive forms and positivity preserving semigroups. Let $(E, \mathcal{B}, m)$ be a measure space. The inner product and the norm of $L^2(E;m) := L^2(E, \mathcal{B}, m)$ are denoted by $(\cdot,\cdot)$ and $\|\cdot\|$ respectively. For a real valued function u on E, we write $u^+ := u \vee 0$, $u^- := (-u) \vee 0$. Let $L^2_+(E;m) := \{u \in L^2(E;m) \mid u \geq 0$ m-a.e.$\}$. The domain of a linear operator T on $L^2(E;m)$ is always denoted by $D(T)$. We recall that a bilinear form $\mathcal{E}$ with dense domain $D(\mathcal{E}) \subset L^2(E;m)$ is called a *coercive closed form* on $L^2(E;m)$ if conditions (i) and (ii) below hold.

(i) $\big(\tilde{\mathcal{E}}, D(\mathcal{E})\big)$ is positive definite and closed on $L^2(E;m)$.

(ii) (Weak sector condition). For one (hence all) $\alpha \in]0,\infty[$ there exists a constant $K_\alpha > 0$ such that $|\mathcal{E}_\alpha(u,v)| \leq K_\alpha \mathcal{E}_\alpha(u,u)^{1/2}\mathcal{E}_\alpha(v,v)^{1/2}$ for all $u, v \in D(\mathcal{E})$. (K_α is called *continuity constant*).

For a coercive closed form $\big(\mathcal{E}, D(\mathcal{E})\big)$ on $L^2(E;m)$, we always denote the corresponding semigroup, resolvent and generator by $(T_t)_{t>0}$, $(G_\alpha)_{\alpha>0}$, and L respectively. More precisely, if L is the generator of $(T_t)_{t>0}$, then $D(L) \subset D(\mathcal{E})$ and

$$\mathcal{E}(u,v) = (-Lu, v) \quad \text{for all } u \in D(L),\ v \in D(\mathcal{E}). \tag{1.1}$$

It is known that there is a one-to-one correspondence between $\big(\mathcal{E}, D(\mathcal{E})\big)$, $(T_t)_{t>0}$, $(G_\alpha)_{\alpha>0}$ and L. For a detailed discussion of the correspondences among $\big(\mathcal{E}, D(\mathcal{E})\big)$, $(T_t)_{t>0}$, $(G_\alpha)_{\alpha>0}$, and L we refer to [MR 92, Chapter I]. We set $\tilde{\mathcal{E}}(u,v) := \frac{1}{2}\big(\mathcal{E}(u,v) + \mathcal{E}(v,u)\big)$, $u, v \in D(\mathcal{E})$ and $\mathcal{E}_\alpha := \mathcal{E} + \alpha(\ ,\ )$ for $\alpha > 0$.

DEFINITION 1.1. A coercive closed form $\big(\mathcal{E}, D(\mathcal{E})\big)$ on $L^2(E;m)$ is called a *positivity preserving coercive form*, if

$$u \in D(\mathcal{E}) \Rightarrow u^+ \in D(\mathcal{E}) \quad \text{and} \quad \mathcal{E}(u,u^+) \geq 0. \tag{1.2}$$

The following will be useful in applications. It gives us a "smoothed version" of (1.2).

PROPOSITION 1.2. *Let $\big(\mathcal{E}, D(\mathcal{E})\big)$ be a coercive closed form on $L^2(E;m)$.*

(i) Let $u \in D(\mathcal{E})$, then the property: $u^+ \in D(\mathcal{E})$ with $\mathcal{E}(u,u^+) \geq 0$, is equivalent to the following condition:

(S) for every $\varepsilon > 0$ there exists $\varphi_\varepsilon \colon \mathbb{R} \longrightarrow [-\varepsilon, \infty[$ such that $\varphi_\varepsilon(t) = t$ for all $t \in [0,\infty[$, $0 \leq \varphi_\varepsilon(t_2) - \varphi_\varepsilon(t_1) \leq t_2 - t_1$ if $t_1 \leq t_2$, $\varphi_\varepsilon \circ u \in D(\mathcal{E})$, $\sup_{\varepsilon>0} \mathcal{E}(\varphi_\varepsilon \circ u, \varphi_\varepsilon \circ u) < \infty$, and $\limsup_{\varepsilon\to 0} \mathcal{E}(u, \varphi_\varepsilon \circ u) \geq 0$.

(ii) Let D be a subset of $D(\mathcal{E})$, dense w.r.t. $\tilde{\mathcal{E}}_1^{1/2}$ *such that for every $\varepsilon > 0$, $u \in D$, (S) holds and, in addition, $\mathcal{E}(\varphi_\varepsilon \circ u, \varphi_\varepsilon \circ u) \leq c\,\mathcal{E}(u,u)$ for some $c \in]0,\infty[$ independent of ε and u. Then $\big(\mathcal{E}, D(\mathcal{E})\big)$ is positivity preserving.*

PROOF. (i) Assume that (S) holds. Since clearly $\varphi_\varepsilon \circ u \longrightarrow_{\varepsilon\downarrow 0} u^+$ in $L^2(E;m)$, it follows by (S) and [MR 92, I. 2.12] that $\varphi_{\varepsilon_n} \circ u \longrightarrow_{n\to\infty} u^+$ weakly in $\big(D(\mathcal{E}), \tilde{\mathcal{E}}_1\big)$ for some sequence $\varepsilon_n \downarrow 0$ for which also $\limsup_{\varepsilon\to 0} \mathcal{E}_\alpha\big(u, \varphi_\varepsilon(u)\big) = \lim_{n\to\infty} \mathcal{E}_\alpha\big(u, \varphi_{\varepsilon_n}(u)\big)$. Hence by (S)

$$\mathcal{E}(u,u^+) = \lim_{n\to\infty} \mathcal{E}(u, \varphi_{\varepsilon_n} \circ u) = \limsup_{\varepsilon\to 0} \mathcal{E}(u, \varphi_\varepsilon \circ u) \geq 0.$$

Conversely, assume that $u^+ \in D(\mathcal{E})$ with $\mathcal{E}(u,u^+) \geq 0$. Then (S) clearly follows by taking $\varphi_\varepsilon(t) := t \vee 0$, $t \in \mathbb{R}$, for all $\varepsilon > 0$.

(ii) Let $u \in D$. Because of the weak convergence of $(\varphi_{\varepsilon_n} \circ u)_{n\in\mathbb{N}}$ to u^+ in $\big(D(\mathcal{E}), \tilde{\mathcal{E}}_1\big)$ proved in part (i), it follows by assumption that

$$\mathcal{E}(u^+,u^+) \leq c\,\mathcal{E}(u,u).$$

Therefore, if $u \in D(\mathcal{E})$ and $u_n \in D, n \in \mathbb{N}$, such that $u_n \longrightarrow_{n\to\infty} u$ w.r.t. $\tilde{\mathcal{E}}^{1/2}$, then $u_n^+ \longrightarrow_{n\to\infty} u^+$ in $L^2(E;m)$ and hence by [MR 92, I. 2.12] $u_n^+ \longrightarrow_{n\to\infty} u^+ \in D(\mathcal{E})$ weakly in $\big(D(\mathcal{E}), \tilde{\mathcal{E}}_1\big)$. Hence $\mathcal{E}(u,u^+) = lim_{n\to\infty}\mathcal{E}(u_n, u_n^+) \geq 0$ by assumption (S) and part (i). ■

PROPOSITION 1.3. *(i) Let $\big(\mathcal{E}, D(\mathcal{E})\big)$ be a coercive closed form. Then (1.2) is equivalent to any of the statements (1.3), (1.4) below.*

$$u \in D(\mathcal{E}) \Rightarrow u^+,\ u^- \in D(\mathcal{E}) \quad \textit{and} \quad \mathcal{E}(u^+,u^-) \leq 0, \tag{1.3}$$

$$u \in D(\mathcal{E}) \Rightarrow |u| \in D(\mathcal{E}) \quad \textit{and} \quad \mathcal{E}(u,|u|) \leq \mathcal{E}(u,u). \tag{1.4}$$

(ii) Let $\big(\mathcal{E}, D(\mathcal{E})\big)$ be a positivity preserving coercive form, then

$$u \in D(\mathcal{E}) \Rightarrow |u| \in D(\mathcal{E}) \quad \textit{and} \quad \mathcal{E}(|u|,|u|) \leq \mathcal{E}(u,u). \tag{1.5}$$

(iii) Let $\big(\mathcal{E}, D(\mathcal{E})\big)$ be a symmetric closed form (cf. [MR 92, I.2.3] for the definition). Then all the above statements (1.2)–(1.5) are equivalent.

REMARK 1.4. (i) From (1.3) we see that if $\big(\mathcal{E}, D(\mathcal{E})\big)$ is a positivity preserving coercive form and $\hat{\mathcal{E}}(u,v) := \mathcal{E}(v,u)$ for $u, v \in D(\mathcal{E})$, then $\big(\hat{\mathcal{E}}, D(\mathcal{E})\big)$ and $\big(\tilde{\mathcal{E}}, D(\mathcal{E})\big)$ are also positivity preserving coercive forms.

(ii) In the literature statement (1.5) is rephrased as "the modulus contraction operates on $\big(\mathcal{E}, D(\mathcal{E})\big)$", which was proved to be equivalent to that $\big(\mathcal{E}, D(\mathcal{E})\big)$ satisfies the "infimum envelope principle", and also equivalent to that $\big(\mathcal{E}, D(\mathcal{E})\big)$ satisfies the "reduit principle". See P. A. Ancona [A 75], [Bl 71].

(iii) We recall that a coercive closed form $\big(\mathcal{E}, D(\mathcal{E})\big)$ is called a *semi-Dirichlet form* (*cf.* [MOR 93]) if

$$u \in D(\mathcal{E}) \Rightarrow u^+ \wedge 1 \in D(\mathcal{E}) \quad \text{and} \quad \mathcal{E}(u + u^+ \wedge 1, u - u^+ \wedge 1) \geq 0.$$

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$(\mathcal{E}, D(\mathcal{E}))$ is called a *Dirichlet form* (*cf.* [MR 92]) if

$$u \in D(\mathcal{E}) \Rightarrow u^+ \wedge 1 \in D(\mathcal{E}) \quad \text{and} \quad \mathcal{E}(u \pm u^+ \wedge 1, u \mp u^+ \wedge 1) \geq 0.$$

By [MR 92, I. 4.4] $(\mathcal{E}, D(\mathcal{E}))$ is a semi-Dirichlet form if and only if

$$u \in D(\mathcal{E}),\ \alpha \geq 0 \Rightarrow u \wedge \alpha \in D(\mathcal{E}) \quad \text{and} \quad \mathcal{E}(u \wedge \alpha, u - u \wedge \alpha) \geq 0.$$

Since in this case $-u^+ = (-u) \wedge 0 \in D(\mathcal{E})$ and

$$\begin{aligned}\mathcal{E}(u^+, u) &= \mathcal{E}(-u^+, -u) \\ &= \mathcal{E}\Big((-u) \wedge 0, (-u) - \big((-u) \wedge 0\big)\Big) + \mathcal{E}\big((-u) \wedge 0, (-u) \wedge 0\big) \geq 0,\end{aligned}$$

it follows by (i) that $(\mathcal{E}, D(\mathcal{E}))$ is positivity preserving.

PROOF OF PROPOSITION 1.3. First note that (1.2) is equivalent to

$$\text{(a)} \qquad \mathcal{E}(u, u^-) \leq 0, \quad \text{for all } u \in D(\mathcal{E}).$$

(1.2) $\Rightarrow$ (1.3): For $u \in D(\mathcal{E})$ and $\alpha > 0$ we set $u_\alpha := u^+ - \alpha u^-$. Applying (a) we obtain that $\alpha\mathcal{E}(u_\alpha, u^-) = \mathcal{E}(u_\alpha, u_\alpha^-) \leq 0$, which implies that

$$\mathcal{E}(u^+, u^-) \leq \alpha\mathcal{E}(u^-, u^-).$$

Since $\alpha > 0$ was arbitrary, we obtain (1.3).

(1.3) $\Rightarrow$ (1.2): $\mathcal{E}(u, u^-) = \mathcal{E}(u^+, u^-) - \mathcal{E}(u^-, u^-) \leq 0$.

(1.2) $\Leftrightarrow$ (1.4): $\mathcal{E}(u, |u|) = \mathcal{E}(u, u) + 2\mathcal{E}(u, u^-)$. Hence $\mathcal{E}(u, |u|) \leq \mathcal{E}(u, u)$ if and only if $\mathcal{E}(u, u^-) \leq 0$.

(1.3) $\Rightarrow$ (1.5): For any $u \in D(\mathcal{E})$ we have

$$\text{(b)} \qquad \mathcal{E}(|u|, |u|) = \mathcal{E}(u, u) + 2\mathcal{E}(u^+, u^-) + 2\mathcal{E}\big((-u)^+, (-u)^-\big).$$

Hence (1.3) implies (1.5). If $\mathcal{E}$ is symmetric, then (1.5) and (b) implies (1.3), proving Proposition 1.3(iii). ■

The following theorem shows the importance of property (1.2). Recall that a bounded linear operator T on $L^2(E; m)$ is said to be *positivity preserving* if $u \in L^2_+(E; m) \Rightarrow Tu \in L^2_+(E; m)$.

THEOREM 1.5. *Let $(\mathcal{E}, D(\mathcal{E}))$ be a coercive closed form on $L^2(E; m)$. Then $(\mathcal{E}, D(\mathcal{E}))$ is a positivity preserving closed form if and only if its associated semigroup $(T_t)_{t>0}$ is positivity preserving (i.e., T_t is positivity preserving for all $t > 0$).*

REMARK 1.6. Theorem 1.5 extends the first Beurling-Deny criterion which says that if $(T_t)_{t>0}$ is the associated semigroup of a symmetric closed form, then $(T_t)_{t>0}$ is positivity preserving if and only if $(\mathcal{E}, D(\mathcal{E}))$ satisfies (1.5). See *e.g.* [ReS 78, XIII.50].

We mention that the positivity preserving property is of importance in the study of semigroups. For the proof of Theorem 1.5 we need Theorem 1.7 below which holds for any strongly continuous contraction semigroup that is not necessarily associated with a coercive closed form (*i.e.*, is not necessarily the restriction of a holomorphic contraction semigroup, *cf.* [MR 92, I.2.21]).

THEOREM 1.7. *Let $(T_t)_{t>0}$ be a strongly continuous contraction semigroup on $L^2(E;m)$ with resolvent $(G_\alpha)_{\alpha>0}$ and generator L. Then the following statements (1.6)–(1.9) are equivalent.*

(1.6) $$(T_t)_{t>0} \text{ is positivity preserving,}$$

(1.7) $$(G_\alpha)_{\alpha>0} \text{ is positivity preserving,}$$

(1.8) $$(Lu, u^+) \leq 0 \quad \text{for all } u \in D(L),$$

(1.9) $$(Lu, u) \leq (Lu, |u|) \quad \text{for all } u \in D(L).$$

PROOF. (1.6) $\Rightarrow$ (1.7): The assertion follows from

$$G_\alpha u = \int_0^\infty e^{-\alpha s} T_s u \, ds, \qquad \text{for all } u \in L^2(E;m),\ \alpha > 0$$

(see *e.g.* [MR 92, I.1.4]).

(1.7) $\Rightarrow$ (1.6): Let $t > 0$. If $u \in D(L)$, then (see *e.g.* [MR 92, I.1.12])

$$T_t u = \lim_{\alpha \to \infty} e^{-t\alpha} \sum_{n=0}^{\infty} \frac{(t\alpha)^n}{n!} (\alpha G_\alpha)^n u.$$

Since L^2-convergence implies m-a.e.-convergence of a subsequence, we obtain that

$$u \in D(L) \cap L^2_+(E;m) \Rightarrow T_t u \in L^2_+(E;m).$$

The general assertion follows from the fact that $u = \lim_{\beta \to \infty} \beta G_\beta u$ in $L^2(E;m)$ for all $u \in L^2(E;m)$.

(1.7) $\Rightarrow$ (1.8): If (1.7) holds, then for all $u \in L^2(E;m)$, we have

(c) $$\beta\big((1 - \beta G_\beta) u^-, u^+\big) = -\beta^2 \int_E u^+ (G_\beta u^-) \, dm \leq 0.$$

On the other hand for all $u \in L^2(E;m)$ we have

(d) $$\beta\big((1 - \beta G_\beta) u, u\big) = \beta\big(\|u\|^2 - (\beta G_\beta u, u)\big) \geq 0.$$

(c) and (d) imply

(e) $$\beta\big((1 - \beta G_\beta) u, u^+\big) = \beta\big((1 - \beta G_\beta) u^+, u^+\big) - \beta\big((1 - \beta G_\beta) u^-, u^+\big) \geq 0.$$

If $u \in D(L)$ then

$$(Lu, u^+) = \lim_{\beta \to \infty} -\beta\big((1 - \beta G_\beta) u, u^+\big) \leq 0.$$

(1.8) $\Rightarrow$ (1.7): For $f \in L^2_+(E;m)$ we set $u := \alpha G_\alpha f$. By (1.8) we have that

$$0 \geq \big(L(-u),(-u)^+\big) = \big((\alpha - L)u, u^-\big) - \alpha(u,u^-) = \alpha(f-u,u^-)$$
$$= \alpha \int f u^- \, dm + \alpha \|u^-\|^2.$$

Since $fu^- \geq 0$ m-a.e. it follows that $u^- = 0$, *i.e.*, $G_\alpha f \in L^2_+(E;m)$.

(1.8) $\Leftrightarrow$ (1.9): The assertion follows by the argument that proved (1.2) $\Leftrightarrow$ (1.4). ∎

PROOF OF THEOREM 1.5. Since (by (1.1)) (1.2) $\Rightarrow$ (1.8) and Theorem 1.7 has already been proved, we only need to show that (1.7) $\Rightarrow$ (1.3). To this end let

$$\mathcal{E}^{(\beta)}(u,v) := \beta\big((1-\beta G_\beta)u, v\big) \quad \text{for all } u,v \in L^2(E;m).$$

By (c) we have

$$\text{(f)} \qquad \mathcal{E}^{(\beta)}(u^-,u^+) \leq 0.$$

Hence for $\mathcal{E}_1^{(\beta)} := \mathcal{E}^{(\beta)} + (\ ,\ )$,

$$\mathcal{E}_1^{(\beta)}(u^+,u^+) = \mathcal{E}_1^{(\beta)}(u,u^+) + \mathcal{E}^{(\beta)}(u^-,u^+)$$
$$\leq \mathcal{E}_1^{(\beta)}(u,u^+) \leq (K+1)\mathcal{E}_1(u,u)^{1/2}\mathcal{E}_1^{(\beta)}(u^+,u^+)^{1/2}$$

where the last step follows by [MR 92, I.2.11(iii)] and K is the continuity constant specified by [MR 92, I.(2.3)]. Consequently,

$$\sup_{\beta>0} \mathcal{E}^{(\beta)}(u^+,u^+) < \infty$$

and thus by [MR 92, I.2.13(i)] $u^+ \in D(\mathcal{E})$. Hence $u^- \in D(\mathcal{E})$ and (1.3) now follows from (f) and [MR 92, I.2.13(iii)]. ∎

REMARK 1.8. (i) For the analogues of Theorems 1.5, 1.7 in the case $\big(\mathcal{E}, D(\mathcal{E})\big)$ is a semi-Dirichlet form we refer to [MR 92, I. 4.3 and I. 4.4].

(ii) Suppose that $(T_t)_{t>0}$ is a strongly continuous semigroup on $L^2(E;m)$ such that there exists a constant $\beta \geq 0$ satisfying

$$\text{(1.10)} \qquad \|T_t f\| \leq e^{\beta t}\|f\| \quad \text{for all } f \in L^2(E;m),\ t > 0.$$

Then (1.6) and (1.7) are still equivalent, and they are equivalent to any of the conditions (1.6′), (1.7′) below.

$$\text{(1.6′)} \qquad (Lu,u^+) \leq \alpha\|u^+\|^2 \quad \text{for all } u \in D(L).$$

$$\text{(1.7′)} \qquad (Lu,u) \leq (Lu,|u|) + 2\beta\|u^-\|^2 \quad \text{for all } u \in D(L).$$

FINAL REMARK 1.9. All results above are more generally true in the case where $L^2(E;m)$ is replaced by a Riesz space H with Hilbert structure satisfying (1.11), (1.12) below.

$$\text{(1.11)} \qquad (u^+,v^+)_H \geq 0 \quad \text{for all } u,v \in H.$$

$$\text{(1.12)} \qquad (u^+,u^-)_H = 0 \quad \text{for all } u \in H.$$

2. Examples.

2.1. *The finite dimensional case.* The following is an extension of [RS 93, Section 1] to the case of positivity preserving coercive forms. For the convenience of the reader we repeat the set-up. For the terminology we refer to [MR 92, Chapters I, II].

Let U be an open (not necessarily bounded) set in $\mathbb{R}^d$, $d \geq 3$, with Borel σ-algebra $\mathcal{B}(U)$.

Let $\sigma, \rho \in L^1_{\mathrm{loc}}(U; dx)$, $\sigma, \rho > 0$ dx-a.e. where dx denotes Lebesgue measure. The following symmetric form will serve as a "reference form". Set for $u, v \in C_0^\infty(U)$ (:= all infinitely differentiable functions with compact support in U)

$$\mathcal{E}_\rho(u,v) = \sum_{i,j=1}^{d} \int \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\rho\, dx. \tag{2.1}$$

Assume that

$$\big(\mathcal{E}_\rho, C_0^\infty(U)\big) \text{ is closable on } L^2(U;\sigma\, dx). \tag{2.2}$$

REMARK 2.1. A sufficient condition for (2.2) to hold is that ρ, σ satisfy *Hamza's condition* (see [MR 92, Chapter II, Subsection 2a)]). We recall that a $\mathcal{B}(U)$-measurable function $f\colon \mathbb{R}^d \longrightarrow [0,\infty)$ satisfies *Hamza's condition* if for dx-a.e. $x \in U$, $f(x) > 0$ implies that for some $\varepsilon > 0$

$$\int_{\{y:\|y-x\|\leq\varepsilon\}} \big(f(y)\big)^{-1}\, dy < \infty, \tag{2.3}$$

where we set $\frac{1}{0} := +\infty$ and $\|\cdot\|$ denotes Euclidean distance in $\mathbb{R}^d$. In particular, σ, ρ may have zeros, and (2.2) holds if, for example, σ, ρ are lower semi-continuous. However, there is also a generalized version, a kind of "Hamza condition on rays", which, if it is fulfilled for σ, ρ, also implies (2.2) (*cf.* [AR 90, (5.7)] and [AR 91, Theorem 2.4]). In particular, if σ, ρ are weakly differentiable then (2.2) holds.

Now let $a_{ij}, b_i, d_i, c \in L^1_{\mathrm{loc}}(U; dx)$, $1 \leq i, j \leq d$, and define for $u, v \in C_0^\infty(U)$

$$\begin{aligned}\mathcal{E}(u,v) = {} & \sum_{i,j=1}^{d} \int \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} a_{ij}\, dx + \sum_{i=1}^{d} \int \frac{\partial u}{\partial x_i} v b_i\, dx \\ & + \sum_{i=1}^{d} \int u \frac{\partial v}{\partial x_i} d_i\, dx + \int uvc\, dx.\end{aligned} \tag{2.4}$$

Then $\big(\mathcal{E}, C_0^\infty(U)\big)$ is a densely defined bilinear form on $L^2(U;\sigma\, dx)$. Set $\tilde{a}_{ij} := \frac{1}{2}(a_{ij} + a_{ji})$, $\check{a}_{ij} := \frac{1}{2}(a_{ij} - a_{ji})$, $\underline{b} := (b_1, \ldots, b_d)$, and $\underline{d} := (d_1, \ldots, d_d)$. We define F to be the set of all functions $g \in L^1_{\mathrm{loc}}(U; dx)$ such that the distributional derivatives $\frac{\partial g}{\partial x_i}$, $1 \leq i \leq d$, are in $L^1_{\mathrm{loc}}(U; dx)$ with $\|\nabla g\|(g\sigma)^{-1/2} \in L^\infty(U; dx)$ or $\|\nabla g\|^p(g^{p+1}\sigma^{p/q})^{-1/2} \in L^d(U; dx)$ for some $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$, $p < \infty$. We say that a $\mathcal{B}(U)$-measurable function f has property $(A_{\rho,\sigma})$ if $f = f_1 + f_2$ where f_1, f_2 are $\mathcal{B}(U)$-measurable functions such that

$$f_1(\rho\sigma)^{-1/2} \in L^\infty(U; dx)$$

and

$$f_2^p(\rho^{p+1}\sigma^{p/q})^{-1/2} \in L^d(U;dx)$$

for some $p, q \in [1,\infty]$ with $\frac{1}{p}+\frac{1}{q}=1, p<\infty$, and $\rho \in F$.

Note that if ρ, σ are constant and U is bounded, f has property $(A_{\rho,\sigma})$ if and only if $f = f_1 + f_2$ with $f_1 \in L^\infty(U;dx)$ and $f_2 \in L^p(U;dx)$ for some $p \geq d$.

THEOREM 2.2. *Suppose that*

$$\|\underline{\xi}\|_{\tilde{a}}^2 := \sum_{i,j=1}^d \tilde{a}_{ij}\xi_i\xi_j \geq \rho\|\underline{\xi}\|^2 \; dx\text{-a.e.} \quad \text{for all } \underline{\xi}=(\xi_1,\dots,\xi_d)\in\mathbb{R}^d. \tag{2.5}$$

$$\check{a}_{ij}\rho^{-1} \in L^\infty(U;dx). \tag{2.6}$$

(2.7) $\|\underline{b}-\underline{d}\|$ *has property* $(A_{\rho,\sigma})$ *and* $1_K\|\underline{b}+\underline{d}\|$, $1_K c^{1/2}$ *have property* $(A_{\rho,\sigma})$ *for all compact* $K \subset U$.

(2.8) $\underline{b} = \underline{b}^{(1)} + \underline{b}^{(2)}$ *and* $\underline{d} = \underline{d}^{(1)} + \underline{d}^{(2)}$, $\|\underline{b}^{(i)}\|, \|\underline{d}^{(i)}\| \in L^1_{\text{loc}}(U;dx)$, $i = 1,2$, *such that* $\underline{b}^{(1)}$, $\underline{d}^{(1)}$ *satisfy* $(A_{\rho,\sigma})$ *and* $(c+\alpha_0\sigma)\,dx - \sum_{i=1}^d \frac{\partial b_i^{(2)}}{\partial x_i} \geq 0$ *and* $(c+\alpha_0\sigma)\,dx - \sum_{i=1}^d \frac{\partial d_i^{(2)}}{\partial x_i} \geq 0$ *(in the sense of Schwarz-distributions) for some* $\alpha_0 \in \,]0,\infty[$.

Then there exists $\alpha \in [0,\infty[$ *such that* $\left(\mathcal{E}_\alpha, C_0^\infty(U)\right)$ *is closable on* $L^2(U;\sigma\,dx)$ *and its closure* $\left(\mathcal{E}_\alpha, D(\mathcal{E}_\alpha)\right)$ *is a positivity preserving coercive form on* $L^2(U;\sigma\,dx)$.

REMARK 2.3. (i) For examples where ρ, σ are not necessarily constants we refer to [RS 93, 1.4].

(ii) One ingredient in the proof of Theorem 2.2 is the classical Sobolev Lemma (*cf. e.g.* [Da 89, 1.7.1]), *i.e.*, if $\lambda := \frac{2(d-1)}{(d-2)d^{1/2}}$, then

$$\|u\|_q \leq \lambda \Big\| \|\nabla u\| \Big\|_2 \quad \text{for all } u \in C_0^\infty(U), \tag{2.9}$$

where $\frac{1}{q}+\frac{1}{d}=\frac{1}{2}$ and for $p \geq 1$, $\|\;\|_p$ denotes the usual norm in $L^p(U;dx)$.

(iii) We stress that in the situation of Theorem 2.2 we can replace U by a Riemannian manifold M as long as condition (2.9) or more generally the following inequality holds for some $\alpha > 0$

$$\|u\|_{\frac{2d}{d-2}} \leq \text{const}\,\|(-\Delta+\alpha)^{1/2}u\|_2 \quad \text{for all } u \in C_0^\infty(M),$$

where Δ is the Laplacian on M. We refer *e.g.* to [VSC 92] for examples.

The proof of Theorem 2.2 is close to that of [RS 93, 1.2]. We only describe the modifications needed. We shall give, however, a full proof of the fact that $\left(\mathcal{E}_\alpha, D(\mathcal{E}_\alpha)\right)$ is positivity preserving. We need two lemmas.

LEMMA 2.4. *Let f be a $\mathcal{B}(U)$-measurable function having property $(A_{\rho,\sigma})$. Then there exists $\delta, \eta \in \,]0,\infty[$, with δ arbitrarily small, such that for all $u \in C_0^\infty(U)$,*

$$\int f^2\rho^{-1}u^2\,dx \leq \delta\int\|\nabla u\|^2\rho\,dx + \eta\int u^2\sigma\,dx.$$

PROOF. Realizing that if the assertion holds for f and $\hat{f}$ then it hold for $f+\hat{f}$, one deduces the assertion from [RS 93, 1.5]. ■

LEMMA 2.5. *Consider the situation of Theorem 2.2. Then for any $\varepsilon \in]0,1[$ there exists $\alpha \in [\alpha_0,\infty[$ such that for all $u \in C_0^\infty(U)$*

$$\int (|\langle \nabla u, \underline{b}^{(1)}\rangle| + |\langle \nabla u, \underline{d}^{(1)}\rangle|)u\,dx \le \varepsilon\Big(\mathcal{E}_\alpha(u,u) - \int \langle \nabla u, \underline{b}^{(1)}\rangle u\,dx - \int \langle \nabla u, \underline{d}^{(1)}\rangle u\,dx\Big)$$

(where $\langle\ ,\ \rangle$ denotes the Euclidean inner product on $\mathbb{R}^d$).

PROOF. We first note that for all $u \in C_0^\infty(U)$

$$\begin{aligned}\mathcal{E}_{\alpha_0}(u,u) &- \int \langle \nabla u, \underline{b}^{(1)}\rangle u\,dx - \int \langle \nabla u, \underline{d}^{(1)}\rangle u\,dx\\ &= \int \|\nabla u\|_{\tilde{a}}^2\,dx + \frac{1}{2}\int [\langle \nabla u^2, \underline{b}^{(2)} + \underline{d}^{(2)}\rangle + 2(c+\alpha_0\sigma)u^2]\,dx \qquad (2.10)\\ &\ge \int \|\nabla u\|_{\tilde{a}}^2\,dx.\end{aligned}$$

Furthermore, by (2.8) and Lemma 2.4 for all $\delta,\delta' \in]0,1[$ and $u \in C_0^\infty(U)$, if $\underline{\beta} := \underline{b}^{(1)}$ or $\underline{\beta} := \underline{d}^{(1)}$,

$$\begin{aligned}\int |\langle \nabla u, \underline{\beta}\rangle u|\,dx &\le \frac{1}{2}\int \Big(\delta'\rho\|\nabla u\|^2 + \frac{1}{\delta'}\|\underline{\beta}\|^2\rho^{-1}u^2\Big)\,dx\\ &\le \frac{1}{2}\Big(\delta' + \frac{\delta}{\delta'}\Big)\int \|\nabla u\|_{\tilde{a}}^2\,dx + \frac{\eta}{2\delta'}\int u^2\sigma\,dx,\end{aligned}$$

for some $\eta \in]0,\infty[$. Now the assertion follows by (2.10). ■

PROOF OF THEOREM 2.2. Let ε, α be as in Lemma 2.5. Since $\varepsilon < 1$ the positive definiteness of $\big(\mathcal{E}_\alpha, C_0^\infty(U)\big)$ is obvious by Lemma 2.5. To prove closability of $\big(\mathcal{E}_\alpha, C_0^\infty(U)\big)$ on $L^2(U;\sigma dx)$ first note that by Lemma 2.5, for all $u \in C_0^\infty(U)$

$$\begin{aligned}(1+\varepsilon)^{-1}\mathcal{E}_\alpha(u,u) &\le \mathcal{E}_\alpha(u,u) - \int \langle \nabla u, \underline{b}^{(1)}\rangle u\,dx - \int \langle \nabla u, \underline{d}^{(1)}\rangle u\,dx \qquad (2.11)\\ &\le (1-\varepsilon)^{-1}\mathcal{E}_\alpha(u,u).\end{aligned}$$

Hence it suffices to consider the case $\underline{b}^{(1)} \equiv \underline{d}^{(1)} \equiv 0$. Now the proof of closability is exactly the same as that in the proof of [RS 93, 1.2]. The same holds for the weak sector condition. We now prove that $\big(\mathcal{E}_\alpha, D(\mathcal{E}_\alpha)\big)$ is positivity preserving using Proposition 1.2. Let $\varepsilon > 0$ and let $\varphi_\varepsilon\colon \mathbb{R} \longrightarrow [-\varepsilon,\infty[$ such that $\varphi_\varepsilon \in C^\infty(\mathbb{R})$, $\varphi_\varepsilon(t) = t$ for all $t \in [0,\infty[$, $0 < \varphi'_\varepsilon \le 1$, and $\varphi_\varepsilon(t) = -\varepsilon$ for $t \in]-\infty,-2\varepsilon]$. To show that there exists $c \in]0,\infty[$ such that

$$\sup_{\varepsilon>0} \mathcal{E}_\alpha(\varphi_\varepsilon \circ u, \varphi_\varepsilon \circ u) \le c\,\mathcal{E}_\alpha(u,u) \quad \text{for all } u \in C_0^\infty(U), \qquad (2.12)$$

by (2.11) we may assume that $\underline{b}^{(1)} \equiv \underline{d}^{(1)} \equiv 0$. But in this case $\big(\mathcal{E}_\alpha, D(\mathcal{E}_\alpha)\big)$ is a Dirichlet form by [RS 93, 1.2]. Furthermore, each $\varphi_\varepsilon \circ u$ is a normal contraction of u. Hence (2.12) holds by [MR 92, I. 4.11]. Fix $u \in C_0^\infty$. To prove that

$$\limsup_{\varepsilon\longrightarrow 0} \mathcal{E}_\alpha(u, \varphi_\varepsilon \circ u) \ge 0 \qquad (2.13)$$

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we have to consider the general case $\underline{b} = \underline{b}^{(1)} + \underline{b}^{(2)}$, $\underline{d} = \underline{d}^{(1)} + \underline{d}^{(2)}$. Realizing that $\varphi_\varepsilon(u) \longrightarrow_{\varepsilon\downarrow 0} 1_{\{u\geq 0\}}u$ and $\varphi'_\varepsilon(u) \longrightarrow_{\varepsilon\downarrow 0} 1_{\{u\geq 0\}}$ we conclude by the chain rule and Lebesgue's dominated convergence theorem that, if $\varepsilon_n := \frac{1}{n}$, $n \in \mathbb{N}$, and $A := (a_{ij})_{i,j}$, then

$$\begin{aligned}\limsup_{\varepsilon\longrightarrow 0} \mathcal{E}_\alpha(u, \varphi_\varepsilon \circ u) &\geq \int 1_{\{u\geq 0\}}\Big(\langle A\nabla u, \nabla u\rangle + \langle \underline{b} + \underline{d}, \nabla u\rangle u + u^2(c+\alpha)\Big)\, dx \\ &= \lim_{n\longrightarrow\infty}\Big(\int \big(\varphi'_{\varepsilon_n}(u)\big)^2 \langle A\nabla u, \nabla u\rangle\, dx \\ &\qquad + \int \varphi'_{\varepsilon_n}(u)\varphi_{\varepsilon_n}(u)\langle \underline{b}+\underline{d}, \nabla u\rangle\, dx + \int \varphi_{\varepsilon_n}(u)^2(c+\alpha)\, dx\Big) \\ &= \lim_{n\longrightarrow\infty} \mathcal{E}_\alpha(\varphi_{\varepsilon_n}\circ u, \varphi_{\varepsilon_n}\circ u) \geq 0.\end{aligned}$$

■

REMARK 2.6. If in (2.8) $\underline{d}^{(1)} \equiv 0$ resp. $\underline{b}^{(1)} \equiv 0$, then $\big(\mathcal{E}_\alpha, D(\mathcal{E}_\alpha)\big)$ resp. $\big(\hat{\mathcal{E}}_\alpha, D(\mathcal{E}_\alpha)\big)$ is a semi-Dirichlet form. Finally, if $\underline{d}^{(1)} \equiv 0 \equiv \underline{b}^{(1)}$, then $\big(\mathcal{E}_\alpha, D(\mathcal{E}_\alpha)\big)$ is a Dirichlet form. Both statements were proved in [RS 93, 1.2]. We just mention, however, that for both statements one has to show (2.12). This is clear in the latter case since [MR 92, I. Condition (4.7)] is easy to check (*cf.* [MR 92, II. p. 49]) and this implies (2.12) (*cf.* proof of [MR 92, I. 4.7]). In the first case the argument to prove (2.12) is the same as in the proof of Theorem 2.2 above.

2.2. *Positivity preserving coercive forms of gradient type on Banach space.* Let E be a (real) separable Banach space with dual E' and μ a finite positive measure on its Borel σ-algebra $\mathcal{B}(E)$ which charges every weakly open set. Define a linear space of functions on E by

$$\mathcal{F}C_b^\infty := \{f(l_1, \ldots, l_m) \mid m \in \mathbb{N}, f \in C_b^\infty(\mathbb{R}^m), l_1, \ldots, l_m \in E'\}.$$

By the Hahn-Banach theorem $\mathcal{F}C_b^\infty$ separates the points of E. The support condition on μ means that we can regard $\mathcal{F}C_b^\infty$ as a subspace of $L^2(E;\mu)$, and a monotone class argument then shows that it is dense in $L^2(E;\mu)$. Define for $u \in \mathcal{F}C_b^\infty$ and $k \in E$

$$\frac{\partial u}{\partial k}(z) := \frac{d}{ds}u(z+sk)|_{s=0}, \quad z \in E.$$

Let us assume that there is a separable real Hilbert space $(H, \langle\ ,\ \rangle_H)$ densely and continuously embedded into E. Fix $z \in E$. Then $k \longmapsto \frac{\partial u}{\partial k}(z)$, $k \in H$, is a continuous linear functional on $(H, \langle\ ,\ \rangle_H)$, hence we can define $\nabla u(z) \in H$ by

$$\langle \nabla u(z), k\rangle_H = \frac{\partial u}{\partial k}(z), \quad k \in H.$$

Define a bilinear form by

$$\mathcal{E}_\mu(u,v) := \int \langle \nabla u(z), \nabla v(z)\rangle_H \mu(dz); \quad u, v \in \mathcal{F}C_b^\infty. \tag{2.14}$$

ASSUMPTION 2.7. We assume that the form $(\mathcal{E}_\mu, \mathcal{F}C_b^\infty)$ in (2.14) is closable on $L^2(E;\mu)$.

For conditions and examples where Assumption 2.7 holds we refer to [AR 90] and [MR 92, III. Subsections 3a)–d)].

Let $\mathcal{L}_\infty(H)$ denote the set of all bounded linear operators on H with operator norm $\| \, \|_\infty$. Suppose $z \longmapsto A(z)$, $z \in E$, is a map from E to $\mathcal{L}_\infty(H)$ such that $z \longmapsto \langle A(z)h_1, h_2\rangle_H$ is $\mathcal{B}(E)$-measurable for all $h_1, h_2 \in H$. Furthermore, assume that

(2.15) there exists $\gamma \in]0, \infty[$ such that $\langle A(z)h, h\rangle_H \geq \gamma \|h\|_H^2$ for all $h \in H, z \in E$,

and that $\|\tilde{A}\|_\infty \in L^1(E;\mu)$ and $\|\check{A}\|_\infty \in L^\infty(E;\mu)$, where $\tilde{A} := \frac{1}{2}(A + \hat{A})$, $\check{A} := \frac{1}{2}(A - \hat{A})$ and $\hat{A}(z)$ denotes the adjoint of $A(z)$, $z \in E$. Let $c \in L^\infty(E;\mu)$ and $\underline{b}, \underline{d} \in L^\infty(E \longrightarrow H;\mu)$. For $u, v \in \mathcal{F}C_b^\infty$ let

$$\mathcal{E}_A(u,v) := \int \langle A(z)\nabla u(z), \nabla v(z)\rangle_H \mu(dz), \tag{2.16}$$

and

$$\mathcal{E}(u,v) := \mathcal{E}_A(u,v) + \int \langle \underline{b}, \nabla u\rangle_H v\, d\mu + \int u\langle \underline{d}, \nabla v\rangle_H\, d\mu + \int uvc\, d\mu. \tag{2.17}$$

LEMMA 2.8. *(i) Let $\underline{\beta} := \underline{b}$ or $\underline{d}$ and $c_0 := \sup_{z\in E} \|\underline{\beta}(z)\|_H$. Then for all $u, v \in \mathcal{F}C_b^\infty$*

$$|\langle \underline{\beta}, \nabla v\rangle_H v| \leq \Big(\frac{1}{2\gamma}\langle A\nabla u, \nabla u\rangle_H + 2c_0^2 u^2\Big)^{1/2}\Big(\frac{1}{2\gamma}\langle A\nabla v, \nabla v\rangle_H + 2c_0^2 v^2\Big)^{1/2}.$$

(ii) There exist $\alpha, \kappa \in]0, \infty[$ such that

$$\kappa^{-1}\mathcal{E}_{A,1}(u,u) \leq \mathcal{E}_\alpha(u,u) \leq \kappa\mathcal{E}_{A,1}(u,u) \quad \textit{for all } u \in \mathcal{F}C_b^\infty.$$

PROOF. (i) We have for $u, v \in \mathcal{F}C_b^\infty$

$$\begin{aligned}|\langle \underline{\beta}, \nabla u\rangle_H v| &\leq \|\underline{\beta}\|_H \|\nabla u\|_H |v| \\ &\leq \Big(\frac{1}{2\gamma}\langle A\nabla u, \nabla u\rangle_H + 2c_0 u^2\Big)^{1/2}\Big(\frac{1}{2\gamma}\langle A\nabla v, \nabla v\rangle_H + 2c_0 v^2\Big)^{1/2}.\end{aligned}$$

(ii) is an immediate consequence of (i). ■

By [MR 92, II. 3.9] $(\mathcal{E}_A, \mathcal{F}C_b^\infty)$ is closable on $L^2(E;\mu)$ hence so is $(\mathcal{E}_\alpha, \mathcal{F}C_b^\infty)$ by Lemma 2.8(ii). Lemma 2.8(ii) also implies that the closure $\big(\mathcal{E}_\alpha, D(\mathcal{E}_\alpha)\big)$ is positive definite. The sector condition follows immediately from Lemma 2.8(i). Hence $\big(\mathcal{E}_\alpha, D(\mathcal{E}_\alpha)\big)$ is a coercive closed from on $L^2(E;\mu)$. The fact that it is positivity preserving is proved in exactly the same way as in the previous subsection.

REMARK 2.9. As in the previous subsection $\big(\mathcal{E}_\alpha, D(\mathcal{E}_\alpha)\big)$ is a Dirichlet form if, in addition, for all $u \in \mathcal{F}C_b^\infty$, $u \geq 0$,

$$\int \big(\langle \underline{b}, \nabla u\rangle_H + (c+\alpha)u\big)\, d\mu \geq 0 \quad \text{and} \quad \int \big(\langle \underline{d}, \nabla u\rangle_H + (c+\alpha)u\big)\, d\mu \geq 0$$

and $\big(\hat{\mathcal{E}}_\alpha, D(\mathcal{E}_\alpha)\big)$ resp. $\big(\mathcal{E}_\alpha, D(\mathcal{E}_\alpha)\big)$ are semi-Dirichlet forms if the first resp. the second inequality holds (*cf.* [MR 92, II. Subsection 3e)].

2.3. *Positivity preserving coercive forms having a space of measures as state space.* For the following framework we refer to [ORS 93]. Let $E := \mathcal{M}_1(S)$ be the space of probability measures on a Polish space S with its Borel σ-algebra $\mathcal{B}(S)$. We equip E with the weak topology and its Borel σ-algebra $\mathcal{B}(E)$. Note that E is then also a Polish space. If f, g are bounded $\mathcal{B}(S)$-measurable functions on S and $\mu \in E$, we define

$$\mu(f) := \int f\, d\mu,$$
$$\langle f, g\rangle_\mu := \int fg\, d\mu - \int f\, d\mu \int g\, d\mu = \mathrm{cov}_\mu(f, g),$$

and

$$\|f\|_\mu = \sqrt{\langle f, f\rangle_\mu}.$$

Similarly, as in the previous subsection the set $\mathcal{F}C_b^\infty$ of *finitely based smooth functions* on E is defined by

$$u \in \mathcal{F}C_b^\infty \Leftrightarrow \begin{array}{l} u(\mu) = \varphi\big(\mu(f_1), \dots, \mu(f_k)\big), \\ k \in \mathbb{N}, f_i \in C_b(S),\ 1 \le i \le k,\ \varphi \in C_b^\infty(\mathbb{R}^k). \end{array}$$

Let m be a finite (positive) measure on $\big(E, \mathcal{B}(E)\big)$, where $\mathcal{B}(E)$ is the Borel σ-algebra of E. We suppose that $\mathrm{supp}[m] = E$. Finally, let

$$\underline{b}(\cdot,\cdot), \underline{d}(\cdot,\cdot) : S \times E \to \mathbb{R}$$

be measurable functions such that

$$\sup_{\mu\in E} \|\underline{b}(\mu)\|_\mu, \sup_{\mu\in E} \|\underline{d}(\mu)\|_\mu < \infty, \tag{2.18}$$

where

$$\underline{b}(\mu)(x) := \underline{b}(x, \mu).$$

Let $c \in L^\infty(E; m)$. For $u, v \in \mathcal{F}C_b^\infty$ let

$$\begin{aligned} \mathcal{E}(u, v) := \int \Big(&\langle \nabla u(\mu), \nabla v(\mu)\rangle_\mu + \langle \underline{b}(\mu), \nabla u(\mu)\rangle_\mu v(\mu) \\ &+ u(\mu)\langle \underline{d}(\mu), \nabla v(\mu)\rangle_\mu + u(\mu)v(\mu)c(\mu)\Big) m(d\mu) \end{aligned} \tag{2.19}$$

where

$$\nabla u(\mu) := \big(\nabla_x u(\mu)\big)_{x\in S} = \Big(\frac{\partial u}{\partial \varepsilon_x}(\mu)\Big)_{x\in S}, \tag{2.20}$$

and

$$\nabla_x u(\mu) := \frac{\partial u}{\partial \varepsilon_x}(\mu) := \frac{d}{ds} u(\mu + s\varepsilon_x)|_{s=0} = \sum_{i=1}^k \frac{\partial \varphi}{\partial y_i}\big(\mu(f_1), \dots, \mu(f_k)\big) f_i(x),$$
$$u(\mu) = \varphi\big(\mu(f_1), \dots, \mu(f_k)\big),\ \varepsilon_x := \text{Dirac measure at } x.$$

Here we consider the natural extension of $u \in \mathcal{F}C_b^\infty$ to all finite positive measures on $\big(S, \mathcal{B}(S)\big)$.

REMARK 2.10. (i) Apart from concrete examples the motivation to study forms of type (2.19) is that they fit into the following "geometrical set-up". One should consider E as "the manifold" and $L^2(S;\mu)$ as the "tangent space" at $\mu \in E$. The "Riemannian structure" is given by $\langle\, ,\, \rangle_\mu$, $\mu \in E$. In this spirit ∇u is then "the gradient vector field" of the smooth function u on E. In these terms in the previous subsection the "manifold" was the Banach space E and the "tangent space" the Hilbert space H with inner product $\langle A\cdot, \cdot\rangle_H$. But in contrast to the present case the "tangent space" H did not vary with the points in E.

(ii) Below, for two functions $\underline{b}_1, \underline{b}_2 : S \times E \longrightarrow \mathbb{R}$ we denote the function $\mu \mapsto \langle \underline{b}_1(\mu), \underline{b}_2(\mu)\rangle_\mu$ (if this is defined) by $\langle \underline{b}_1, \underline{b}_2\rangle$.

ASSUMPTION 2.11. Assume that

$$\mathcal{E}^0(u,v) := \int \langle \nabla u(\mu), \nabla v(\mu)\rangle_\mu m(d\mu); \quad u, v \in \mathcal{F}C_b^\infty$$

is closable on $L^2(E;m)$.

Assumption 2.11 has *e.g.* been verified for m equal to the stationary reversible measure of the Fleming-Viot process in [ORS 93, Subsection 5.1] on the basis of results in [EK 93]. If Assumption 2.11 holds an analogue of Lemma 2.8 can be proved in exactly the same way as in the previous section which then implies that for some $\alpha > 0$ $(\mathcal{E}_\alpha, \mathcal{F}C_b^\infty)$ is closable and positive definite, and that its closure $\big(\mathcal{E}_\alpha, D(\mathcal{E}_\alpha)\big)$ is a coercive closed form on $L^2(E;m)$. The fact that $\big(\mathcal{E}_\alpha, D(\mathcal{E}_\alpha)\big)$ is positivity preserving can be shown in exactly the same way as in Subsection 2.1. For a detailed study of $\big(\mathcal{E}_\alpha, D(\mathcal{E}_\alpha)\big)$ and applications in case it is a semi-Dirichlet form we refer to [ORS 93].

3. ***h*-transform of a positivity preserving coercive form.** The analytic potential theory of positivity preserving coercive forms can be developed along the lines of [MR 92] and [MOR 93] almost word for word. Below we shall clearly indicate where modifications are necessary and otherwise refer to [MR 92], [MOR 93]. We shall also use the "h-transformation", by which any positivity preserving coercive form $\big(\mathcal{E}, D(\mathcal{E})\big)$ on $L^2(E;m)$ is isomorphic to a semi-Dirichlet form on a weighted Hilbert space $L^2(E; h^2\cdot m)$, provided m is σ-finite on E. Therefore, all known results for the potential theory of semi-Dirichlet forms can be transfered to the case of positivity preserving coercive forms. We mention that the *h-transformation* (or its spirit) has been used by many authors in different contexts. See *e.g.* [D 84, X], [DM 88, IX.25.e], [BlHa 86, II 7.9], [Sh 88, (62.2)], [AMR 92].

DEFINITION 3.1. Let $\big(\mathcal{E}, D(\mathcal{E})\big)$ be a coercive closed form on $L^2(E;m)$ and $h \in L^2(E;m)$, $h > 0$ m-a.e. Define

$$D(\mathcal{E}^h) := \{u \in L^2(E; h^2\cdot m) \mid uh \in D(\mathcal{E})\}, \tag{3.1}$$

$$\mathcal{E}^h(u,v) = \mathcal{E}(uh, vh) \quad \text{for } u, v \in D(\mathcal{E}^h). \tag{3.2}$$

$\big(\mathcal{E}^h, D(\mathcal{E}^h)\big)$ is called the *h-transform* of $\big(\mathcal{E}, D(\mathcal{E})\big)$.

REMARK 3.2. The following facts are easy to check.

(i) Define $i: L^2(E; h^2 \cdot m) \longrightarrow L^2(E; m)$ by $i(u) = uh$ m-a.e. Then i is an isometry from $L^2(E; h^2 \cdot m)$ onto $L^2(E; m)$, and the restriction of i to $D(\mathcal{E}^h)$ is an isometry from the Hilbert space $\big(D(\mathcal{E}^h), \tilde{\mathcal{E}}_1^h\big)$ onto the Hilbert space $\big(D(\mathcal{E}), \tilde{\mathcal{E}}_1\big)$.

(ii) $\big(\mathcal{E}^h, D(\mathcal{E}^h)\big)$ is a coercive closed form on $L^2(E; h^2 \cdot m)$. $\big(\mathcal{E}^h, D(\mathcal{E}^h)\big)$ is a positivity preserving coercive form if and only if so is $\big(\mathcal{E}, D(\mathcal{E})\big)$. $\big(\mathcal{E}^h, D(\mathcal{E}^h)\big)$ is symmetric if and only if so is $\big(\mathcal{E}, D(\mathcal{E})\big)$.

(iii) Let $(T_t)_{t>0}$, $(G_\alpha)_{\alpha>0}$, L be the associated semigroup, resolvent resp. generator of $\big(\mathcal{E}, D(\mathcal{E})\big)$, and $(T_t^h)_{t>0}$, $(G_\alpha^h)_{\alpha>0}$, L^h be the corresponding objects associated with $\big(\mathcal{E}^h, D(\mathcal{E}^h)\big)$. Then

$$T_t^h f = h^{-1} T_t(fh) \quad h^2 \cdot m\text{-a.e.} \quad \text{for all } f \in L^2(E; h^2 \cdot m). \tag{3.3}$$

$$G_\alpha^h f = h^{-1} G_\alpha(fh) \quad h^2 \cdot m\text{-a.e.} \quad \text{for all } f \in L^2(E; h^2 \cdot m). \tag{3.4}$$

$$D(L^h) = \{f \in L^2(E; h^2 \cdot m) \mid fh \in D(L)\} \quad \text{and} \quad L^h f = h^{-1} L(fh) \quad \text{for all } f \in D(L^h). \tag{3.5}$$

In what follows we assume that $\big(\mathcal{E}, D(\mathcal{E})\big)$ is a positivity preserving coercive form on $L^2(E; m)$ and m is σ-finite on E.

DEFINITION 3.3. Let $f \in L^2(E; m)$ and $\alpha \geq 0$. f is called *α-excessive* if $f \geq 0$ m-a.e. and

$$e^{-\alpha t} T_t f \leq f \quad m\text{-a.e. for all } t > 0. \tag{3.6}$$

REMARK 3.4. (i) Following the argument of [MR 92, III.1.2], one can check that if $\alpha > 0$, then the condition $f \geq 0$ m-a.e. is implied by (3.6). Moreover, if $f \in D(\mathcal{E}) \cap L_+^2(E; m)$, then (3.6) is equivalent with any of (3.7), (3.8) below.

$$\beta G_{\beta+\alpha} f \leq f \quad m\text{-a.e. for all } \beta \geq 0 \tag{3.7}$$

$$\mathcal{E}_\alpha(f, g) \geq 0 \quad \text{for all } g \in D(\mathcal{E}) \cap L_+^2(E; m). \tag{3.8}$$

(ii) Since $u \wedge h = h - (h - u)^+$, $D(\mathcal{E})$ is always inf-stable, *i.e.*, $u, h \in D(\mathcal{E})$ implies $u \wedge h \in D(\mathcal{E})$ (which implies also $u \vee h \in D(\mathcal{E})$, $|u| \in D(\mathcal{E})$). Following the arguments of [MOR 93, 2.6] we can extend this inf-stability to the case where h is α-excessive and is not necessarily in $D(\mathcal{E})$. More precisely, let $h \in L_+^2(E; m)$ be α-excessive and $u \in D(\mathcal{E})$. Then

$$u \wedge h \in D(\mathcal{E}) \quad \text{and} \quad \mathcal{E}_\alpha(u \wedge h, u \wedge h) \leq \mathcal{E}_\alpha(u \wedge h, u). \tag{3.9}$$

It is easy to see that if h is α-excessive and $h > 0$ m-a.e., then $(e^{-\alpha t} T_t^h)_{t>0}$ is sub-Markovian where T_t^h is defined by (3.3). Thus the following theorem follows *e.g.* from [MR 92, I.4.3 and 4.4].

THEOREM 3.5. *Let h be α-excessive and $h > 0$ m-a.e. Let $\big(\mathcal{E}^h, D(\mathcal{E}^h)\big)$ be defined by (3.1), (3.2). Then $\big(\mathcal{E}_\alpha^h, D(\mathcal{E}^h)\big)$ is a semi-Dirichlet form on $L^2(E; h^2 \cdot m)$.*

The following lemma guarantees the existence of many strictly positive α-excessive functions h.

LEMMA 3.6. *Let $\varphi \in L^2(E;m)$, $\varphi > 0$ m-a.e. and $\alpha > 0$. Define*

$$h := G_\alpha \varphi. \tag{3.10}$$

Then h is α-excessive and $h > 0$ m-a.e.

PROOF. It is clear that h is α-excessive. Let $(\hat{G}_\alpha)_{\alpha>0}$ be the co-associated resolvent of $\big(\mathcal{E}, D(\mathcal{E})\big)$. If $A \in \mathcal{B}$ with $0 < m(A) < \infty$, then $\hat{G}_\alpha 1_A \in L^2_+(E;m)$ and $\|\hat{G}_\alpha 1_A\| \neq 0$, thus $\int_A h\,dm = (\varphi, \hat{G}_\alpha 1_A) > 0$. Since m is σ-finite and $A \in \mathcal{B}$ with $0 < m(A) < \infty$ was arbitrary, $h > 0$ m-a.e. ∎

4. **Quasi-regularity.** In this section we assume that E is a Hausdorff topological space and $\mathcal{B} := \mathcal{B}(E)$ is the Borel σ-algebra of E. As before m is a σ-finite measure on $(E;\mathcal{B})$ and $\big(\mathcal{E}, D(\mathcal{E})\big)$ is a positivity preserving coercive form on $L^2(E;m)$.

For a closed set $F \subset E$ we set

$$D(\mathcal{E})_F := \{u \in D(\mathcal{E}) \mid u = 0 \ m\text{-a.e. on } E \setminus F\}. \tag{4.1}$$

DEFINITION 4.1 (*cf.* [MR 92, III. 2.1]). (i) An increasing sequence $(F_k)_{k\in\mathbb{N}}$ of closed subsets of E is called an *$\mathcal{E}$-nest* if $\bigcup_{k\geq 1} D(E)_{F_k}$ is $\tilde{\mathcal{E}}_1^{1/2}$-dense in $D(\mathcal{E})$.

(ii) A subset $N \subset E$ is called *$\mathcal{E}$-exceptional* if $N \subset \bigcap_{k\geq 1}(E \setminus F_k)$ for some $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$. We say that a property of points in E holds *$\mathcal{E}$-quasi-everywhere* (abbreviated $\mathcal{E}$-q.e.), if the property holds outside some $\mathcal{E}$-exceptional set.

PROPOSITION 4.2. *Let $h \in L^2(E;m)$, $h > 0$ m-a.e., and let $\big(\mathcal{E}^h, D(\mathcal{E}^h)\big)$ be the h-transform defined by (3.1), (3.2). Then $(F_k)_{k\in\mathbb{N}}$ is an $\mathcal{E}$-nest if and only if it is an $\mathcal{E}^h$-nest. A subset $N \subset E$ is $\mathcal{E}$-exceptional if and only if N is $\mathcal{E}^h$-exceptional. In this case $m(N) = 0$.*

PROOF. The assertion follows from Remark 3.2(i). ∎

Given an $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$ we define

$$C(\{F_k\}) := \Big\{f : A \longrightarrow \mathbb{R} \;\Big|\; \bigcup_{k\geq 1} F_k \subset A \subset E,\ f|_{F_k} \text{ is continuous for each } k \in \mathbb{N}\Big\}. \tag{4.2}$$

DEFINITION 4.3. An $\mathcal{E}$-q.e. defined function f on E is called *$\mathcal{E}$-quasi-continuous* if there exists an $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$ such that $f \in C(\{F_k\})$.

Let $h \in D(\mathcal{E})$. For an open set $U \subset E$, we denote by h_U the α-reduced function of h on U. That is,

(4.3)

h_U is the smallest α-excessive function u in $D(\mathcal{E})$ such that $u \geq h$ m-a.e. on U.

For the existence and properties of h_U we refer to [MOR 93, 2.8] or [MR 92, III. 1.5., III. 1.6].

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Let h be a strictly positive m-version of an α-excessive function in $D(\mathcal{E})$ and $\varphi \in L^2(E;m)$, $\varphi > 0$ m-a.e. We define for an open set $U \subset E$

$$(4.4) \qquad (\alpha-)\operatorname{Cap}_{h,\varphi}(U) := (h_U, \varphi) = \mathcal{E}_\alpha(h_U, \hat{G}_\alpha\varphi) = \mathcal{E}_\alpha(h_U, \hat{g}_U),$$

where $(\hat{G}_\beta)_{\beta>0}$ is the co-resolvent associated with $(\mathcal{E}, D(\mathcal{E}))$ and $\hat{g}_U$ is the α-co-reduced function of $g := \hat{G}_\alpha\varphi$ on U. For an arbitrary $A \subset E$ we define

$$(4.5) \qquad (\alpha-)\operatorname{Cap}_{h,\varphi}(A) := \inf\{\operatorname{Cap}_{h,\varphi}(U) \mid A \subset U,\ U \text{ open}\}.$$

One can check that $\operatorname{Cap}_{h,\varphi}$ is a Choquet capacity enjoying countable sub-additivity and $\operatorname{Cap}_{h,\varphi}(A) = 0$ implies $m(A) = 0$ for $A \in \mathcal{B}(E)$. Concerning the proof of both statements we refer to [MR 92, III. 2.8] whose proof extends word for word to this more "general case".

The following theorem improves [MR 92, III. 2.11] and [MOR 93, 2.14] because we do not require that $h = G_\alpha\psi$ for some $\psi \in L^2(E;m)$. Below we set $A^c := E\setminus A$ for $A \subset E$.

THEOREM 4.4. *Let h, φ be as above. An increasing sequence $(F_k)_{k\in\mathbb{N}}$ of closed subsets of E is an $\mathcal{E}$-nest if and only if* $\lim \operatorname{Cap}_{h,\varphi}(F_k^c) = 0$.

PROOF. If $(F_k)_{k\in\mathbb{N}}$ is an $\mathcal{E}$-nest, then $h_{F_k^c} \downarrow 0$ m-a.e. (*cf.* [MOR 93, 2.10(i)]). Therefore,

$$\operatorname{Cap}_{h,\varphi}(F_k^c) = \int h_{F_k^c}\varphi\, dm \longrightarrow_{k\to\infty} 0.$$

To prove the converse, suppose that $\operatorname{Cap}_{h,\varphi}(F_k^c) \longrightarrow_{k\to\infty} 0$ and that $u \in D(\mathcal{E})$ such that $\mathcal{E}_{\alpha+1}(w,u) = 0$ for all $w \in \bigcup_k D(\mathcal{E})_{F_k}$. Since $\tilde{\mathcal{E}}_{\alpha+1}^{1/2}$ is equivalent with $\tilde{\mathcal{E}}_1^{1/2}$, by the theorems of Hahn-Banach and Lax-Milgram it suffices to show that $u = 0$.

Let $g \le h$, $g \in D(\mathcal{E})$. Then by [MR 92, III. 1.6(iii)] $g_U \le h_U$ for every open set $U \subset E$ and consequently

$$(4.6) \qquad 0 \le (g_{F_k^c}, \varphi) \le (h_{F_k^c}, \varphi).$$

Hence by assumption, since $(g_{F_k^c})_{k\in\mathbb{N}}$ is decreasing by [MR 92, III. 1.5(iv)], $g_{F_k^c} \longrightarrow 0$ in $L^2(E;m)$ as $k \longrightarrow \infty$. But $\sup_k \mathcal{E}_{\alpha+1}(g_{F_k^c}, g_{F_k^c}) < \infty$ (by [MR 92, III. 1.5(iv)]); hence [MR 92, I. 2.12] implies that $g_{F_k^c} \longrightarrow 0$ weakly in $(D(\mathcal{E}), \tilde{\mathcal{E}}_{\alpha+1})$. Now we specify g as $G_{\alpha+1}(1_A h)$ where $A \in \mathcal{B}(E)$. Since h is α-excessive, we have that $g \le h$ and that

$$0 = \mathcal{E}_{\alpha+1}(g - g_{F_k^c}, u) \longrightarrow \mathcal{E}_{\alpha+1}(g, u) = \int_A hu\, dm.$$

Because $A \in \mathcal{B}(E)$ was arbitrary and $h > 0$ m-a.e., it follows that $u = 0$. ∎

COROLLARY 4.5. *(i) Let S be a countable family of $\mathcal{E}$-quasi-continuous functions on E. Then there exists an $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$ such that $S \subset C(\{F_k\})$.*

(ii) Let f be an $\mathcal{E}$-quasi-continuous function such that $f \ge 0$ m-a.e. on an open set U. Then $f \ge 0$ $\mathcal{E}$-q.e. on U.

PROOF. (i) One uses Theorem 4.4 and follows the argument of [F 80, Theorem 3.1.2(i)] (see also [MR 92, III. 3.3]).

(ii) One uses Theorem 4.4 and follows the argument of [BH 91, Proposition 8.1.6]. ∎

We introduce the following assumption for the discussion below.

ASSUMPTION 4.6. There exists an m-a.e. strictly positive m-version h of an α-excessive function in $D(\mathcal{E})$ and an $\mathcal{E}$-quasi-continuous function g such that $h \leq g$ m-a.e.

REMARK. Assumption 4.6 is trivially fulfilled if h is bounded or h itself is $\mathcal{E}$-quasi-continuous. In particular, any semi-Dirichlet form satisfies Assumption 4.6 with *e.g.* $h := G_1\varphi$ where $\varphi \in L^2(E;m)$, $0 < \varphi \leq 1$ m-a.e. and $g \equiv 1$.

LEMMA 4.7. *Let $\varphi \in L^2(E;m)$, $\varphi > 0$ m-a.e. Assume that h and g are specified as in Assumption 4.6. Let $u \in D(\mathcal{E})$ such that it has an $\mathcal{E}$-quasi-continuous m-version $\tilde{u}$. Then for all $\lambda > 0$*

$$\mathrm{Cap}_{h,\varphi}(\{|\tilde{u}| > \lambda g\}) \leq \frac{K_\alpha^2}{\lambda}\mathcal{E}_\alpha(u,u)^{1/2}\mathcal{E}_\alpha(\hat{G}_\alpha\varphi, \hat{G}_\alpha\varphi). \tag{4.7}$$

PROOF. Let $(F_k)_{k\in\mathbb{N}}$ be an $\mathcal{E}$-nest such that $\tilde{u}, g \in C(\{F_k\})$. For $\lambda > 0$ we set $U_k := \{|\tilde{u}| > \lambda g\} \cup F_k^c$ and $u_k := \lambda^{-1}|u| + h_{F_k^c}$, $k \in \mathbb{N}$. Then U_k is open, $u_k \in D(\mathcal{E})$ and $u_k \geq h$ m-a.e. on U_k. Now (4.7) is verified by the same argument as in the proof of [MR 92, III. 3.4]. ∎

The following is crucial for our further discussion.

PROPOSITION 4.8. *Suppose that Assumption 4.6 is fulfilled. Let $u_n \in D(\mathcal{E})$ which have $\mathcal{E}$-quasi-continuous m-version $\tilde{u}_n$, $n \in \mathbb{N}$, such that $u_n \longrightarrow_{n\to\infty} u \in D(\mathcal{E})$* w.r.t. $\tilde{\mathcal{E}}_1^{1/2}$-*norm. Then there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ and an $\mathcal{E}$-quasi-continuous m-version $\tilde{u}$ of u such that $(\tilde{u}_{n_k})_{k\in\mathbb{N}}$ converges $\mathcal{E}$-q.e. to $\tilde{u}$. If there exists an $\mathcal{E}$-nest consisting of compact sets then $(\tilde{u}_{n_k})_{k\in\mathbb{N}}$ converges $\mathcal{E}$-quasi-uniformly to $\tilde{u}$.*

PROOF. Let h and g be specified as in Assumption 4.6. Without loss of generality we may assume that $g \geq 1$ (otherwise we may use $g \vee 1$ in place of g). By (4.7) we may choose a subsequence $(u_{n_i})_{i\in\mathbb{N}}$ such that

$$\mathrm{Cap}_{h,\varphi}\{|\tilde{u}_{n_{i+1}}g^{-1} - \tilde{u}_{n_i}g^{-1}| > 2^{-i}\} \leq 2^{-i} \quad \text{for all } i \in \mathbb{N}.$$

Let $A_k := \bigcup_{i\geq k}\{|\tilde{u}_{n_{i+1}}g^{-1} - \tilde{u}_{n_i}g^{-1}| > 2^{-i}\}$, $k \in \mathbb{N}$. Let $(F'_k)_{k\in\mathbb{N}}$ be an $\mathcal{E}$-nest such that $g, \tilde{u}_n \in C(\{F'_k\})$ for all $n \in \mathbb{N}$. We define

$$F_k := F'_k \cap A_k^c, \quad k \in \mathbb{N}.$$

Then by Theorem 4.4 $(F_k)_{k\in\mathbb{N}}$ is an $\mathcal{E}$-nest and $(\tilde{u}_{n_i}g^{-1})$ converges uniformly on each F_k. Set

$$f(z) := \begin{cases} \lim_{i\to\infty}\tilde{u}_{n_i}(z)g^{-1}(z) & \text{if } z \in \bigcup_{k\geq 1} F_k \\ 0 & \text{else}. \end{cases}$$

Then f is continuous on each F_k. Let $\tilde{u} := fg$. Then $\tilde{u}$ is $\mathcal{E}$-quasi-continuous and $(\tilde{u}_{n_i})_{i\in\mathbb{N}}$ converges to $\tilde{u}$ $\mathcal{E}$-q.e. Since $\tilde{\mathcal{E}}_1^{1/2}$-convergence implies the m-a.e. convergence of a subsequence and $m(\bigcap_{k\geq 1} F_k^c) = 0$ by Theorem 4.4, $\tilde{u}$ is an m-version of u. If an $\mathcal{E}$-nest consisting of compact sets exists, we can choose F'_k, $k \in \mathbb{N}$, above compact. Hence each F_k is compact and g is bounded on each F_k. Hence $\tilde{u}_{n_i} \longrightarrow_{i\to\infty} \tilde{u}$ $\mathcal{E}$-quasi-uniformly. ∎

DEFINITION 4.9 (*cf.* [MR 92, IV. 3.1]). A positivity preserving coercive form $\big(\mathcal{E}, D(\mathcal{E})\big)$ on $L^2(E;m)$ is called *quasi-regular* if:

(i) There exists an $\mathcal{E}$-nest $(E_k)_{k\in\mathbb{N}}$ consisting of compact sets.

(ii) There exists an $\tilde{\mathcal{E}}_1^{1/2}$-dense subset of $D(\mathcal{E})$ whose elements have $\mathcal{E}$-quasi-continuous m-versions.

(iii) There exist $u_n \in D(\mathcal{E})$, $n \in \mathbb{N}$, having $\mathcal{E}$-quasi-continuous m-versions $\tilde{u}_n$, $n \in \mathbb{N}$, and an $\mathcal{E}$-exceptional set $N \subset E$ such that $\{\tilde{u}_n \mid n \in \mathbb{N}\}$ separates the point of $E \setminus N$.

(iv) There exists an $\mathcal{E}$-q.e. strictly positive $\mathcal{E}$-quasi-continuous m-version h of an α-excessive function in $D(\mathcal{E})$ for some $\alpha \in]0,\infty[$.

REMARK 4.10. If $\big(\mathcal{E}, D(\mathcal{E})\big)$ is a semi-Dirichlet form, then condition (iv) in Definition 4.9 is a consequence of Definition 4.9(i)–(iii). See [MOR 93, 2.18] or [MR 92, III. 3.6, IV. 3.3].

Now we give an alternative description of condition (iv) in Definition 4.9.

PROPOSITION 4.11. *Let $\big(\mathcal{E}, D(\mathcal{E})\big)$ be a positivity preserving coercive form on $L^2(E;m)$ satisfying Definition 4.9(i)–(iii). Then $\big(\mathcal{E}, D(\mathcal{E})\big)$ is quasi-regular if and only if Assumption 4.6 is fulfilled and*

(4.8) *one can choose $(u_n)_{n\in\mathbb{N}}$ and $N \subset E$ in Definition 4.9(iii) such that* $(E \setminus N) \subset \bigcup_{n\geq 1}\{\tilde{u}_n \neq 0\}$.

PROOF. The "only if"-part is clear. The "if"-part is a consequence of the following lemma which also is of its own interest. ∎

LEMMA 4.12. *Let $\big(\mathcal{E}, D(\mathcal{E})\big)$ be a positivity preserving coercive form.*

(i) If $\big(\mathcal{E}, D(\mathcal{E})\big)$ satisfies Definition 4.9(ii) and Assumption 4.6, then every $u \in D(\mathcal{E})$ has an $\mathcal{E}$-quasi-continuous m-version $\tilde{u}$.

(ii) If $\big(\mathcal{E}, D(\mathcal{E})\big)$ satisfies Definition 4.9(ii), Assumption 4.6, and (4.8), then for any $\alpha > 0$ we can find a function h satisfying Definition 4.9(iv).

PROOF. (i) follows from Proposition 4.8. We now prove (ii). Let $u_n, \tilde{u}_n$ be as specified in (4.8). We set

$$f_n := \sum_{i=1}^{n} 2^{-i}\big(\mathcal{E}_1(u_i,u_i)^{1/2} + 1\big)^{-1}|u_i|, \quad n \in \mathbb{N},$$
$$\tilde{f}_n := \sum_{i=1}^{n} 2^{-i}\big(\mathcal{E}_1(u_i,u_i)^{1/2} + 1\big)^{-1}|\tilde{u}_i|, \quad n \in \mathbb{N}.$$

Clearly, $f_n \in D(\mathcal{E})$ and $\tilde{f}_n$ is an $\mathcal{E}$-quasi-continuous m-version of f_n. Moreover, $(f_n)_{n\in\mathbb{N}}$ converges to some $f \in D(\mathcal{E})$ w.r.t. $\tilde{\mathcal{E}}_1^{1/2}$-norm. $(\tilde{f}_n)_{n\in\mathbb{N}}$ is increasing pointwise on E. Therefore, $\tilde{f}_n \uparrow \tilde{f} := \sum_{i=0}^{\infty} 2^{-i}(\mathcal{E}_1(u_i,u_i)^{1/2}+1)^{-1}|\tilde{u}_i|$ pointwise. On the other hand, by Proposition 4.8 there exists a subsequence $(\tilde{f}_{n_k})_{k\in\mathbb{N}}$ such that $(\tilde{f}_{n_k})_{k\in\mathbb{N}}$ $\mathcal{E}$-q.e.-converges to an $\mathcal{E}$-quasi-continuous m-version of f. Therefore, $\tilde{f}$ is an $\mathcal{E}$-quasi-continuous m-version of f. Clearly, $\tilde{f} > 0$ on $E \setminus N$ where N is an $\mathcal{E}$-exceptional set as specified in (4.8). Let $g := G_\alpha f$. Then $g \in D(\mathcal{E})$ and g is α-excessive. Since $\beta G_{\beta+\alpha} f \longrightarrow_{\beta\to\infty} f$ w.r.t. $\tilde{\mathcal{E}}_1^{1/2}$, by Proposition 4.8 there exists a sequence $(\beta_n)_{n\in\mathbb{N}}$ such that for $g_n := \beta_n G_{\beta_n+\alpha} f$, $n \in \mathbb{N}$, the corresponding $\mathcal{E}$-quasi-continuous m-versions $\tilde{g}_n$ converge $\mathcal{E}$-q.e. to $\tilde{f}$. By the resolvent equation $G_\alpha f \geq G_{\beta+\alpha} f$ m-a.e. and hence by Corollary 4.5(ii)

$$h := \tilde{g} \geq \beta_n^{-1}\tilde{g}_n \quad \mathcal{E}\text{-q.e.}$$

Consequently $h > 0$ $\mathcal{E}$-q.e. and hence h satisfies Definition 4.9(iv). ∎

The following proposition shows that for a given quasi-regular positivity preserving coercive form there are many different choices of functions h satisfying Definition 4.9(iv).

PROPOSITION 4.13. *Let $\big(\mathcal{E}, D(\mathcal{E})\big)$ be a positivity preserving coercive form such that there exists h satisfying Definition 4.9(iv). Then for any α-excessive function g in $D(\mathcal{E})$, $g > 0$ m-a.e., which has an $\mathcal{E}$-quasi-continuous m-version $\tilde{g}$, it follows that $\tilde{g} > 0$ $\mathcal{E}$-q.e.*

PROOF. Let $(F_k)_{k\in\mathbb{N}}$ be an $\mathcal{E}$-nest such that $h, g \in C(\{F_k\})$. For $k \in \mathbb{N}$ we set $U_k := \{\tilde{g} < \frac{1}{k}h\} \cup F_k^c$. Then U_k is open. Without loss of generality we may assume that g is strictly positive pointwise. Let $g_k := \frac{1}{k}h + g_{F_k^c}$ where $g_{F_k^c}$ is the $(\alpha+1)$-reduced function of g on F_k^c specified by (4.3). Then $g_k \geq g$ m-a.e. on U_k and g_k is an $(\alpha+1)$-excessive function in $D(\mathcal{E})$. Therefore, by (4.3) $g_k \geq g_{U_k}$ m-a.e. and by (4.4) and Theorem 4.4 for $\varphi \in L^2(E;m)$, $\varphi > 0$ m-a.e.,

$$\mathrm{Cap}_{g,\varphi}(U_k) = (g_{U_k}, \varphi) \leq \frac{1}{k}(h,\varphi) + \mathrm{Cap}_{g,\varphi}(F_k^c) \longrightarrow 0 \quad \text{as } k \longrightarrow \infty.$$

(Note that here $\mathrm{Cap}_{g,\varphi} = (\alpha+1)\text{-}\mathrm{Cap}_{g,\varphi}$; *cf.* (4.4), (4.5)). Set $F_k' := E \setminus U_k$, $k \in \mathbb{N}$. It follows from Theorem 4.4 that $(F_k')_{k\in\mathbb{N}}$ is an $\mathcal{E}$-nest. Obviously, we have that $\tilde{g} \geq \frac{1}{k}h > 0$ on F_k'. Therefore, $\tilde{g} > 0$ $\mathcal{E}$-q.e. ∎

The following theorem is now an easy consequence, but will be important below.

THEOREM 4.14. *A positivity preserving coercive form $\big(\mathcal{E}, D(\mathcal{E})\big)$ on $L^2(E;m)$ is quasi-regular if and only if for one (hence every) m-a.e. strictly positive function $h \in D(\mathcal{E})$, which is α-excessive for some $\alpha > 0$, there exists an $\mathcal{E}$-q.e. strictly positive $\mathcal{E}$-quasi-continuous m-version $\tilde{h}$ and the corresponding h-transform $\big(\mathcal{E}^h, D(\mathcal{E}^h)\big)$ is quasi-regular.*

PROOF. The assertion follows from the definition of quasi-regularity, Proposition 4.13 and Proposition 4.2. ∎

5. **h-associated processes and examples.** For the concept of an *(m-tight) special standard process* occurring in Definition 5.1 below we refer *e.g.* to [MR 92, IV. 1.13]. We assume from now on that $\mathcal{B}(E)$ is generated by the continuous functions on E.

DEFINITION 5.1. Let $\big(\mathcal{E}, D(\mathcal{E})\big)$ be a positivity preserving coercive form on $L^2(E;m)$ and let h be a strictly positive and finite valued m-version of an α-excessive function in $D(\mathcal{E})$ (for some $\alpha \geq 0$). $\big(\mathcal{E}, D(\mathcal{E})\big)$ is called *h-associated* with a special standard process $\mathbf{M} := (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (P_z)_{z\in E_\Delta})$ if for all $t > 0$

(5.1) $he^{\alpha t}E.[(\hat{f}h^{-1})(X_t)]$ is an m-version of $T_t f$ for any m-version $\hat{f}$ of $f \in L^2(E;m)$.

$\big(\mathcal{E}, D(\mathcal{E})\big)$ is called *properly h-associated* with $\mathbf{M}$ if, in addition, for all $t > 0$

(5.2) $hE.[(\hat{f}h^{-1})(X_t)]$ is $\mathcal{E}$-quasi-continuous for any m-version $\hat{f}$ of $f \in L^2(E;m)$.

Here E_z denotes the expectation w.r.t. P_z, $z \in E$.

Recall that according to [MR 92, Chapter IV] $f(\Delta) := 0$ for $f: E \longrightarrow \mathbb{R}$, where Δ is the cemetery. We can now prove the main result of this paper.

THEOREM 5.2. *Let $\big(\mathcal{E}, D(\mathcal{E})\big)$ be a positivity preserving coercive form on $L^2(E;m)$. Then a necessary and sufficient condition for $\big(\mathcal{E}, D(\mathcal{E})\big)$ to be properly $\tilde{h}$-associated with an m-tight special standard process for some strictly positive and finite valued m-version $\tilde{h}$ of some (hence every) m-a.e. strictly positive α-excessive function $h \in D(\mathcal{E})$, is that $\big(\mathcal{E}, D(\mathcal{E})\big)$ is quasi-regular. In this case $\tilde{h}$ is $\mathcal{E}$-quasi-continuous and may be replaced by any other $\mathcal{E}$-quasi-continuous m-version of h.*

PROOF. Suppose that $\big(\mathcal{E}, D(\mathcal{E})\big)$ is quasi-regular. Then by Theorem 4.14 for any $\alpha > 0$ and any m-a.e. strictly positive α-excessive function $h \in D(\mathcal{E})$ we can find a strictly positive and finite valued $\mathcal{E}$-quasi-continuous m-version $\tilde{h}$. By Theorems 3.5 and 4.14 $\big(\mathcal{E}^h_\alpha, D(\mathcal{E}^h)\big)$ is a quasi-regular semi-Dirichlet form, hence by [MOR 93, 3.8] there exists an m-tight special standard process $\mathbf{M} := \big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (P_z)_{z\in E_\Delta}\big)$ properly associated with $\big(\mathcal{E}^h_\alpha, D(\mathcal{E}^h)\big)$. By (3.3) $\mathbf{M}$ satisfies (5.1) and also (5.2). Hence $\big(\mathcal{E}, D(\mathcal{E})\big)$ is properly $\tilde{h}$-associated with $\mathbf{M}$. Conversely, assume that for some strictly positive and finite valued m-version h of an m-a.e. strictly positive α-excessive function in $D(\mathcal{E})$, $\big(\mathcal{E}, D(\mathcal{E})\big)$ is properly h-associated with an m-tight special standard process $\mathbf{M} := \big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (P_z)_{z\in E_\Delta}\big)$. Then by [MOR 93, 3.9] $\big(\mathcal{E}^h_\alpha, D(\mathcal{E}^h)\big)$ is a quasi-regular semi-Dirichlet form on $L^2(E; h^2 \cdot m)$ and $\mathbf{M}$ is properly associated with $\big(\mathcal{E}^h_\alpha, D(\mathcal{E}^h)\big)$, that is,

(5.3) $E.[\hat{f}(X_t)]$ is an $\mathcal{E}^h_\alpha$-quasi-continuous $h^2 \cdot m$-version of $e^{-\alpha t}T^h_t f$ for any $(h^2 \cdot m)$-version $\hat{f}$ of $f \in L^2(E; h^2 \cdot m)$.

Therefore, if we can prove that h is $\mathcal{E}$-quasi-continuous, then by Theorem 4.14 $\big(\mathcal{E}, D(\mathcal{E})\big)$ is quasi-regular. To this end we set $t_n := \frac{1}{n}$, $f_n := hE.[hh^{-1}(X_{t_n})] = hE.[1_E(X_{t_n})]$, and $g_n := E.[1_E(X_{t_n})]$. Then by (5.2), (5.3) and Proposition 4.2, $f_n, g_n, n \in \mathbb{N}$ are all $\mathcal{E}$-quasi-continuous. By Corollary 4.5 (i) we may take an $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$ such that $f_n, g_n \in C(\{F_k\})$

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for all $n \in \mathbb{N}$. We claim now that $h \in C(\{F_k\})$. To show this, fix k and let $z \in F_k$ be arbitrary. Since $E_z[1_E(X_0)] = 1$, by the right continuity of $\mathbf{M}$ we can find t_n such that $g_n(z) := E_z[1_E(X_{t_n})] \geq \varepsilon > 0$. Thus, by continuity there is an open neighbourhood U of z such that $g_n \geq \frac{\varepsilon}{2}$ on $U \cap F_k$. Consequently, the restriction of h to $U \cap F_k$ is continuous because $h|_{U\cap F_k} = f_n g_n^{-1}|_{U\cap F_k}$. Since $z \in F_k$ was arbitrary, h is continuous on F_k, which completes the proof. ∎

REMARK 5.3. The same methods that led to the proof of Theorem 5.2 allow a detailed study of the process h-associated with a quasi-regular positivity preserving coercive form using the results in [MR 92] (*cf.* in particular also [AMR 93b]).

EXAMPLES 5.4. (i) Consider the example discussed in Subsection 2.1. We want to show that $\big(\mathcal{E}_\alpha, D(\mathcal{E}_\alpha)\big)$, specified in Theorem 2.2, is quasi-regular. Taking $E_k \subset U$, E_k compact such that E_k is contained in the interior of E_{k+1} for every $k \in \mathbb{N}$ and such that $\bigcup E_k = U$, we see that Definition 4.9(i) is satisfied. Definition 4.9(ii), (iii) hold since $C_0^\infty(U)$ is $\tilde{\mathcal{E}}_\alpha^{1/2}$-dense in $D(\mathcal{E}_\alpha)$ resp. $C_0^\infty(U)$ is separable and separates the points of U. To show Definition 4.9(iv) by Lemma 4.12 we only have to check whether Assumption 4.6 holds since (4.8) is obvious since $C_0^\infty(U) \subset D(\mathcal{E}_\alpha)$. So let $h \in D(\mathcal{E}_\alpha)$, h $(\alpha+1)$-excessive and strictly positive m-a.e. It suffices to show that h has an $\mathcal{E}_\alpha$-quasi-continuous dx-version $\tilde{h}$. Consider the Dirichlet form $\big(\mathcal{E}^1, D(\mathcal{E}^1)\big)$ on $L^2(U; \sigma dx)$ defined by

$$\mathcal{E}^1(u,v) := \mathcal{E}_\alpha(u,v) - \int \big(\langle \nabla u, \underline{b}^{(1)}\rangle v + u\langle \nabla v, \underline{d}^{(1)}\rangle\big)\, dx;$$
$$u, v \in D(\mathcal{E}^1) := D(\mathcal{E}_\alpha)$$

(*cf.* Remark 2.9). By the same arguments as above and Remark 4.10 $\big(\mathcal{E}^1, D(\mathcal{E}^1)\big)$ is quasi-regular. In particular, $h \in D(\mathcal{E}_\alpha) = D(\mathcal{E}^1)$ has an $\mathcal{E}^1$-quasi-continuous dx-version $\tilde{h}$. But (2.11) implies that $\tilde{h}$ is also $\mathcal{E}_\alpha$-quasi-continuous. Therefore, $\big(\mathcal{E}_\alpha, D(\mathcal{E}_\alpha)\big)$ is quasi-regular and Theorem 5.2 applies.

(ii) Consider the example in Subsection 2.2. We first note that the closure $\big(\mathcal{E}_A, D(\mathcal{E}_A)\big)$ of $\big(\mathcal{E}_A, \mathcal{F}C_b^\infty\big)$ on $L^2(E;\mu)$ is by [MR 92, IV. 4.3] a quasi-regular Dirichlet form. Hence Lemma 2.8(ii) implies that $\big(\mathcal{E}_\alpha, D(\mathcal{E}_\alpha)\big)$, specified in Subsection 2.2, has the properties in Definition 4.9(i)–(iii) and (4.8) (since clearly $D(\mathcal{E}_\alpha) = D(\mathcal{E}_A)$). Consequently, to show Definition 4.9(iv) it remains to check Assumption 4.6. But again since $D(\mathcal{E}_\alpha) = D(\mathcal{E}_A)$ and by Lemma 2.8(ii) every α-excessive, m-a.e. strictly positive $h \in D(\mathcal{E}_\alpha)$ has an $\mathcal{E}_A$-, hence an $\mathcal{E}_\alpha$-quasi-continuous dx-version $\tilde{h}$. Therefore, $\big(\mathcal{E}_\alpha, D(\mathcal{E}_\alpha)\big)$ is quasi-regular and Theorem 5.2 applies.

(iii) Consider the example in Subsection 2.3 and let $\big(\mathcal{E}^0, D(\mathcal{E}^0)\big)$ and $\big(\mathcal{E}_\alpha, D(\mathcal{E}_\alpha)\big)$ be as specified there. By [RS 93, Subsection 4(c)] the Dirichlet form $\big(\mathcal{E}^0, D(\mathcal{E}^0)\big)$ is quasi-regular. Since we have an analogue of Lemma 2.8 in this case, exactly the same arguments as in (ii) above show that $\big(\mathcal{E}_\alpha, D(\mathcal{E}_\alpha)\big)$ is quasi-regular and Theorem 5.2 applies.

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Institute of Applied Mathematics
Academia Sinica
Beijng
China

Fakultät für Mathematik
Universität Bielefeld
Postfach 100131
D-33501 Bielefeld
Germany