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GIBBS RANDOM FIELDS WITH UNBOUNDED SPINS ON UNBOUNDED DEGREE GRAPHS

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Abstract

Gibbs fields are constructed and studied which correspond to systems of real-valued spins (e.g. systems of interacting anharmonic oscillators) indexed by the vertices of unbounded degree graphs of a certain type, for which the Gaussian Gibbs fields need not be existing. In these graphs, the vertex degree growth is controlled by a summability requirement formulated with the help of a generalized Randić index. In particular, it is proven that the Gibbs fields obey uniform integrability estimates, which are then used in the study of the topological properties of the set of Gibbs fields. In the second part, a class of graphs is introduced in which the mentioned summability is obtained by assuming that the vertices of large degree are located at large distances from each other. This is a stronger version of the metric property employed in Bassalygo and Dobrushin (1986).

Keywords: Generalized Randić index; Gibbs measure; Gibbs specification; unbounded spin; unbounded degree graph; DLR equation

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1. Introduction and paper overview

1.1. Introduction

The theory of Gibbs random fields has its roots in statistical physics where they serve as mathematical models of phase transitions, e.g. in ferromagnets; see [10]. Recently, the interest in Gibbs fields has been stimulated by applications in probabilistic combinatorics, statistical inference, and image processing. Typically, such a field is a collection of dependent random variables, called *spins*, indexed by the elements of a discrete metric space (e.g. of a graph). Their joint probability distributions are defined by the families of local conditional distributions constructed by means of interaction potentials. We quote the monographs [10] and [11] as standard sources in the theory of Gibbs fields. Each spin takes values in the corresponding *single-spin* space, say $\mathcal{S}_x$. Most of the Gibbs fields constructed on general graphs have finite single-spin spaces. Perhaps, the best studied example is the Ising model where $\mathcal{S}_x = \{-1, 1\}$ for all x. By the compactness of $\mathcal{S}_x$, such Gibbs fields exist for arbitrary graphs; see [11], [12], [13], [17], and [18]. Their properties are closely related to those of random walks or corresponding percolation models; see, e.g. [12], [17], and [18]. The development of the theory of Gibbs random fields with unbounded spins, started in the late seventies in the pioneering works of Ruelle [20], Lebowitz and Presutti [16], and Cassandro *et al.* [6], was strongly motivated by

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physical applications, especially in Euclidean quantum field theory (see, e.g. [21]). Since that time, such random fields have been extensively studied; see, e.g. the bibliographical notes in [19]. However, the results obtained in all these works were restricted to the case where the underlying metric space is a simple cubic lattice $\mathbb{Z}^d$. In [15] and [19], the theory of Gibbs random fields was extended to unbounded spin systems living on more general discrete metric spaces, including graphs of bounded degree. In this context, we also mention the paper by Hattori *et al.* [14], who studied a Gaussian field on a bounded degree graph. However, in the case where both the spins and the vertex degrees are unbounded, all methods used in the abovementioned works cannot be applied.

In the present paper we develop a new method of constructing Gibbs random fields with unbounded spins (we take $\mathcal{S}_x = \mathbb{R}$ for all x), which can also be applied to unbounded degree graphs of a certain kind. In such graphs, the degree growth is controlled by a summability requirement formulated with the help of a generalized Randić index; see [7]. By means of this method we construct and study such random fields and analyze the influence of the graph geometry on their stability. To the best of the authors' knowledge, the present study is the first attempt in such a direction. Our motivation behind doing this study is as follows.

- Random fields on Riemannian manifolds, especially those associated with the corresponding Laplace–Beltrami operators (cf. [8]) can be approximated by their discrete versions living on appropriate graphs [9]. This also includes the case of quantum fields in curved spacetime; see [1] and [22].
- As the degree of the graph can be related to the curvature of the corresponding manifold, the use of unbounded degree graphs essentially extends the class of manifolds that can be approximated in the above sense.
- Another application is the description of systems of interacting oscillators located at vertices of an infinite graph—the so called oscillating networks; see Section 14 of [4]. We also refer the reader to survey [5], where other relevant physical models can be found.

In the present paper we construct Gibbs random fields, derive exponential integrability estimates and support properties for such fields, and present a concrete family of unbounded degree graphs, which can serve as underlying graphs for our model. The essential property of these graphs is that the vertices of large degrees are located at large path distances from each other. Similar graphs were employed in [3] to study Gibbs random fields on $\mathbb{Z}^d$ with finite single-spin spaces and random interactions. We plan to continue investigating the model introduced here in forthcoming papers. In particular, we are going to study the problem of uniqueness of Gibbs random fields, as well as the ergodicity properties of the corresponding stochastic dynamics. Another direction where the technique developed here can be of use is the study of Gibbs fields with unbounded spins and unbounded random interactions.

1.2. Paper overview

The model we deal with is the triple (G, W, V), where $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ is a graph, and $W : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $V : \mathbb{R} \to \mathbb{R}$ are functions (potentials). The properties of the triple (G, W, V) are specified below in Assumption 1. This triple determines the heuristic Hamiltonian

$$H(\omega) = \sum_{\langle x,y\rangle} W(\omega(x), \omega(y)) + \sum_{x} V(\omega(x)), \tag{1}$$

where the first and second sums are taken over all edges and vertices, respectively, of the graph.

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For this model, Gibbs random fields are defined as probability measures on the configuration space $\Omega = \mathbb{R}^{\mathsf{V}}$. In contrast to the case of bounded spins, it is unrealistic to describe all Gibbs measures of an unbounded spin system without assuming a priori any of their properties. Thus, among all Gibbs measures corresponding to (1) we distinguish those that have a prescribed support property, i.e. such that $\mu(\Omega^{\mathrm{t}}) = 1$ for an a priori chosen $\Omega^{\mathrm{t}} \subset \Omega$. We introduce a scale of such sets Ω^{t}, which are weighted L^p spaces on the vertex set V. In Theorem 1, we show that the sets of Gibbs measures supported by such L^p spaces of configurations are nonvoid and locally setwise compact. Here we also show that each Gibbs measure obeys important integrability estimates, the same for all such measures. In Theorems 2 and 3, these results are modified and extended. First we prove that the sets of Gibbs measures obtained in Theorem 1 are also weakly compact provided that the interaction potential $W\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous. Then in Theorem 3 we make precise the support properties of the Gibbs measures. These results are valid for any graph possessing the summability specified in Assumption 1. To provide a nontrivial example of unbounded degree graphs with this property, in the second part of the paper we introduce a new class of such graphs, which we believe is interesting in its own right. This class is characterized by the following property (cf. (46) and (45)). For vertices x and y, such that their degrees, $n(x)$ and $n(y)$, exceed some threshold value, the path distance is supposed to obey a 'repulsion' condition,

$$\rho(x, y) \geq \phi(\max\{n(x), n(y)\}), \tag{2}$$

where ϕ is a given increasing function. In such graphs, every vertex x has the property that

$$\sup_{\{y\colon \rho(x,y)\leq N\}} n(y) \leq \phi^{-1}(2N),$$

whenever N exceeds some integer N_x, specific for this x. By means of this property, for $\phi(b) = \upsilon \log b[\log\log b]^{1+\varepsilon}$, $\upsilon, \varepsilon > 0$, we obtain the estimate

$$\sum_{\{y\colon \rho(x,y)=N\}} [n(y)]^{1+\theta} \leq \exp(aN),$$

which holds for any $\theta > 0$ and an appropriate $a > 0$, whenever $N \geq N_x$. In Theorem 4, we show that the latter estimate implies the required summability (5).

The rest of the paper is organized as follows. In the first part, we place emphasis on the probabilistic nature of the problem, whereas the second part, Section 4, is devoted to the graph-theoretical aspects of the problem. In Section 2 we specify the class of models by imposing conditions on the graph and on the potentials. The only essential condition imposed on G is the summability (5). The potentials are supposed to obey quite standard stability requirements only. We note, however, that the stability condition (7) is slightly stronger than that with $q = 2$, typical for graphs of bounded degree. In view of this fact, the Gaussian case is not covered by our theory. Thereafter, we formulate Theorem 1 and its modifications—Theorems 2 and 3. The proof of Theorem 3 follows from the estimates obtained in Theorem 1. The proof of Theorem 1, which is the main technical component of the first part of the paper, is given in Section 3, where we also discuss the proof of Theorem 2. The proof of Theorem 1 is preceded by a number of lemmas, in which we elaborate the corresponding tools. The key element here is Lemma 2, the proof of which crucially employs the summability (5). In Section 4 we introduce and describe the class of graphs possessing property (2), which, by Theorem 4, can serve as underlying graphs for our model.

2. The setup and the main results

2.1. The model

The underlying graph $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ of model (1) is supposed to be undirected and countable. Two adjacent vertices $x, y \in \mathsf{V}$ are also called neighbors. In this case, we write $x \sim y$ and $\langle x, y\rangle \in \mathsf{E}$. The *degree* of $x \in \mathsf{V}$, denoted by $n(x)$, is the cardinality of the neighborhood of x, that is, of the set $\{y : y \sim x\}$. We use the shorthand

$$\sum_x = \sum_{x\in\mathsf{V}}, \qquad \sup_x = \sup_{x\in\mathsf{V}}, \qquad \sum_{y\sim x} = \sum_{\{y\in\mathsf{V}\colon y\sim x\}}.$$

The graph is assumed to be *locally finite*, which means that $n(x) \in \mathbb{N}$ for any x. At the same time, we assume that $\sup_x n(x) = +\infty$, which is reflected in the title of the paper. Of course, our results are trivially valid for bounded degree graphs.

A sequence $\vartheta = \{x_0, x_1, \dots, x_n\}$, such that $x_k \sim x_{k+1}$ for all $k = 0, \dots, n-1$, is called a *path*. Herein, some of the vertices may be repeated. The path connects its *endpoints* x_0 and x_n; it *leaves* the vertices $x_0, \dots, x_{n-1}$ and *enters* $x_1, \dots, x_n$. The number of left vertices, denoted by $\|\vartheta\|$, is called the *length* of the path. For $x, y \in \mathsf{V}$, by $\vartheta(x, y)$ we denote a path, whose endpoints are x and y. We assume that G is connected, which means that there exists a path $\vartheta(x, y)$ for every x and y. The path distance $\rho(x, y)$ is set to be the length of the shortest $\vartheta(x, y)$. It is a metric on G by means of which, for a certain $o \in \mathsf{V}$ and $\alpha > 0$, we introduce

$$w_\alpha(x) = \exp(-\alpha\rho(o, x)), \qquad x \in \mathsf{V}. \tag{3}$$

For $\theta > 0$, we also set

$$m_\theta(x) = \sum_{y\sim x}[n(x)n(y)]^\theta, \qquad x \in \mathsf{V}. \tag{4}$$

In mathematical chemistry, the sum of terms $[n(x)n(y)]^\theta$ taken over the edges $\langle x, y\rangle$ of a finite tree is known under the name *generalized Randić* or *connectivity* index; see, e.g. [7].

The remaining properties of the model are now summarized.

Assumption 1. *The triple* (G, W, V) *is subject to the following conditions:*

(i) *the graph* G *is such that*

$$\Theta(\alpha, \theta) := \sum_x m_\theta(x)w_\alpha(x) < \infty \tag{5}$$

for some positive α *and* θ*;*

(ii) $W\colon \mathbb{R}^2 \to \mathbb{R}$ *is measurable, symmetric, and such that*

$$|W(u, v)| \le \tfrac{1}{2}(I_W + J_W(u^r + v^r)) \tag{6}$$

for some positive I_W, J_W, r, *and all* $u, v \in \mathbb{R}$*;*

(iii) $V\colon \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ *is measurable, the set* $\{u\colon V(u) < +\infty\}$ *is of positive Lebesgue measure, and the estimate*

$$V(u) \ge a_V|u|^q - c_V \tag{7}$$

holds for all $u \in \mathbb{R}$ *and some* $a_V, c_V > 0$, $q > r + r/\theta$, *with* θ *being the same as in (i).*

A necessary condition for a Gibbs random field to exist is that the restriction of the corresponding Hamiltonian (1) to any finite $\Lambda \subset \mathsf{V}$ be bounded below, uniformly in Λ. This property is often referred to as the (global) stability of the model. Assumption 1(ii) and (iii) provide such a stability. For bounded degree graphs, it is enough to demand that $q > r$, in contrast to $q > r + r/\theta$ in Assumption 1(iii). To illustrate the destabilizing effect of the graph, let us consider the Gaussian Gibbs random field, which corresponds to the Hamiltonian

$$H(\omega) = \sum_{\langle x,y\rangle} J\omega(x)\omega(y) + \frac{a}{2}\sum_{x}[\omega(x)]^2, \qquad a > 0.$$

This field exists only if all local restrictions of the above quadratic form are positive definite. This in turn is possible only if the vertex degree is bounded. Otherwise, we have to make the second term $[\omega(x)]^q$, with q bigger than 2. Then the global stability will be secured for big enough q, depending on the graph. For our graphs, it is enough to take $q > 2 + 2/\theta$.

Let us give now an example of a graph, which has property (5). Herein, a ray is an infinite sequence, $\{x_0, x_1, \ldots\}$, such that each two consecutive vertices are adjacent.

Example 1. Let $\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ be an increasing sequence. The graph G is supposed to be a tree, which consists of the 'main' ray $\{x_1, x_2, \ldots\}$, comprising 'main' vertices for which $n(x_k) = n_k$, and of the 'auxiliary' rays $\{x_k, y_{k1}, y_{k2}, \ldots\}$, $k \in \mathbb{N}$, such that $n(y_{kl}) = 2$. The vertex x_1 is the root for $n_1 - 1$ 'auxiliary' rays, whereas, for the remaining x_k, this number is $n_k - 2$. For such a graph, condition (5) is equivalent to

$$\sum_{k=1}^{\infty}(n_k n_{k+1})^{\theta}\mathrm{e}^{-\alpha k} < \infty \quad \text{and} \quad \sum_{k=1}^{\infty} n_k^{1+\theta}\mathrm{e}^{-\alpha k} < \infty.$$

If $n_k \le n_0 \exp(\beta k)$, the above conditions are satisfied provided that

$$\beta < \min\left\{\frac{\alpha}{2\theta}; \frac{\alpha}{1+\theta}\right\}.$$

2.2. The basic result

Following the standard Dobrushin–Lanford–Ruelle (DLR) route (see [10]), Gibbs random fields for our model are defined as probability measures on the measurable space $(\Omega, \mathcal{B}(\Omega))$. Here $\Omega = \mathbb{R}^{\mathsf{V}}$ is the configuration space, equipped with the product topology and with the corresponding Borel σ-field $\mathcal{B}(\Omega)$. By $\mathcal{P}(\Omega)$ we denote the set of all probability measures on $(\Omega, \mathcal{B}(\Omega))$.

In the sequel, by writing $\Lambda \Subset \mathsf{V}$ we mean that Λ is a finite and nonvoid set of vertices. A property related to such a subset is called *local*. As usual, $\Lambda^{\mathrm{c}} = \mathsf{V} \setminus \Lambda$ stands for the complement of $\Lambda \subset \mathsf{V}$. For $\Lambda \Subset \mathsf{V}$ and $\omega \in \Omega$, by ω_Λ we denote the restriction of ω to Λ, and use the decomposition $\omega = \omega_\Lambda \times \omega_{\Lambda^{\mathrm{c}}}$. For such Λ, we set $\Omega_\Lambda = \mathbb{R}^{|\Lambda|}$ and denote by $\mathcal{B}(\Omega_\Lambda)$ the corresponding Borel σ-field. A function $f\colon \Omega \to \mathbb{R}$ is said to be local if it is $\mathcal{B}(\Omega_\Lambda)/\mathcal{B}(\mathbb{R})$-measurable for some $\Lambda \Subset \mathsf{V}$. By $\mathcal{F}_{\mathrm{loc}}$ we denote the set of all bounded local functions. The algebra of local events is

$$\mathcal{B}_{\mathrm{loc}} = \bigcup_{\Lambda \Subset \mathsf{V}} \mathcal{B}(\Omega_\Lambda).$$

For an appropriate $f\colon \Omega \to \mathbb{R}$ and $\mu \in \mathcal{P}(\Omega)$, we write

$$\mu(f) = \int_{\Omega} f(\omega)\mu(\mathrm{d}\omega), \tag{8}$$

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if the integral makes sense. In the sequel, we mostly use the following topology on $\mathcal{P}(\Omega)$, cf. Definition 4.2 of [10, p. 59].

Definition 1. The local setwise topology $\mathcal{T}_{\mathrm{loc}}$ is the weakest topology on the set $\mathcal{P}(\Omega)$ for which the evaluation maps $\mathcal{P}(\Omega) \ni \mu \mapsto \mu(A),\ A \in \mathcal{B}_{\mathrm{loc}}$, are continuous. Equivalently, a net $\{\mu_\iota\}_{\iota \in \mathcal{I}} \subset \mathcal{P}(\Omega)$ is $\mathcal{T}_{\mathrm{loc}}$-convergent to some μ if and only if $\mu_\iota(f) \to \mu(f)$ for all $f \in \mathcal{F}_{\mathrm{loc}}$.

We observe that $\mathcal{T}_{\mathrm{loc}}$ is not metrizable. In view of Assumption 1(iii), we can define the probability measure χ on $\mathbb{R}$ by

$$\chi(\mathrm{d}u) = C \exp(-V(u))\,\mathrm{d}u,$$

where $C > 0$ is the corresponding normalizing factor. Thereafter, for $\Lambda \Subset \mathsf{V}$, we set

$$\chi_\Lambda(\mathrm{d}\omega_\Lambda) = \bigotimes_{x \in \Lambda} \chi(\mathrm{d}\omega(x)), \tag{9}$$

which is a probability measure on $(\Omega_\Lambda, \mathcal{B}(\Omega_\Lambda))$. For a given $\Lambda \Subset \mathsf{V}$ and a fixed $\xi \in \Omega$, the relative local interaction energy corresponding to the Hamiltonian (1) is set to be

$$E_\Lambda(\omega_\Lambda \mid \xi) = \sum_{\{\langle x,y\rangle :\, x,y \in \Lambda\}} W(\omega(x), \omega(y)) + \sum_{\{\langle x,y\rangle :\, x \in \Lambda,\, y \in \Lambda^c\}} W(\omega(x), \xi(y)). \tag{10}$$

Then by means of this energy, for such Λ, ξ, and $A \in \mathcal{B}(\Omega)$, we define

$$\begin{aligned} \pi_\Lambda(A \mid \xi) &= \frac{1}{Z_\Lambda(\xi)} \int_{\mathbb{R}^{|\Lambda|}} \mathbf{1}_A(\omega_\Lambda \times \xi_{\Lambda^c}) \exp(-E_\Lambda(\omega_\Lambda \mid \xi)) \chi_\Lambda(\mathrm{d}\omega_\Lambda), \\ Z_\Lambda(\xi) &= \int_{\mathbb{R}^{|\Lambda|}} \exp(-E_\Lambda(\omega_\Lambda \mid \xi)) \chi_\Lambda(\mathrm{d}\omega_\Lambda), \end{aligned} \tag{11}$$

where $\mathbf{1}_A$ is the indicator function. Hence, each $\pi_\Lambda(\cdot \mid \xi) \in \mathcal{P}(\Omega)$. The family $\{\pi_\Lambda\}_{\Lambda \Subset \mathsf{V}}$ is called the *local Gibbs specification* for the model we consider. Directly from definition (11), we can verify that this family is consistent in the sense that

$$\int_\Omega \pi_\Delta(A \mid \omega) \pi_\Lambda(\mathrm{d}\omega \mid \xi) = \pi_\Lambda(A \mid \xi), \tag{12}$$

which holds for all $A \in \mathcal{B}(\Omega)$, all $\Delta \subset \Lambda$, and all $\Lambda \Subset \mathsf{V}$.

Definition 2. A measure $\mu \in \mathcal{P}(\Omega)$ is said to be a Gibbs random field corresponding to the Hamiltonian (1) if it solves the following (DLR) equation:

$$\mu(A) = \int_\Omega \pi_\Lambda(A \mid \omega) \mu(\mathrm{d}\omega), \tag{13}$$

for all $A \in \mathcal{B}(\Omega)$ and $\Lambda \Subset \mathsf{V}$.

Let $\mathcal{G}$ stand for the set of all solutions of (13). As is typical for unbounded spin systems, it is far from obvious whether $\mathcal{G}$ is nonvoid. But if it is, the description of the properties possessed by all of the elements of $\mathcal{G}$ is rather unrealistic. Thus, we construct and study a subset of $\mathcal{G}$, consisting of the measures possessing a prescribed (support) property. Such measures are called *tempered*.

For $p \geq 1$ and $\alpha > 0$, we set

$$\|\omega\|_{p,\alpha} = \left[\sum_{x \in \mathsf{V}} |\omega(x)|^p w_\alpha(x)\right]^{1/p}, \tag{14}$$

where the weights w_α are defined in (5). Then

$$L^p(\mathsf{V}, w_\alpha) = \{\omega \in \mathbb{R}^{\mathsf{V}} : \|\omega\|_{p,\alpha} < \infty\}$$

is a Banach space. Next, we define

$$\underline{\alpha} = \inf\{\alpha : \Theta(\alpha, \theta) < \infty\},$$

and let $\overline{\alpha} > \underline{\alpha}$ be such that (5) holds for all $\alpha \in (\underline{\alpha}, \overline{\alpha}]$. Finally, we set

$$p_0 = r + \frac{r}{\theta}.$$

For $\alpha', \alpha \in (\underline{\alpha}, \overline{\alpha}]$ and $p', p \in [p_0, q)$, by (14) we have

$$L^{p'}(\mathsf{V}, w_{\alpha'}) \hookrightarrow L^p(\mathsf{V}, w_\alpha) \quad \text{whenever } \alpha' < \alpha \text{ and } p' \geq p. \tag{15}$$

Notably, the above embedding is compact. Then, for $\alpha \in (\underline{\alpha}, \overline{\alpha}]$ and $p \in [p_0, q)$, we set

$$\mathcal{G}_{p,\alpha} = \{\mu \in \mathcal{G} \mid \mu[L^p(\mathsf{V}, w_\alpha)] = 1\}.$$

Clearly,

$$\mathcal{G}_{p',\alpha'} \subset \mathcal{G}_{p,\alpha} \quad \text{whenever } \alpha' \leq \alpha \text{ and } p' \geq p. \tag{16}$$

The following statement is the main result of the first part of the paper.

Theorem 1. *For every $\alpha \in (\underline{\alpha}, \overline{\alpha}]$ and $p \in [p_0, q)$, the set $\mathcal{G}_{p,\alpha}$ is nonvoid and $\mathcal{T}_{\mathrm{loc}}$-compact. For every $\lambda > 0$ and $x \in \mathsf{V}$, there exists a positive constant $C(p, \alpha; \lambda, x)$ such that, for all $\mu \in \mathcal{G}_{p,\alpha}$,*

$$\int_\Omega \exp(\lambda |\omega(x)|^p)\mu(\mathrm{d}\omega) \leq C(p, \alpha; \lambda, x). \tag{17}$$

Furthermore, for every $\lambda > 0$, there exists a positive constant $C(p, \alpha; \lambda)$ such that, for all $\mu \in \mathcal{G}_{p,\alpha}$,

$$\int_\Omega \exp(\lambda \|\omega\|_{p,\alpha}^p)\mu(\mathrm{d}\omega) \leq C(p, \alpha; \lambda). \tag{18}$$

The proof of Theorem 1 will be given in Section 3. Now let us make some comments.

- For our graphs, we cannot expect that the constants $C(p, \alpha; \lambda, x)$ in (17) are bounded uniformly in x. This could be the case if the quantities $\Theta(\alpha, \theta)$ were bounded uniformly with respect to the choice of the root o.

- Both estimates (17) and (18) also hold for $p = q$, but not for all λ, which ought to be small enough in this case.

- The interval $[p_0, q)$ is nonvoid if $q > r + r/\theta$, i.e. if the stabilizing effect of the potential V is stronger than the destabilizing effects of the interaction and of the underlying graph, caused by its degree property. If the graph is of bounded degree $\bar{n} = \sup_x n(x)$, condition (5) is satisfied for any $\theta > 0$ and $\alpha > \log \bar{n}$. In this case, we can take θ arbitrarily large and obtain $q > r$ (or $q \geq r$ for small λ), which is typical for such situations.

- According to (14) and (16), the stronger the estimates we want to get, the smaller the class of tempered Gibbs random fields we obtain.
- In view of the specific features of the graph geometry, such as the degree unboundedness and the lack of transitivity, the two basic statistical-mechanical tools—Ruelle's superstability method and Dobrushin's existence and uniqueness criteria—are not applicable to our model.

2.3. Modifications and extensions

One of the natural properties of the interaction potential W which we, however, do not mention in Assumption 1, is continuity. Thus, if we add it to Assumption 1(ii) then the kernels in (11) acquire the so-called *Feller* property. This merely means that if $f\colon \Omega \to \mathbb{R}$ is bounded and continuous then the function $\pi_\Lambda(f \mid \cdot)$, cf. (8), is bounded and continuous as well. The proof of such a property is quite standard; see, e.g. the proof of Lemma 2.10 of [15]. Let $C_{\mathrm{b}}(\Omega)$ be the set of all functions just mentioned. Then the weak topology $\mathcal{T}_{\mathrm{weak}}$ on $\mathcal{P}(\Omega)$ is defined to be the weakest topology in which the maps $\mu \mapsto \mu(f)$ are continuous for all $f \in C_{\mathrm{b}}(\Omega)$.

Theorem 2. *If, in addition to Assumption 1, we assume that the interaction potential $W\colon \mathbb{R}^2 \to \mathbb{R}$ is jointly continuous, then, for every $\alpha \in (\underline{\alpha}, \overline{\alpha}]$ and $p \in [p_0, q)$, the set $\mathcal{G}_{p,\alpha}$ is $\mathcal{T}_{\mathrm{weak}}$-compact.*

The proof of this theorem will be discussed in Section 3. Now we present its extension. Taking into account (15), we define the set of tempered configurations by

$$\tilde{\Omega}^{\mathrm{t}} = \bigcap_{p\in[p_0,q)} \bigcap_{\alpha\in(\underline{\alpha},\overline{\alpha}]} L^p(\mathsf{V}, w_\alpha). \tag{19}$$

This set can be endowed with the projective limit topology and thereby turned into a Fréchet space. By standard arguments, its Borel σ-field $\mathcal{B}(\tilde{\Omega}^{\mathrm{t}})$ has the property

$$\mathcal{B}(\tilde{\Omega}^{\mathrm{t}}) = \{A \cap \tilde{\Omega}^{\mathrm{t}}\colon A \in \mathcal{B}(\Omega)\}, \tag{20}$$

in view of which we can define

$$\tilde{\mathcal{G}}^{\mathrm{t}} = \{\mu \in \mathcal{G}\colon \mu(\tilde{\Omega}^{\mathrm{t}}) = 1\}.$$

The elements of the latter set have the smallest support we have managed to establish. In view of (20), they can be redefined as probability measures on $(\tilde{\Omega}^{\mathrm{t}}, \mathcal{B}(\tilde{\Omega}^{\mathrm{t}}))$. Let $\tilde{\mathcal{T}}_{\mathrm{weak}}$ be the weak topology on the set of all probability measures $\mathcal{P}(\tilde{\Omega}^{\mathrm{t}})$. Clearly, $\tilde{\mathcal{T}}_{\mathrm{weak}}$ is stronger than the topology induced on $\tilde{\Omega}^{\mathrm{t}}$ by $\mathcal{T}_{\mathrm{weak}}$.

Theorem 3. *Let the assumptions of Theorem 1 or Theorem 2 hold. Then the set $\tilde{\mathcal{G}}^{\mathrm{t}}$ is nonvoid and $\mathcal{T}_{\mathrm{loc}}$-compact or, respectively, $\tilde{\mathcal{T}}_{\mathrm{weak}}$-compact.*

Proof. Let $\tilde{\mathcal{G}}$ be the intersection of all $\mathcal{G}_{p,\alpha}$, with $\alpha \in (\underline{\alpha}, \overline{\alpha}]$ and $p \in [p_0, q)$; see (19). By the compactness established in Theorem 1, the set $\tilde{\mathcal{G}}$ is nonvoid. Obviously, all its elements belong to $\tilde{\mathcal{G}}^{\mathrm{t}}$ and, hence, these two sets coincide. Furthermore, the elements of $\tilde{\mathcal{G}}$ obey estimates (17) and (18) with all $\alpha \in (\underline{\alpha}, \overline{\alpha}]$ and $p \in [p_0, q)$. The $\mathcal{T}_{\mathrm{loc}}$-compactness has already been mentioned. To prove the $\tilde{\mathcal{T}}_{\mathrm{weak}}$-compactness, we consider the balls

$$B_{p,\alpha}(R) = \{\omega\colon \|\omega\|_{p,\alpha} \le R\}, \qquad R > 0, \tag{21}$$

and fix two monotone sequences $\alpha_k \downarrow \underline{\alpha}$ and $p_k \uparrow q$ as $k \to +\infty$. In view of (18), for any $k \in \mathbb{N}$ and $\varepsilon > 0$, we can pick $R_{k,\varepsilon} > 0$ such that

$$\mu[B_{p_k,\alpha_k}(R_{k,\varepsilon})] \geq 1 - \frac{\varepsilon}{2^k}$$

uniformly for all $\mu \in \mathcal{G}_{p_k,\alpha_k}$, and, hence, for all $\mu \in \tilde{\mathcal{G}}^{\mathrm{t}}$. By the compactness of the embedding (15), the set

$$B = \bigcap_{k \in \mathbb{N}} B_{p_k,\alpha_k}(R_{k,\varepsilon})$$

is compact in $\tilde{\Omega}^{\mathrm{t}}$, and is such that $\mu(B) \geq 1 - \varepsilon$ for all $\mu \in \tilde{\mathcal{G}}^{\mathrm{t}}$. Thereafter, the $\tilde{\mathcal{T}}^{\mathrm{weak}}$-compactness of $\tilde{\mathcal{G}}^{\mathrm{t}}$ follows by the renowned Prokhorov theorem.

Finally, let us mention one more possible extension of Theorems 1 and 2. The Gibbs random fields constructed above can serve as equilibrium thermodynamic states of systems of one-dimensional anharmonic oscillators, indexed by the vertices of G and interacting with each other along the edges by the potential W (oscillating networks). Obviously, the above results hold true if we replace the single-spin space $\mathbb{R}$ with $\mathbb{R}^\nu$, $\nu \in \mathbb{N}$, which would correspond to multidimensional oscillators. Furthermore, by means of the technique developed in [2], [15], and [19], our theorems can also be extended to the case where the single-spin spaces are copies of C_β—the Banach space of continuous functions (temperature loops) $\omega\colon [0, \beta] \to \mathbb{R}^\nu$, $\beta > 0$, such that $\omega(0) = \omega(\beta)$. In this case, the Gibbs random fields correspond to the so-called *Euclidean* thermodynamic Gibbs states of a system of interacting ν-dimensional quantum anharmonic oscillators, for which β^{-1} is the temperature.

3. Properties of the local Gibbs specification

In this section we prove that estimate (18) also holds for all $\pi_\Lambda(\cdot \mid \xi)$. This will imply all the properties of the family $\{\pi_\Lambda\}_{\Lambda \Subset \mathsf{V}}$ which we need to prove Theorem 1. We begin by deriving a basic estimate, which allows us to control the ξ-dependence of the moments of π_Λ with one-point sets $\Lambda = \{x\}$. Its extension to arbitrary sets will be obtained by means of the consistency property (12).

3.1. Moment estimates

From (6), by an easy calculation we obtain

$$|W(u, v)| \leq \varkappa(|u|^p + |v|^p) + \frac{I_W}{2} + 2(p - r)\left(\frac{J_W}{2p}\right)^{p/(p-r)}\left(\frac{r}{\varkappa}\right)^{r/(p-r)}, \tag{22}$$

which holds for all $u, v \in \mathbb{R}$, $\varkappa > 0$, and $p > r$. We will use this estimate with $\varkappa = \beta/n(x)n(y)$, $x, y \in \mathsf{V}$, where $\beta > 0$ will be chosen later in (27). For $p \in [p_0, q)$, we set

$$\begin{aligned} \Gamma_{xy}(\beta, p) &= \gamma(\beta, p)[n(x)n(y)]^{r/(p-r)}, \\ \gamma(\beta, p) &= I_W + 4(p - r)\left(\frac{J_W}{2p}\right)^{p/(p-r)}\left(\frac{r}{\beta}\right)^{r/(p-r)}, \end{aligned} \tag{23}$$

and

$$\begin{aligned} C(\beta, \lambda, p) = c_V &+ \log\left\{\int_{\mathbb{R}} \exp((\lambda + \beta)|u|^p - a_V|u|^q)\,\mathrm{d}u\right\} \\ &- \log\left\{\int_{\mathbb{R}} \exp(-\beta|u|^p - V(u))\,\mathrm{d}u\right\}, \end{aligned}$$

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where $\lambda > 0$ and a_V, c_V, and q are the same as in (7). Note that the integral in the latter line is positive. In the lemma below, π_x stands for the corresponding objects defined in (11) with $\Lambda = \{x\}$.

Lemma 1. *For every $\lambda > 0$, $p \in [p_0, q)$, $x \in \mathsf{V}$, and $\xi \in \Omega$, the following estimate holds:*

$$\int_\Omega \exp(\lambda|\omega(x)|^p)\pi_x(\mathrm{d}\omega \mid \xi) \le \exp\biggl(C(\beta,\lambda,p) + \sum_{y\sim x}\frac{2\beta|\xi(y)|^p}{n(x)n(y)} + \sum_{y\sim x}\Gamma_{xy}(\beta,p)\biggr). \quad (24)$$

Proof. According to (11), for the left-hand side of (24), we have

$$\mathrm{LHS}(24) = \frac{1}{Y_x(\xi)}\int_{\mathbb{R}} \exp\biggl(\lambda|u|^p - \sum_{y\sim x} W(u,\xi(y)) - V(u)\biggr)\,\mathrm{d}u,$$

where

$$Y_x(\xi) = \int_{\mathbb{R}} \exp\biggl(-\sum_{y\sim x} W(u,\xi(y)) - V(u)\biggr)\,\mathrm{d}u.$$

By (22), with $\varkappa = \beta/n(x)n(y)$, and (23), we obtain

$$\begin{aligned}
-\sum_{y\sim x}&\biggl[\frac{\beta}{n(x)n(y)}(|u|^p + |\xi(y)|^p) + \frac{1}{2}\Gamma_{xy}(\beta,p)\biggr] - V(u)\\
&\le -\sum_{y\sim x} W(u,\xi(y)) - V(u)\\
&\le \sum_{y\sim x}\biggl[\frac{\beta}{n(x)n(y)}(|u|^p + |\xi(y)|^p) + \frac{1}{2}\Gamma_{xy}(\beta,p)\biggr] - a_V|u|^q + c_V.
\end{aligned}$$

Then

$$Y_x(\xi) \ge \exp\biggl(-\sum_{y\sim x}\biggl[\frac{\beta|\xi(y)|^p}{n(x)n(y)} + \frac{1}{2}\Gamma_{xy}(\beta,p)\biggr]\biggr)\int_{\mathbb{R}}\exp(-\beta|u|^p - V(u))\,\mathrm{d}u,$$

and

$$\begin{aligned}
\int_{\mathbb{R}} \exp\biggl(\lambda|u|^p - \sum_{y\sim x} W(u,\xi(y)) - V(u)\biggr)\,\mathrm{d}u &\le \exp\biggl(c_V + \sum_{y\sim x}\biggl[\frac{\beta|\xi(y)|^p}{n(x)n(y)} + \frac{1}{2}\Gamma_{xy}(\beta,p)\biggr]\biggr)\\
&\quad\times\int_{\mathbb{R}}\exp((\lambda+\beta)|u|^p - a_V|u|^q)\,\mathrm{d}u,
\end{aligned}$$

which clearly yields (24).

Now, for $\lambda > 0$, $p \in [p_0, q)$, $\Lambda \Subset \mathsf{V}$, and a fixed $x \in \Lambda$, we set

$$M_x(\lambda,p,\Lambda;\xi) = \log\biggl\{\int_\Omega \exp(\lambda|\omega(x)|^p)\pi_\Lambda(\mathrm{d}\omega \mid \xi)\biggr\}, \quad (25)$$

which is obviously finite. Our aim is to find an upper bound for this quantity. Integrating both sides of (24) with respect to $\pi_\Lambda(\cdot \mid \xi)$ and taking into account (12), we obtain

$$\begin{aligned}
\exp(M_x(\lambda,p,\Lambda;\xi)) \le \exp\biggl(&C(\beta,\lambda,p) + \sum_{y\sim x}\Gamma_{xy}(\beta,p) + \sum_{y\sim x,\, y\in\Lambda^c}\frac{2\beta|\xi(y)|^p}{n(x)n(y)}\biggr)\\
&\times\int_\Omega \exp\biggl(\sum_{y\sim x,\, y\in\Lambda}\frac{2\beta|\omega(y)|^p}{n(x)n(y)}\biggr)\pi_\Lambda(\mathrm{d}\omega \mid \xi). \quad (26)
\end{aligned}$$

In the sequel, the parameter $\overline{\alpha}$ will be fixed. Then, for a given λ, the parameter β will always be chosen in such a way that

$$2\beta \mathrm{e}^{\overline{\alpha}} < \lambda, \tag{27}$$

which, in particular, yields

$$\sum_{y\sim x} \frac{2\beta}{\lambda n(x)n(y)} \leq 1. \tag{28}$$

To estimate the integral in (26), we use the multiple Hölder inequality,

$$\int \bigg(\prod_{i=1}^{n} \varphi_i^{\alpha_i}\bigg) \,\mathrm{d}\mu \leq \prod_{i=1}^{n} \bigg(\int \varphi_i \,\mathrm{d}\mu\bigg)^{\alpha_i}, \tag{29}$$

in which μ is a probability measure, $\varphi_i \geq 0$, $i = 1, \dots, n$, are integrable functions, and $\alpha_i \geq 0$ $i = 1, \dots, n$, are numbers such that $\sum_{i=1}^{n} \alpha_i \leq 1$. Applying this inequality to (26) and taking into account (28) we arrive at

$$\begin{aligned} M_x(\lambda, p, \Lambda; \xi) \leq C(\beta, \lambda, p) + \sum_{y\sim x} \Gamma_{xy}(\beta, p) + \sum_{y\sim x,\, y\in\Lambda^{\mathrm{c}}} \frac{2\beta |\xi(y)|^p}{n(x)n(y)} \\ + \sum_{y\sim x,\, y\in\Lambda} \frac{2\beta}{\lambda n(x)n(y)} M_y(\lambda, p, \Lambda; \xi). \end{aligned} \tag{30}$$

As the quantity we want to estimate appears on both sides of the above estimate, we proceed as follows. For $\alpha \in (\underline{\alpha}, \overline{\alpha}]$, we set (cf. (3) and (14))

$$\|M(\lambda, p, \Lambda; \xi)\|_\alpha = \sum_{x\in\Lambda} M_x(\lambda, p, \Lambda; \xi) \exp(-\alpha\rho(o, x)), \tag{31}$$

and obtain an upper bound for $\|M(\lambda, p, \Lambda; \xi)\|_\alpha$. To this end, we multiply both sides of (30) by $\exp(-\alpha\rho(o, x))$, sum over $x \in \Lambda$, and obtain

$$\|M(\lambda, p, \Lambda; \xi)\|_\alpha \leq \Upsilon_1^\alpha + \Upsilon_2^\alpha + \Upsilon_3^\alpha(\Lambda) + \Upsilon_4^\alpha(\Lambda).$$

Here

$$\Upsilon_1^\alpha = C(\beta, \lambda, p) \sum_{x} \exp(-\alpha\rho(o, x))$$

and

$$\Upsilon_2^\alpha = \gamma(\beta, p)\Theta\bigg(\alpha; \frac{r}{p-r}\bigg) \leq \gamma(\beta, p)\Theta(\alpha; \theta).$$

The latter estimate holds since $p \geq p_0 = r + r/\theta$. The term corresponding to the third summand in (30) is estimated as follows:

$$\begin{aligned} \sum_{x\in\Lambda} \exp(-\alpha\rho(o, x)) \sum_{y\sim x,\, y\in\Lambda^{\mathrm{c}}} \frac{2\beta}{n(x)n(y)} |\xi(y)|^p &\leq \Upsilon_3^\alpha(\Lambda) \\ &:= 2\beta \mathrm{e}^{\alpha} \sum_{x\in\Lambda^{\mathrm{c}}} \exp(-\alpha\rho(o, x)) |\xi(x)|^p, \end{aligned} \tag{32}$$

which is finite whenever $\xi \in L^p(\mathsf{V}, w_\alpha)$, and tends to 0 as $\Lambda \to \mathsf{V}$. In a similar way, we obtain

$$\sum_{x\in\Lambda} \exp(-\alpha\rho(o,x)) \sum_{y\sim x,\, y\in\Lambda} \frac{2\beta}{\lambda n(x)n(y)} M_y(\lambda, p, \Lambda; \xi) \le \Upsilon_4^\alpha(\Lambda)$$
$$:= \frac{2\beta \mathrm{e}^\alpha}{\lambda} \|M(\lambda, p, \Lambda; \xi)\|_\alpha. \quad (33)$$

Recall that β and λ are supposed to obey (27). Then from the estimates obtained above we obtain

$$\|M(\lambda, p, \Lambda; \xi)\|_\alpha \le \frac{\Upsilon_1^\alpha + \Upsilon_2^\alpha + \Upsilon_3^\alpha(\Lambda)}{1 - 2\beta\mathrm{e}^\alpha/\lambda}, \quad (34)$$

which yields

$$M_x(\lambda, p, \Lambda; \xi) \le C_x(\lambda, p, \xi) \quad (35)$$

for some $C_x(\lambda, p, \xi) > 0$, which is independent of Λ, but obviously depends on x and on the choice of the root o.

3.2. Compactness of the local Gibbs specification

The result just obtained allows us to prove the next statement, which is crucial for establishing the relative $\mathcal{T}_{\mathrm{loc}}$-compactness of the family $\{\pi_\Lambda(\cdot \mid \xi)\}_{\Lambda \Subset \mathsf{V}}$, as well as the corresponding integrability estimates.

Lemma 2. *Let $p \in [p_0, q)$ and $\alpha \in (\underline{\alpha}, \overline{\alpha}]$ be fixed. Then, for every $\lambda > 0$ and $\xi \in L^p(\mathsf{V}, w_\alpha)$, we find a positive constant $C(p, \alpha; \lambda, \xi)$ such that, for all $\Lambda \Subset \mathsf{V}$,*

$$\int_\Omega \exp(\lambda\|\omega\|_{p,\alpha}^p)\pi_\Lambda(\mathrm{d}\omega \mid \xi) \le C(p, \alpha; \lambda, \xi). \quad (36)$$

Furthermore, for the same λ, we find a positive constant $C(p, \alpha; \lambda)$ such that, for all $\xi \in L^p(\mathsf{V}, w_\alpha)$,

$$\limsup_{\Lambda\to\mathsf{V}} \int_\Omega \exp(\lambda\|\omega\|_{p,\alpha}^p)\pi_\Lambda(\mathrm{d}\omega \mid \xi) \le C(p, \alpha; \lambda). \quad (37)$$

Proof. By (11) and (14), for any $\delta > 0$, we have

$$\int_\Omega \exp(\lambda\|\omega\|_{p,\alpha}^p)\pi_\Lambda(\mathrm{d}\omega \mid \xi) = \exp\biggl(\lambda \sum_{x\in\Lambda^{\mathrm{c}}} |\xi(x)|^p w_\alpha(x)\biggr)$$
$$\times \int_\Omega \prod_{x\in\Lambda} [\exp(\delta|\omega(x)|^p)]^{\lambda w_\alpha(x)/\delta} \pi_\Lambda(\mathrm{d}\omega \mid \xi). \quad (38)$$

Now we pick δ such that

$$\frac{\lambda}{\delta} \sum_{x\in\Lambda} w_\alpha(x) \le 1,$$

and apply in (38) the Hölder inequality (29). This yields, see (25) and (31),

$$\int_\Omega \exp(\lambda\|\omega\|_{p,\alpha}^p)\pi_\Lambda(\mathrm{d}\omega \mid \xi) \le \exp\biggl(\lambda \sum_{x\in\Lambda^{\mathrm{c}}} |\xi(x)|^p w_\alpha(x)\biggr)$$
$$\times \exp\biggl(\frac{\lambda}{\delta}\|M(\delta, p, \Lambda; \xi)\|_\alpha\biggr). \quad (39)$$

By (34), the set $\{\mathrm{RHS}(39)(\Lambda) \mid \Lambda \Subset \mathsf{V}\}$ is bounded for every fixed $\xi \in L^p(\mathsf{V}, w_\alpha)$. We denote its upper bound by $C(p, \alpha; \lambda, \xi)$ and obtain (36). Estimate (37) follows from (39) by (32), (34), and the fact that $\xi \in L^p(\mathsf{V}, w_\alpha)$.

Lemma 3. *For every $\xi \in L^p(\mathsf{V}, w_\alpha)$, the family $\{\pi_\Lambda(\cdot \mid \xi)\}_{\Lambda \Subset \mathsf{V}} \subset \mathcal{P}(\Omega)$ is relatively $\mathcal{T}_{\mathrm{loc}}$-compact.*

Proof. According to Proposition 4.9 of [10, p. 61], the proof will be complete if we show that the family $\{\pi_\Lambda(\cdot \mid \xi)\}_{\Lambda \Subset \mathsf{V}} \subset \mathcal{P}(\Omega)$ is locally equicontinuous. The latter means that, for each $\Delta \Subset \mathsf{V}$ and any sequence $\{A_k\}_{k \in \mathbb{N}} \subset \mathcal{B}(\Omega_\Delta)$ with $A_k \downarrow \varnothing$, as $k \to +\infty$, we have

$$\lim_{k \to +\infty} \limsup_{\Lambda \to \mathsf{V}} \pi_\Lambda(A_k \mid \xi) = 0. \tag{40}$$

Here, as well as in (37), by $\Lambda \to \mathsf{V}$ we mean the convergence of the corresponding net with the index set $\{\Lambda\}_{\Lambda \Subset \mathsf{V}}$, ordered by inclusion. To obtain (40), we adapt the arguments used in the proof of Theorem 4.12 and Corollary 4.13 of [10, pp. 62, 63]. Let T be a positive number, and let Δ be as above. Set

$$B_T = \{\omega \in \Omega \colon |\omega(x)| \le T \text{ for all } x \in \Delta \cup \partial\Delta\}, \qquad B_T^{\mathrm{c}} = \Omega \setminus B_T,$$

where $\partial\Delta$ is the outer boundary of Δ consisting of those $y \in \Delta^{\mathrm{c}}$ for which $\rho(y, \Delta) = 1$, ρ being the path distance. For a fixed $k \in \mathbb{N}$, we have

$$\limsup_{\Lambda \to \mathsf{V}} \pi_\Lambda(A_k \mid \xi) \le \limsup_{\Lambda \to \mathsf{V}} \pi_\Lambda(A_k \cap B_T \mid \xi) + \limsup_{\Lambda \to \mathsf{V}} \pi_\Lambda(B_T^{\mathrm{c}} \mid \xi). \tag{41}$$

The second summand can be estimated by means of (37), which yields

$$\limsup_{\Lambda \to \mathsf{V}} \pi_\Lambda(B_T^{\mathrm{c}} \mid \xi) \le C(p, \alpha, \lambda) \exp\Big(-\lambda T^p \sum_{x \in \Delta} w_\alpha(x)\Big) < \frac{\varepsilon}{2}, \tag{42}$$

holding for any $\varepsilon > 0$ and sufficiently large T. To handle the first summand in (41), we first estimate $\pi_\Delta(A_k \cap B_T \mid \eta)$, $\eta \in \Omega$, which in view of (11) is nonzero only if $\omega_\Delta \times \eta_{\Delta^{\mathrm{c}}} \in B_T$. For such η, by (10), (11), (6), and (9), we have

$$Z_\Delta(\eta) \ge \exp\big(-\tfrac{1}{2}(I_W|\mathsf{E}_\Delta| + J_W T^r |\partial\Delta|) + \zeta_\Delta\big),$$

where $\mathsf{E}_\Delta \subset \mathsf{E}$ is the set of edges with both ends in Δ, and

$$\mathrm{e}^{\zeta_\Delta} = \prod_{x \in \Delta} \int_{\mathbb{R}} \exp\big(-J_W\big(n(x) + \tfrac{1}{2}\big)|u|^r\big)\chi(\mathrm{d}u).$$

Applying (6) again, this time to the numerator in (11), we arrive at

$$\pi_\Delta(A_k \cap B_T \mid \eta) \le \exp(I_W|\mathsf{E}_\Delta| + J_W T^r C(\Delta) - \zeta_\Delta)\chi_\Delta(A_k) < \frac{\varepsilon}{2},$$

which holds, uniformly in η, for T obeying (42) and sufficiently large k by the continuity of χ_Δ. Then, for any $\Lambda \Subset \mathsf{V}$ that contains Δ, by (12) we obtain

$$\pi_\Lambda(A_k \cap B_T \mid \xi) < \frac{\varepsilon}{2},$$

which when applied in (41) yields (40).

Corollary 1. *For every $p \in [p_0, q)$ and $\alpha \in (\underline{\alpha}, \overline{\alpha}]$, the set $\mathcal{G}_{p,\alpha}$ is nonvoid.*

Proof. For every $\Lambda \Subset \mathsf{V}$ and $\xi \in L^p(\mathsf{V}, w_\alpha)$, by (11), each $\pi_\Lambda(\cdot \mid \xi)$ is supported by the set

$$\{\omega = \omega_\Lambda \times \xi_{\Lambda^c} : \omega_\Lambda \in \Omega_\Lambda\},$$

which yields

$$\pi_\Lambda[L^p(\mathsf{V}, w_\alpha) \mid \xi] = 1.$$

Let us fix some $\xi \in L^p(\mathsf{V}, w_\alpha)$. By Lemma 3, there exists an increasing sequence $\{\Lambda_n\}_{n\in\mathbb{N}}$, which exhausts V, such that the sequence $\{\pi_{\Lambda_n}(\cdot \mid \xi)\}_{n\in\mathbb{N}}$ $\mathcal{T}_{\mathrm{loc}}$-converges to a certain $\mu \in \mathcal{P}(\Omega)$. Let us show that this μ also solves the DLR equation. For any Λ, we find an $n' \in \mathbb{N}$ such that $\Lambda \subset \Lambda_n$ for all $n \geq n'$. For such n and $A \in \mathcal{B}_{\mathrm{loc}}$, by (12) we have

$$\int_\Omega \pi_\Lambda(A \mid \omega)\pi_{\Lambda_n}(\mathrm{d}\omega \mid \xi) = \pi_{\Lambda_n}(A \mid \xi). \tag{43}$$

By (11) we immediately find that the function $\Omega \ni \omega \mapsto \pi_\Lambda(A \mid \omega)$ is in $\mathcal{F}_{\mathrm{loc}}$. Thus, we can pass in (43) to the limit $n \to +\infty$ and obtain $\mu \in \mathcal{G}$; see Definition 1 and Lemma 3. To prove that μ is supported by $L^p(\mathsf{V}, w_\alpha)$, we show that this measure obeys estimate (18). For $\lambda > 0$, we set

$$F_{N,\Delta}(\omega) = \exp\biggl(\lambda \min\biggl\{N; \sum_{x\in\Delta} |\omega(x)|^p w_\alpha(x)\biggr\}\biggr), \qquad N \in \mathbb{N},\ \Delta \Subset \mathsf{V}.$$

Clearly, $F_{N,\Delta} \in \mathcal{F}_{\mathrm{loc}}$. Then, by (37) and the $\mathcal{T}_{\mathrm{loc}}$-convergence $\pi_{\Lambda_n}(\cdot \mid \xi) \to \mu$, we have

$$\int_\Omega F_{N,\Delta}(\omega)\mu(\mathrm{d}\omega) \leq \lim_{n\to+\infty} \int_\Omega F_{N,\Delta}(\omega)\pi_{\Lambda_n}(\mathrm{d}\omega \mid \xi) \leq C(p, \alpha; \lambda),$$

where the latter constant is the same as in (37). Thereafter, the proof of (18), with the same constant, follows by Levi's monotone convergence theorem. Hence, $\mu \in \mathcal{G}_{p,\alpha}$.

Proof of Theorem 1. We proved above that the accumulation points of the family $\{\pi_\Lambda(\cdot \mid \xi)\}$, $\xi \in L^p(\mathsf{V}, w_\alpha)$, obey (37). Let us extend this to all $\mu \in \mathcal{G}_{p,\alpha}$. For such μ, by (13), Fatou's lemma, and estimate (37), we obtain

$$\begin{aligned}
\int_\Omega F_N(\omega)\mu(\mathrm{d}\omega) &= \limsup_{\Lambda\to\mathsf{V}} \int_\Omega \biggl[\int_\Omega F_N(\omega)\pi_\Lambda(\mathrm{d}\omega \mid \xi)\biggr]\mu(\mathrm{d}\xi) \\
&\leq \int_\Omega \biggl[\limsup_{\Lambda\to\mathsf{V}} \int_\Omega F_N(\omega)\pi_\Lambda(\mathrm{d}\omega \mid \xi)\biggr]\mu(\mathrm{d}\xi) \\
&\leq \int_\Omega \biggl[\limsup_{\Lambda\to\mathsf{V}} \int_\Omega \exp(\lambda\|\omega\|_{p,\alpha}^p)\pi_\Lambda(\mathrm{d}\omega \mid \xi)\biggr]\mu(\mathrm{d}\xi) \\
&\leq C(p, \alpha; \lambda).
\end{aligned}$$

Then we again apply Levi's theorem and obtain (18). The proof of (17) follows by (35) along the same line of arguments. Now let $\{\mu_\iota\}_{\iota\in\mathcal{I}} \subset \mathcal{G}_{p,\alpha}$ be any net. Then its relative compactness can be established by the same arguments used in the proof of Lemma 3. As $\mathcal{G}_{p,\alpha}$ is evidently $\mathcal{T}_{\mathrm{loc}}$-closed, the latter fact completes the proof.

Notes on the proof of Theorem 2. By the Feller property of the specification $\{\pi_\Lambda(\cdot \mid \cdot)\}_{\Lambda\Subset\mathsf{V}}$ and (13), we readily find that $\mathcal{G}_{p,\alpha}$ is $\mathcal{T}_{\mathrm{weak}}$-closed. Clearly, the balls $\{\omega \colon \|\omega\|_{p,\alpha} \leq R\}$, $R > 0$, are compact in Ω for any fixed $\alpha \in (\underline{\alpha}, \overline{\alpha}]$ and $p \in [p_0, q)$. Thus, by Prokhorov's theorem, any net $\{\mu_\iota\}_{\iota\in\mathcal{I}} \subset \mathcal{G}_{p,\alpha}$ is relatively $\mathcal{T}_{\mathrm{weak}}$-compact for any $\xi \in L^p(\mathsf{V}, w_\alpha)$.

4. Repulsive graphs

In the remaining part of the paper we present a family of unbounded degree graphs which obey estimate (5). The defining property of such graphs is that vertices of large degree are located at large distances from each other.

4.1. The family of graphs and the main statement

For $n_* \in \mathbb{N}$, we set

$$\mathsf{V}_* = \{x \in \mathsf{V} \colon n(x) \leq n_*\}, \qquad \mathsf{V}_*^{\mathrm{c}} = \mathsf{V} \setminus \mathsf{V}_*. \tag{44}$$

Definition 3. For an integer $n_* > 2$ and a strictly increasing function $\phi \colon (n_*, +\infty) \to (0, +\infty)$, the family $\mathbb{G}(n_*, \phi)$ consists of those connected simple graphs $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ for which the path distance obeys the condition

$$\rho(x, y) \geq \phi[n(x, y)] \quad \text{for all } x, y \in \mathsf{V}_*^{\mathrm{c}}, \tag{45}$$

where

$$n(x, y) = \max\{n(x); n(y)\}. \tag{46}$$

No restrictions are imposed on $\rho(x, y)$ if either x or y belongs to V_*.

Let us make some comments. The graph presented in Example 1 is certainly not in $\mathbb{G}(n_*, \phi)$ for any increasing ϕ. However, in this graph the increase of the degrees is allowed only along a single ray (it is the 'main' ray in that example). That is why it possesses property (5). For graphs in $\mathbb{G}(n_*, \phi)$ with appropriate ϕ, this property also holds; see Theorem 4 below. In such graphs, the vertices of large degree are sparse, but they can appear 'in all directions'. To see this, for a given $x \in \mathsf{V}_*^{\mathrm{c}}$, we set

$$\mathsf{K}(x) = \{y \in \mathsf{V} \colon \rho(y, x) < \phi[n(x)]\}.$$

Then by (45) we have $\mathsf{K}(x) \cap \mathsf{V}_*^{\mathrm{c}} = \{x\}$, i.e. such x 'repels' all vertices $y \in \mathsf{V}_*^{\mathrm{c}}$ from the ball $\mathsf{K}(x)$. For the sake of convenience, we will assume that $\mathsf{K}(x)$ contains the neighborhood of x, which is equivalent to assuming that

$$\phi(n_* + 1) > 1. \tag{47}$$

The graphs introduced and studied in [3] were defined by the condition which can be written in the form (cf. Equations (3.8) and (3.9) of [3])

$$\rho(x, y) \geq \phi[m(x, y)], \qquad m(x, y) := \min\{n(x); n(y)\}. \tag{48}$$

In this case, a vertex x 'repels' from the ball $\{y \colon \rho(y, x) < \phi[n(x)]\}$ only those ys for which $n(y) \geq n(x)$. We employ (45) rather than (48) in view of its application in Lemma 5 below; see Remark 1 below for further comments. The concrete choice of the function ϕ in Theorem 4 is discussed in Remark 2 below.

Theorem 4. *Let* G *be in* $\mathbb{G}(n_*, \phi)$ *with* ϕ *having the form*

$$\phi(b) = \upsilon \log b [\log \log b]^{1+\varepsilon}, \qquad \varepsilon > 0,\ b \geq n_* + 1 \geq \mathrm{e}^{\mathrm{e}}, \tag{49}$$

where $\upsilon > 1/\mathrm{e}$ *and, hence, is such that (47) holds. Then, for any* $\theta > 0$*, there exists* $\underline{\alpha} \geq 0$*, which may depend on* θ*,* n_**,* υ*, and* ε*, such that* $\Theta(\alpha, \theta) < \infty$ *whenever* $\alpha > \underline{\alpha}$*.*

The proof of Theorem 4 is given at the very end of this subsection. It is preceded by and based on Lemmas 4 and 5, which in turn are proven in the remaining part of the paper. For $N \in \mathbb{N}$ and $x \in \mathsf{V}$, we set

$$\mathsf{S}(N, x) = \{y \in \mathsf{V} \colon \rho(x, y) = N\},$$
$$\mathsf{B}(N, x) = \{y \in \mathsf{V} \colon \rho(x, y) \le N\},$$

and

$$T_x(\alpha, \theta) = \sum_y [n(y)]^{1+\theta} \exp(-\alpha\rho(x, y)), \qquad \alpha, \theta > 0. \tag{50}$$

Lemma 4. *Let* G *be in* $\mathbb{G}(n_*, \phi)$ *with* ϕ *obeying (47). Then, for every positive* θ *and* α*, it follows that*

$$\Theta(\alpha, \theta) \le n_*^{\theta}(\mathrm{e}^{\alpha} + 1) T_o(\alpha, \theta). \tag{51}$$

Lemma 5. *Let* G *be as in Theorem 4. Then, for every* $\theta > 0$*, there exists* $a > 0$*, which may also depend on the parameters of the function in (49), such that, for any* $x \in \mathsf{V}$*, there exist* $\tilde{N}_x \in \mathbb{N}$ *for which*

$$G_\theta(N, x) := \sum_{y \in \mathsf{S}(N,x)} [n(y)]^{1+\theta} \le \mathrm{e}^{aN}, \tag{52}$$

whenever $N \ge \tilde{N}_x$.

Remark 1. A condition like (48) could guarantee that estimate (52) holds only for $N = N_k$, $k \in \mathbb{N}$, for some increasing sequence $\{N_k\}_{k \in \mathbb{N}} \subset \mathbb{N}$. This would not be enough to prove Theorem 4.

Proof of Theorem 4. By (50) and (52), we have

$$T_x(\alpha, \theta) \le \sum_{N=0}^{N_x} \exp(-\alpha N) \Bigg(\sum_{y \in \mathsf{S}(N,x)} [n(y)]^{1+\theta} \Bigg) + \sum_{N=N_x+1}^{\infty} \exp(-(\alpha - a)N).$$

Thus, the proof of the theorem follows by (51) with $\underline{\alpha} = a$.

4.2. A property of the balls in repulsive graphs

The proof of Lemma 5 is based on a property of the balls $\mathsf{B}(N, x)$ in the graphs $\mathsf{G} \in \mathbb{G}(n_*, \phi)$, due to which we can control the growth of the maximum degree of $y \in \mathsf{B}(N, x)$. Here we do not suppose that ϕ has the concrete form of (49).

Lemma 6. *Let* $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ *be in* $\mathbb{G}(n_*, \phi)$ *with an arbitrary increasing function*

$$\phi \colon (n_*, +\infty) \to (1, +\infty).$$

Then, for every $x \in \mathsf{V}$*, there exists* $N_x \in \mathbb{N}$ *such that*

$$\max_{y \in \mathsf{B}(N,x)} n(y) \le \phi^{-1}(2N), \tag{53}$$

whenever $N \ge N_x$.

Proof. Given x, let $\tilde{x}$ be the vertex in $\mathsf{V}_*^{\mathrm{c}}$ which is closest to x; see (44). If more than one vertex lies at the same distance from x, we take the vertex with the highest degree. For this $\tilde{x}$, we have the following possibilities: (i) $\rho(x, \tilde{x}) \geq \phi[n(\tilde{x})]/2$; and (ii) $\rho(x, \tilde{x}) < \phi[n(\tilde{x})]/2$. The latter possibility also includes the case in which $\tilde{x} = x$, i.e. where x itself is in $\mathsf{V}_*^{\mathrm{c}}$. In (i), we set $N_x = 1$, which means that (53) holds for all $N \in \mathbb{N}$. Indeed, if $N < \rho(x, \tilde{x})$ then the ball $\mathsf{B}(N, x)$ contains only vertices $y \in \mathsf{V}_*$, for which $n(y) \leq n_* \leq \phi^{-1}(2N)$ for any $N \in \mathbb{N}$. If $N \geq \rho(x, \tilde{x})$ and $\max_{y \in \mathsf{B}(N,x)} n(y) = n(\tilde{x})$ we have $N \geq \rho(x, \tilde{x}) \geq \phi[n(\tilde{x})]/2$, which also yields (53) for this case. Finally, let $\max_{y \in \mathsf{B}(N,x)} n(y) = n(z)$ for some $z \neq \tilde{x}$, which means that $n(z) > n(\tilde{x})$. In this case, by (45) we have $\rho(\tilde{x}, z) \geq \phi[n(z)]$, and

$$2N \geq \rho(x, z) + \rho(x, \tilde{x}) \geq \rho(\tilde{x}, z) \geq \phi[n(z)], \tag{54}$$

which yields (53) for this case as well.

If (ii) holds, we let x_1 be the closest vertex to x, such that $n(x_1) > n(\tilde{x})$. Again, we take the vertex with the highest degree if more than one such vertex lies at the same distance from x. By (45) we have $\rho(\tilde{x}, x_1) \geq \phi[n(x_1)]$. If, for $N \geq N_x := \rho(x, x_1)$, we have $\max_{y \in \mathsf{B}(N,x)} n(y) = n(x_1)$ then

$$N \geq \rho(x, x_1) \geq \phi[n(x_1)] - \rho(x, \tilde{x}) \geq \phi[n(x_1)] - \frac{\phi[n(\tilde{x})]}{2} \geq \frac{\phi[n(x_1)]}{2},$$

which yields (53). Finally, let $\max_{y \in \mathsf{B}(N,x)} n(y) = n(z)$ for some $z \neq x_1$, which means that $n(z) > n(x_1)$. In this case, $\rho(x_1, z) \geq \phi[(n(z)]$, and we obtain (53) by applying (54) with $\tilde{x}$ replaced by x_1.

4.3. Proofs of Lemmas 4 and 5

First we prove an auxiliary statement. Recall that by $\vartheta(x, y)$ we denote a path with endpoints x and y. For a path ϑ, by V_ϑ we denote the set of its vertices. A path is called *simple* if none of its inner vertices are repeated. For $m \leq n$, let $\vartheta' = \{x_0, \dots, x_m\}$ and $\vartheta = \{y_0, \dots, y_n\}$ be such that $x_0 = y_k, x_1 = y_{k+1}, \dots, x_m = y_{k+m}$ for some $k = 0, \dots, n-m$. Then we say that ϑ' is a *subpath* of ϑ, and write $\vartheta' \subset \vartheta$.

Let $\Sigma_N(x)$ denote the family of all simple paths of length N originated at x. Then, for every $y \in \mathsf{S}(N, x)$, there exists $\vartheta \in \Sigma_N(x)$ such that $\vartheta = \vartheta(x, y)$. We use this fact for estimating the cardinality of $\mathsf{S}(N, x)$.

Proposition 1. (Cf. Assertion 6 of [3].) *In any graph* G*, for any* $x \in \mathsf{V}$ *and* $N \in \mathbb{N}$*, we have*

$$|\mathsf{S}(N, x)| \leq |\Sigma_N(x)| \leq \max_{\vartheta \in \Sigma_N(x)} \prod_{y \in \mathsf{V}_\vartheta \setminus \{x_N\}} n(y). \tag{55}$$

Proof. We prove this proposition by induction in N. For $N = 1$, estimate (55) is obvious. For any $N \geq 2$, we have

$$|\Sigma_N(x)| \leq n(x) \max_{y \sim x} |\Sigma_{N-1}^x(y)|,$$

where $\Sigma_{N-1}^x(y)$ is the corresponding family of paths in the graph which we obtain from G be deleting the edge $\langle x, y \rangle$. Every $\vartheta \in \Sigma_N(x)$ can be written in the form $\vartheta = \{x\tilde{\vartheta}\}$ with $\tilde{\vartheta} \in \Sigma_{N-1}^x(y)$ for some $y \sim x$. Then by the inductive assumption we have

$$|\Sigma_N(x)| \leq n(x) \max_{y \sim x} \max_{\tilde{\vartheta} \in \Sigma_{N-1}^x(y)} \prod_{z \in \mathsf{V}_{\tilde{\vartheta}} \setminus \{x_N\}} n(z) \leq \max_{\vartheta \in \Sigma_N(x)} \prod_{z \in \mathsf{V}_\vartheta \setminus \{x_N\}} n(z),$$

which completes the proof.

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Proof of Lemma 5. We will prove that estimate (52) holds with N_x being as in Lemma 6 and a given by

$$a = 2\mathrm{e}(1+\theta) + \log n_* + \frac{3\mathrm{e}}{\upsilon}\sum_{k=1}^{\infty}\frac{1}{k^{1+\varepsilon}}.$$

For any $N \geq N_x$, by (53) and (55) we obtain

$$G_\theta(N,x) \leq \exp\biggl((1+\theta)\log\phi^{-1}(2N) + \max_{\vartheta\in\Sigma_N(x)}\sum_{z\in\mathsf{V}_\vartheta\setminus\{x_N\}}\log n(z)\biggr). \tag{56}$$

By (49) we have

$$\phi^{-1}(2N) \leq \exp(2\mathrm{e}N). \tag{57}$$

If $\mathsf{V}_\vartheta \subset \mathsf{V}_*$ for any $\vartheta \in \Sigma_N(x)$, the second summand in (56) does not exceed $N\log n_*$, which certainly yields (52) for all $N \geq 1$. For $\mathsf{V}_\vartheta \cap \mathsf{V}_*^{\mathrm{c}} \neq \varnothing$, let N_x be as in Lemma 6. Then, for $N \geq \max\{N_x; \phi(n_*+1)/2\}$, we have

$$\{y \in \mathsf{B}(N,x)\colon n_*+1 \leq n(y) \leq \phi^{-1}(2N)\} \neq \varnothing.$$

Let k_* be the least $k \in \mathbb{N}$ such that $c_{k_*} \geq n_*+1$, where $c_k = \exp(\mathrm{e}^k)$, $k \in \mathbb{N}$. Then we set $b_{k_*} = n_*+1$ and $b_k = c_k$ for $k > k_*$. Let k_N be the largest k such that $b_k \leq \phi^{-1}(2N)$. For $k = k_*, \dots, k_N$ and a given $\vartheta \in \Sigma_N(x)$, let m_k^ϑ be the number of vertices $y \in \mathsf{V}_\vartheta$ such that $n(y) \in [b_k, b_{k+1}]$. Given $\tau \in (0,N)$, for any $\vartheta \in \Sigma_N(x)$, the number of vertices in V_ϑ which are at least a distance τ apart is $1+N/\tau$, at most. Therefore,

$$m_k^\vartheta \leq m_k := 1 + \frac{N}{\phi(b_k)} \leq \frac{3N}{\phi(b_k)}.$$

Taking this into account, by (49) we obtain

$$\max_{\vartheta\in\Sigma_N(x)}\sum_{z\in\mathsf{V}_\vartheta\setminus\{x_N\}}\log n(z) \leq N\log n_* + \sum_{k=k_*}^{k_N} m_k\log b_{k+1} \leq N\biggl(\log n_* + \frac{3\mathrm{e}}{\upsilon}\sum_{k=k_*}^{\infty}\frac{1}{k^{1+\varepsilon}}\biggr).$$

Applying (57) and the latter estimate in (56) we obtain (52) in this case also.

Remark 2. Our choice of ϕ made in (49) was predetermined by condition (57), which we used to estimate the first summand in (56), as well as by the condition that

$$\sum_{k=k_*}^{\infty}\frac{\log b_{k+1}}{\phi(b_k)} < \infty, \tag{58}$$

which was employed for estimating the second summand in (56), for a concrete choice of the sequence $\{b_k\}_{k\geq k_*}$ made therein. In principle, any ϕ obeying these two conditions (for some choice of $\{b_k\}_{k\geq k_*}$) can be used. For $b_k = k$, $k \geq k_* = n_*+1$, we can take $\phi(b) = b^{1+\varepsilon}$ for some $\varepsilon > 0$, which obviously obeys (57) and (58), but imposes a stronger repulsion (see (45)). Our choice of ϕ in (49) seems to be optimal.

Proof of Lemma 4. In view of (47), we have $\rho(x, y) \geq 2$ for any $x, y \in \mathsf{V}_*^{\mathrm{c}}$; hence, for two adjacent vertices, at least one should be in V_*. Taking this into account, by (4) and the triangle inequality, we derive

$$\begin{aligned}\Theta(\alpha, \theta) &= \sum_x [n(x)]^\theta \bigg(\sum_{y \sim x} [n(y)]^\theta\bigg) \exp(-\alpha\rho(o, x)) \\ &\leq n_*^\theta \mathrm{e}^\alpha \sum_y [n(y)]^{1+\theta} \exp(-\alpha\rho(o, y)) + n_*^\theta \sum_x [n(x)]^{1+\theta} \exp(-\alpha\rho(o, x)),\end{aligned}$$

which yields (51).

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