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Delta-invariants of complete intersection log del Pezzo surfaces

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We show that complete intersection log del Pezzo surfaces with amplitude one in weighted projective spaces are uniformly K-stable. As a result, they admit an orbifold Kähler–Einstein metric.

Keywords: K-stability; del Pezzo surface; complete intersection; delta invariant

1. Introduction

Throughout the article, the ground field is assumed to be the field of complex numbers. Let S be a codimension c complete intersection of type $(d_1, \dots, d_c)$ in a weighted projective space $\mathbb{P}(a_0, \dots, a_n)$ that is quasi-smooth, well-formed and $a_0 \leqslant a_1 \leqslant \cdots \leqslant a_n < d_1 \leqslant \cdots \leqslant d_c$. Suppose that S is a log del Pezzo surface. Then we have exactly two possibilities:

(A) Either $n = 3$ and $S \subset \mathbb{P}(a_0, a_1, a_2, a_3)$ is a hypersurface of degree

$$d < a_0 + a_1 + a_2 + a_3$$

with amplitude $I = a_0 + a_1 + a_2 + a_3 - d$

(B) Or $n = 4$ and $S \subset \mathbb{P}(a_0, a_1, a_2, a_3, a_4)$ is a complete intersection of two hypersurfaces of degrees d_1 and d_2 such that

$$d_1 + d_2 < a_0 + a_1 + a_2 + a_3 + a_4$$

with amplitude $I = a_0 + a_1 + a_2 + a_3 + a_4 - d_1 - d_2$.

In the case (A), Johnson and Kollár [9] found the complete list of all possibilities for the quintuple (a_0, a_1, a_2, a_3, d) in the case when the amplitude I is one. Moreover, they computed the α-invariants and proved the existence of the orbifold Kähler–Einstein metrics in the case when the quintuple (a_0, a_1, a_2, a_3, d) is not one of the following four quintuples

$$(1, 2, 3, 5, 10), \quad (1, 3, 5, 7, 15), \ (1, 3, 5, 8, 16), \ (2, 3, 5, 9, 18).$$

To prove the above statement they used the criterion that a log del Pezzo surface S admits an orbifold Kähler–Einstein metric whenever the α-invariant of S is bigger than $\frac{2}{3}$. Later, Araujo [1] computed the α-invariants for two of these four cases to show the existence of an orbifold Kähler–Einstein metric when $(a_0, a_1, a_2, a_3, d) = (1, 2, 3, 5, 10)$ or $(a_0, a_1, a_2, a_3, d) = (1, 3, 5, 7, 15)$ and the defining equation contains the monomial yzt where x, y, z and t are coordinates with weights $\mathrm{wt}(x) = a_0$, $\mathrm{wt}(y) = a_1$, $\mathrm{wt}(z) = a_2$ and $\mathrm{wt}(t) = a_3$. Finally, Cheltsov, Park and Shramov [2] computed the α-invariants for the remaining families.

For the case (A) every log del Pezzo surface S admits an orbifold Kähler–Einstein metric except possibly the case when $(a_0, a_1, a_2, a_3, d) = (1, 3, 5, 7, 15)$ and the defining equation does not contain the monomial yzt whose α-invariant is $\frac{8}{15}(< \frac{2}{3})$.

Recently Fujita and Odaka introduced δ-invariant which gives a strong criterion showing the uniform K-stability of $\mathbb{Q}$-Fano varieties (see [8]).

Theorem 1.1. *Let X be a $\mathbb{Q}$-Fano variety. Then X is uniformly K-stable if and only if $\delta(X) > 1$.*

The estimation of the δ-invariant has been investigated on several log del Pezzo surfaces in [4–7, 14, 15]. Moreover Li, Tian and Wang generalized in [13] the result of Chen, Donaldson, Sun and Tian for the K-polystability and the existence of the Kähler–Einstein metric to some singular Fano varieties. In virtue of the δ-invariant method and the result [13], the paper [3] completes the problem of the existence of the (orbifold) Kähler–Einstein metric on del Pezzo hypersurfaces with $I = 1$, case (A):

Theorem 1.2 [3]. *Let S be a quasi-smooth hypersurface in $\mathbb{P}(1, 3, 5, 7)$ of degree 15 such that its defining equation does not contain yzt. Then the surface S admits an orbifold Kähler–Einstein metric.*

Corollary 1.3. *Every quasi-smooth hypersurface with $I = 1$ admits an orbifold Kähler-Einstein metric.*

In [10] and [11], we classified the log del Pezzo surfaces S for the case (B) when $S \subset \mathbb{P}(a_0, a_1, a_2, a_3, a_4)$ are quasi-smooth and well-formed complete intersection log del Pezzo surfaces given by two quasi-homogeneous polynomials of degrees d_1 and d_2 with amplitude 1, and not being the intersection of a linear cone with another hypersurface. Then there are 42 families. We denote family No. i as the number i in the first column Γ of the table which is represented in [11, section 5].

Suppose that the log del Pezzo surface S is not one of the following:

- No. 3 : a complete intersection of two hypersurfaces of degrees 6 and 8 embedded in $\mathbb{P}(1, 2, 3, 4, 5)$ such that the defining equation of the hypersurface of degree 6 does not contain the monomial yt, where y is the coordinate function of weight 2 and t is the coordinate function of weight 4.

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- No. 40 : a complete intersection of two hypersurfaces of degree $2n$ embedded in $\mathbb{P}(1, 1, n, n, 2n-1)$ where n is a positive integer.

Then the α-invariant of S is bigger than $\frac{2}{3}$, in fact they are bigger or equal to one, so that it admits an orbifold Kähler–Einstein metric (see [**10**, theorem 1.9] and [**11**, theorem 1.2]).

The present article completes the existence of the orbifold Kähler–Einstein metric of the remaining two cases.

THEOREM 1.4. *Let S be a quasi-smooth member of family No. i with $i \in \{3, 40\}$. Then the log del Pezzo surface S is uniformly K-stable so that it admits an orbifold Kähler–Einstein metric.*

COROLLARY 1.5. *Every quasi-smooth weighted complete intersection with $I = 1$ admits an orbifold Kähler–Einstein metric.*

2. Preliminary

2.1. Notation

Throughout the paper we use the following notations:

- For positive integers a_0, a_1, a_2, a_3 and a_4, $\mathbb{P}(a_0, a_1, a_2, a_3, a_4)$ is the weighted projective space. We assume that $a_0 \leqslant a_1 \leqslant a_2 \leqslant a_3 \leqslant a_4$.
- We usually write x, y, z, t and w for the weighted homogeneous coordinates of $\mathbb{P}(a_0, a_1, a_2, a_3, a_4)$ with weights $\mathrm{wt}(x) = a_0$, $\mathrm{wt}(y) = a_1$, $\mathrm{wt}(z) = a_2$, $\mathrm{wt}(t) = a_3$ and $\mathrm{wt}(w) = a_4$.
- $S \subset \mathbb{P}(a_0, a_1, a_2, a_3, a_4)$ denotes a quasi-smooth complete intersection log del Pezzo surface given by quasi-homogeneous polynomials of degrees d_1 and d_2.
- The integer $I = a_0 + a_1 + a_2 + a_3 + a_4 - d_1 - d_2$ is called the amplitude of S.
- H_* is the hyperplane section on the log del Pezzo surface S cut out by the equation $* = 0$.
- p_x denotes the point on S given by $y = z = t = w = 0$. The points p_y, p_z, p_t and p_w are defined in a similar way.
- $-K_S$ denotes the anti-canonical divisor of S.

2.2. Foundation

X is $\mathbb{Q}$-Fano variety, i.e., a normal projective $\mathbb{Q}$-factorial variety with at most terminal singularities such that $-K_X$ is ample.

DEFINITION 2.1. *Let (X, D) be a pair, that is, D is an effective $\mathbb{Q}$-divisor, and let $\mathsf{p} \in X$ be a point. We define the log canonical threshold (LCT, for short) of (X, D)*

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and the log canonical threshold of (X, D) *at* p *to be the numbers*

$$\mathrm{lct}(X, D) = \sup\{\, c \mid (X, cD) \text{ is log canonical}\},$$
$$\mathrm{lct}_{\mathsf{p}}(X, D) = \sup\{\, c \mid (X, cD) \text{ is log canonical at } \mathsf{p}\},$$

respectively. We define

$$\mathrm{lct}_{\mathsf{p}}(X) = \inf\{\, \mathrm{lct}_{\mathsf{p}}(X, D) \mid D \text{ is an effective } \mathbb{Q}\text{-divisor}, D \equiv -K_X\},$$

and for a subset $\Sigma \subset X$*, we define*

$$\mathrm{lct}_{\Sigma}(X) = \inf\{\mathrm{lct}_{\mathsf{p}}(X) \mid \mathsf{p} \in \Sigma\}.$$

The number $\alpha(X) := \mathrm{lct}_X(X)$ *is called the global log canonical threshold (GLCT, for short) or the* α*-invariant of* X

Let S be a surface with at most cyclic quotient singularities, and let D be an effective $\mathbb{Q}$-divisor on X.

LEMMA 2.2 [**12**]. *Let* p *be a smooth point of* S*. Suppose that the log pair* (S, D) *is not log canonical at the point* p*. Then* $\mathrm{mult}_{\mathsf{p}}(D) > 1$*.*

Suppose that S has a cyclic quotient singular point q of type $\frac{1}{r}(a, b)$. Then there is an orbifold chart $\pi\colon \bar{U} \to U$ for some open set $\mathsf{q} \in U$ on S such that $\bar{U}$ is smooth and π is a cyclic cover of degree r branched over q.

LEMMA 2.3 [**12**]. *Let* $\bar{\mathsf{q}} \in \bar{U}$ *be the point such that* $\pi(\bar{\mathsf{q}}) = \mathsf{q}$*. Then the log pair* $(U, D|_U)$ *is log canonical at the point* q *if and only if the log pair* $(\bar{U}, \bar{D}|_{\bar{U}})$ *is log canonical at the point* $\bar{\mathsf{q}}$ *where* $\bar{D} = \pi^*(D|_U)$*.*

DEFINITION 2.4 [**8**]. *Let* k *be a positive integer. We set* $h = h^0(S, -kK_S)$*. Given any basis*

$$s_1, \ldots, s_h$$

of $H^0(S, -kK_S)$*, taking the corresponding divisors* $D_1, \ldots, D_h$ *with* $D_i \sim -kK_S$*, we get an anti-canonical* $\mathbb{Q}$*-divisor*

$$D := \frac{D_1 + \ldots + D_h}{kh}.$$

We call this kind of anti-canonical $\mathbb{Q}$*-divisor an anti-canonical* $\mathbb{Q}$*-divisor of* k*-basis type.*

Then we can define the δ-invariant of S using an anti-canonical $\mathbb{Q}$-divisor of k-basis type. The definition of the δ-invariant of a Fano variety is the following.

DEFINITION 2.5 [**8**]. *For* $k \in \mathbb{Z}_{>0}$*, set*

$$\delta_k(S) := \inf\{\ \mathrm{lct}(S, D) \mid D \text{ is of } k \text{ -basis type }\}.$$

Moreover, we define

$$\delta(S) := \limsup_{k \to \infty} \delta_k(S).$$

It is called the δ*-invariant of* S*.*

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Definition 2.6. *Let X be an irreducible projective variety of dimension n, and let D be a Cartier divisor on X. The volume of D is defined to be the non-negative real number*

$$\mathrm{vol}(D) = \mathrm{vol}_X(D) = \limsup_{m\to\infty} \frac{h^0(X, \mathcal{O}_X(mD))}{m^n/n!}.$$

For a $\mathbb{Q}$-divisor D on the surface S we can define its volume using the identity

$$\mathrm{vol}(D) = \frac{\mathrm{vol}(\lambda D)}{\lambda^2}$$

for an appropriate positive rational number λ.

Let D be an anti-canonical $\mathbb{Q}$-divisor of k-basis type with $k \gg 1$, and let C be an irreducible reduced curve on S. We write

$$D = aC + \Delta$$

where a is non-negative real number and Δ is an effective $\mathbb{Q}$-divisor such that $C \not\subset \mathrm{Supp}(\Delta)$. Let

$$\tau = \sup\{\ x \in \mathbb{R}_{>0} \mid D - xC \text{ is pseudoeffective } \}.$$

In the case that D is an ample $\mathbb{Q}$-divisor of k-basis type with $k \gg 1$ we can find a better bound for a. One such estimate is given by the following very special case of [8, lemma 2.2].

Theorem 2.7 [3, theorem 2.9]. *Suppose that D is a big $\mathbb{Q}$-divisor of k-basis type for $k \gg 1$. Then*

$$a \leqslant \int_0^\tau \mathrm{vol}(D - xC)dx + \epsilon_k$$

where ϵ_k is a small constant depending on k such that $\epsilon_k \to 0$ as $k \to \infty$.

Corollary 2.8 [3, corollary 2.10]. *Suppose that D is a big $\mathbb{Q}$-divisor of k-basis type for $k \gg 0$, and*

$$C \sim_{\mathbb{Q}} \mu D$$

for some positive rational number μ. Then

$$a \leqslant \frac{1}{3\mu} + \epsilon_k,$$

where ϵ_k is a small constant depending on k such that $\epsilon_k \to 0$ as $k \to \infty$.

3. Family No. 3

In this section we prove the following theorem:

Theorem 3.1. *Let S be a quasi-smooth member of family No. 3. Then $\delta(S) \geqslant \frac{5}{4}$. Moreover, S admits an orbifold Kähler–Einstein metric.*

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Proof. Let D be an anti-canonical $\mathbb{Q}$-divisor of k-basis type on S with $k \gg 0$. By lemmas 3.2–3.4 the log pair $(S, \frac{5}{4}D)$ is log canonical. Therefore $\delta(S) \geqslant \frac{5}{4}$. □

We divide the proof of the above theorem into a sequence of lemmas. Let $S \subset \mathbb{P}(1, 2, 3, 4, 5)$ be a quasi-smooth complete intersection log del Pezzo surface given by two quasi-homogeneous polynomials of degrees 6 and 8. By suitable coordinate change we may assume that S is given by

$$\begin{aligned} &wx + \xi ty + z^2 + y^3 = 0,\\ &wz + t^2 + g(x, y) = 0, \end{aligned}$$

where ξ is a constant and $g(x, y)$ is a quasi-homogeneous polynomial of degree 8. Then S is singular only at the point p_w, which is a cyclic quotient singularity of type $\frac{1}{5}(4, 3)$. Since the defining equation of degree 6 of a member of family No. 3 does not contain the monomial ty, $\xi = 0$. Thus S is given by

$$\begin{aligned} F &= wx + z^2 + y^3 = 0,\\ G &= wz + t^2 + g(x, y) = 0. \end{aligned}$$

Let H_x be the hyperplane section given by $x = 0$. Then it is isomorphic to the variety embedded in $\mathbb{P}(2, 3, 4, 5)$ given by

$$\begin{aligned} &z^2 + y^3 = 0,\\ &wz + t^2 + \zeta y^4 = 0, \end{aligned}$$

where $\zeta = g(0, 1)$. We consider the open set $U = S \setminus H_w$ where H_w is the hyperplane section given by $w = 0$. $H_x|_U$ is isomorphic to the $\mathbb{Z}_5$-quotient of the affine curve given by

$$(t^2 + \zeta y^4)^2 + y^3 = 0 \tag{3.1}$$

in $\mathbb{A}^2$. From the equation (3.1), we can see that H_x is irreduciblyreduced and singular at the point p_w. Also, we have $\mathrm{lct}(S, H_x) = \frac{7}{12}$.

Let D be an anti-canonical $\mathbb{Q}$-divisor of k-basis type on S with $k \gg 0$. We put $\lambda = \frac{5}{4}$.

Lemma 3.2. *The log pair $(S, \lambda D)$ is log canonical along $H_x \setminus \{\mathsf{p}_w\}$.*

Proof. Suppose that the log pair $(S, \lambda D)$ is not log canonical at some point $\mathsf{p} \in H_x \setminus \{\mathsf{p}_w\}$. We write

$$D = aH_x + \Delta$$

where a is non-negative rational number and Δ is an effective divisor such that $H_x \not\subset \mathrm{Supp}(\Delta)$. By corollary 2.8 we have $a \leqslant \frac{1}{3} + \epsilon_k < \frac{9}{25}$ for $k \gg 0$. Since $\lambda a \leqslant 1$ the log pair $(S, H_x + \lambda\Delta)$ is not log canonical at the point p. By the inversion of adjunction formula the log pair $(H_x, \lambda\Delta|_{H_x})$ is not log canonical at point p. We have the inequalities

$$\frac{1}{\lambda} < \mathrm{mult}_{\mathsf{p}}(\Delta|_{H_x}) \leqslant \Delta \cdot H_x = (D - aH_x) \cdot H_x = \frac{2}{5} - \frac{2}{5}a,$$

which imply that $a < -1$. This is impossible. Therefore the log pair $(S, \lambda D)$ is log canonical along $H_x \setminus \{\mathsf{p}_w\}$. □

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LEMMA 3.3. *The log pair $(S, \lambda D)$ is log canonical long $S \setminus H_x$.*

Proof. Suppose that the log pair $(S, \lambda D)$ is not log canonical at some point $\mathsf{p} \in S \setminus H_x$. By suitable coordinate change we can assume that $\mathsf{p} = \mathsf{p}_x$.

Let C be the curve on S cut out by the equation $y = 0$. Then C passes through the point p. Since the curve C is smooth at p_w and $C \cdot H_x = \frac{4}{5}$, it is irreducible and reduced. Let $\mathcal{L}$ be the pencil cut out by the equations $\alpha xy + \beta z = 0$ where $[\alpha : \beta] \in \mathbb{P}^1$. The base locus of $\mathcal{L}$ is given by $z = yx = 0$. Since $S \cap H_x \cap H_z = \{\mathsf{p}_y\}$ and $S \cap H_y \cap H_z = \{\mathsf{p}_x, \mathsf{p}_w\}$ we have $\mathrm{BS}(\mathcal{L}) = \{\mathsf{p}_x, \mathsf{p}_y, \mathsf{p}_w\}$. Thus there is a general member $M \in \mathcal{L}$ such that $\mathsf{p} \in M$ and $C \not\subset \mathrm{Supp}(M)$. We have

$$\mathrm{mult}_{\mathsf{p}}(M)\,\mathrm{mult}_{\mathsf{p}}(C) \leqslant M \cdot C = \frac{12}{5}.$$

It implies that $\mathrm{mult}_{\mathsf{p}}(C)$ is either 1 or 2. We write

$$D = bC + \Sigma$$

where b is non-negative rational number and Σ is an effective $\mathbb{Q}$-divisor such that $C \not\subset \mathrm{Supp}(\Sigma)$. By Corollary 2.8, we have $b \leqslant \frac{1}{6} + \epsilon_k < \frac{1}{3}$ for $k \gg 0$.

We assume that $\mathrm{mult}_{\mathsf{p}}(C) = 1$. Since $\lambda b \leqslant 1$ the log pair $(S, C + \lambda\Sigma)$ is not log canonical at the point p. By the inversion of adjunction formula the log pair $(C, \lambda\Sigma|_C)$ is not log canonical at the point p. We have the inequalities

$$\frac{1}{\lambda} < \mathrm{mult}_{\mathsf{p}}(\Sigma|_C) \leqslant \Sigma \cdot C = (D - bC) \cdot C = \frac{4}{5} - \frac{8}{5}b.$$

They imply that $b < 0$. It is impossible. Thus $\mathrm{mult}_{\mathsf{p}}(C) = 2$. From lemma 2.2 we have the following inequalities

$$2\left(\frac{1}{\lambda} - 2b\right) < \mathrm{mult}_{\mathsf{p}}(C)\,\mathrm{mult}_{\mathsf{p}}(D - bC) \leqslant C \cdot (D - bC) = \frac{4}{5} - \frac{8}{5}b.$$

Then we have $\frac{1}{3} < b$. It is impossible. Thus the log pair $(S, \lambda D)$ is log canonical along $S \setminus H_x$. □

LEMMA 3.4. *The log pair $(S, \lambda D)$ is log canonical at p_w.*

Proof. Suppose that the log pair $(S, \lambda D)$ is not log canonical at p_w. We consider the open set U given by $w \neq 0$. Then we may regard y and t are local coordinates with weights $\mathrm{wt}(y) = 4$ and $\mathrm{wt}(t) = 3$ in U. Let $\pi \colon \bar{S} \to S$ be the weighted blow-up at p_w with weights $\mathrm{wt}(y) = 4$ and $\mathrm{wt}(t) = 3$. Then $\bar{S}$ has the singular points q_1 and q_2 of types $\frac{1}{4}(1, 1)$ and $\frac{1}{3}(1, 1)$, respectively. We have

$$K_{\bar{S}} \sim_{\mathbb{Q}} \pi^*(K_S) + \frac{2}{5}E, \quad \bar{H_x} \sim_{\mathbb{Q}} \pi^*(H_x) - \frac{12}{5}E$$

where $\bar{H_x}$ is the strict transform of H_x and E is the exceptional divisor of π. We write

$$D = aH_x + \Delta$$

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where a is a non-negative rational number and Δ is an effective $\mathbb{Q}$-divisor such that $H_x \not\subset \mathrm{Supp}(\Delta)$. By corollary 2.8, we have

$$a \leqslant \frac{9}{25} \tag{3.2}$$

for $k \gg 0$. We also have

$$\bar{\Delta} \sim_{\mathbb{Q}} \pi^*(\Delta) - mE$$

where $\bar{\Delta}$ is the strict transform of Δ and m is a non-negative rational number. To obtain a bound of m we consider the inequality

$$0 \leqslant \bar{\Delta} \cdot \bar{H}_x = (\pi^*(\Delta) - mE) \cdot \left(\pi^*(H_x) - \frac{12}{5}E\right) = \Delta \cdot H_x + \frac{12}{5}mE^2.$$

Since $\Delta \cdot H_x = (D - aH_x) \cdot H_x = \frac{2}{5} - \frac{2}{5}a$ and $E^2 = -\frac{5}{12}$, we have

$$m \leqslant \frac{2}{5} - \frac{2}{5}a. \tag{3.3}$$

Meanwhile, we have

$$K_{\bar{S}} + \lambda(a\bar{H}_x + \bar{\Delta}) + \mu E \sim_{\mathbb{Q}} \pi^*(K_S + \lambda D)$$

where

$$\mu = \lambda\left(\frac{12}{5}a + m\right) - \frac{2}{5}.$$

It implies that the log pair $(\bar{S},\, \lambda(a\bar{H}_x + \bar{\Delta}) + \mu E)$ is not log canonical at some point $\mathsf{q} \in E$. From the inequalities (3.2) and (3.3) we have $\mu \leqslant 1$. It implies that the log pair $(\bar{S},\, \lambda(a\bar{H}_x + \bar{\Delta}) + E)$ is not log canonical at the point q. We consider the case that E is smooth at the point q. By the inversion of adjunction formula the log pair $(E,\, \lambda(a\bar{H}_x + \bar{\Delta})|_E)$ is not log canonical at q. If $\mathsf{q} \notin \bar{H}_x$ then the log pair $(E,\, \lambda\bar{\Delta}|_E)$ is not log canonical at q. From this we have the inequalities

$$\frac{1}{\lambda} < \mathrm{mult}_{\mathsf{q}}(\bar{\Delta}|_E) \leqslant \bar{\Delta} \cdot E = -mE^2 = \frac{5}{12}m.$$

They imply that $\frac{48}{25} < m$. From the inequality (3.3), it is impossible. Thus $\mathsf{q} \in \bar{H}_x$. From lemma 2.2 and the inequality (3.3) we have the inequalities

$$\frac{1}{\lambda} < \mathrm{mult}_{\mathsf{q}}((a\bar{H}_x + \bar{\Delta})|_E) \leqslant (a\bar{H}_x + \bar{\Delta}) \cdot E = a + \frac{5}{12}m \leqslant \frac{1+5a}{6}.$$

They imply that $\frac{19}{25} < a$. From the inequality (3.2), it is impossible. Thus E is singular at the point q. Also, the point q is either q_1 or q_2.

Suppose that $\mathsf{q} = \mathsf{q}_1$. Then there is a cyclic cover $\varphi\colon \tilde{U} \to \bar{U}$ of degree 4 branched over q for some open set $\mathsf{q} \in \bar{U}$ on $\bar{S}$ such that $\tilde{U}$ is smooth. From lemma 2.3, the log pair $(\tilde{U},\, \lambda\tilde{\Delta} + \tilde{E})$ is not log canonical at some point $\tilde{\mathsf{q}}$ where $\tilde{\Delta} = \varphi^*(\Delta|_U)$,

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$\tilde{E} = \varphi^*(E|_U)$ and $\varphi(\tilde{\mathsf{q}}) = \mathsf{q}$. By the inversion of adjunction formula the log pair $(\tilde{E},\, \lambda\tilde{\Delta}|_{\tilde{E}})$ is not log canonical at the point $\tilde{\mathsf{q}}$. From this we have the inequalities

$$\frac{1}{\lambda} < \mathrm{mult}_{\tilde{\mathsf{q}}}(\tilde{\Delta}|_{\tilde{E}}) \leqslant 4\bar{\Delta} \cdot E = -4mE^2 = \frac{5}{3}m.$$

They imply that $\frac{12}{25} < m$. From the inequality (3.3), it is impossible. Thus $\mathsf{q} = \mathsf{q}_2$. Similarly, we can see that this case is impossible. Therefore the log pair $(S,\, \lambda D)$ is log canonical at the point p_w. □

By the above lemmas we prove that the log pair $(S,\, \lambda D)$ is log canonical.

4. On smooth points of family No. 40

Let $S_n \subset \mathbb{P}(1,\, 1,\, n,\, n,\, 2n-1)$ be a quasi-smooth complete intersection log del Pezzo surface given by two quasi-homogeneous polynomials of degree $2n$, where n is a positive integer bigger than 1. By suitable coordinate change we may assume that S_n is given by

$$\begin{aligned} &wx + z^2 + zf_n(x,y) + t\hat{f}_n(x,y) + f_{2n}(x,y) = 0,\\ &wy + t^2 + zg_n(x,y) + t\hat{g}_n(x,y) + g_{2n}(x,y) = 0 \end{aligned}$$

where f_i, $\hat{f}_i$, g_i and $\hat{g}_i$ are homogeneous polynomials of degree i. Then S_n is only singular at the point p_w of type $\frac{1}{2n-1}(1,\, 1)$. In the paper [**10**], we have $\alpha(S_2) = 7/10$. It implies that S_2 admits an orbifold Kähler–Einstein metric. Thus we only consider the cases that $n \geqslant 3$.

Let D be an anti-canonical $\mathbb{Q}$-divisor of k-basis type on S_n with $k \gg 0$. We set $\lambda = \frac{6n}{4n+3}$. To prove that $\delta(S_n) > 1$ along the smooth points of S_n, we consider the following.

Lemma 4.1. *The log pair $(S_n,\, \lambda D)$ is log canonical along $S_n \setminus \{\mathsf{p}_w\}$*

Proof. For the convenience, we set $S = S_n$. Suppose that the log pair $(S,\, \lambda D)$ is not log canonical at some point $\mathsf{p} \in S \setminus \{\mathsf{p}_w\}$. Let $\mathcal{L} = |-K_S|$ be the pencil cut out on S by the equations $\alpha x + \beta y = 0$ where $[\alpha : \beta] \in \mathbb{P}^1$. Since the point p is not the point p_w, there is the unique curve $C \in \mathcal{L}$ passing through p. Without loss of generality we can assume that p is contained in the open set U_x given by $x = 1$. Then C is given by the equation $y = \xi x$ on S where ξ is a constant. On the open set U_x, the affine curve $C|_{U_x}$ is given by

$$\begin{aligned} &w + z^2 + zf_n(1,\xi) + t\hat{f}_n(1,\xi) + f_{2n}(1,xi) = 0,\\ &\xi w + t^2 + zg_n(1,\xi) + t\hat{g}_n(1,\xi) + g_{2n}(1,\xi) = 0 \end{aligned}$$

Thus it is isomorphic to the variety given by

$$\xi_1 z^2 + t^2 + \xi_2 z + \xi_3 t + \xi_4 = 0 \tag{4.1}$$

where $\xi_1 \ldots,\, \xi_4$ are constants. Since S is quasi-smooth at least one ξ_i in $i \in \{1,\, 2,\, 3,\, 4\}$ is non-zero. It implies that the rank of the quadratic equation (4.1)

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is either 1 or 2. We assume that C is irreducible. By the quadratic equation (4.1), C is smooth at the point p. We write

$$D = aC + \Delta$$

where Δ is an effective $\mathbb{Q}$-divisor such that $C \not\subset \mathrm{Supp}(\Delta)$ and a is a non-negative constant. By corollary 2.8 we have $\lambda a \leqslant 1$. By the inversion of adjunction formula, the log pair $(C, \lambda\Delta|_C)$ is not log canonical at p. Then we have the inequalities

$$\frac{1}{\lambda} < \mathrm{mult}_{\mathsf{p}}(\Delta|_C) \leqslant \Delta \cdot C = \frac{4}{2n-1} - \frac{4a}{2n-1}.$$

The above inequalities imply that a is negative. This is impossible. Thus C is reducible. We now turn to the case that C is the sum of two irreducible curves L_1 and L_2, that is, we write

$$C = L_1 + L_2.$$

Then L_1 and L_2 satisfy the following intersection numbers:

$$L_1 \cdot (-K_S) = L_2 \cdot (-K_S) = \frac{2}{2n-1}, \quad L_1 \cdot L_2 = \frac{2n}{2n-1}, \quad L_1^2 = L_2^2 = -\frac{2n-2}{2n-1}.$$

Without loss of generality we can assume that $\mathsf{p} \in L_1$. We write

$$D = bL_1 + \Sigma$$

where Σ is an effective $\mathbb{Q}$-divisor such that $L_1 \not\subset \mathrm{Supp}(\Sigma)$ and b is a non-negative number. By theorem 2.7, we have

$$b \leqslant \frac{1}{D^2} \int_0^{\tau(L_1)} \mathrm{vol}(D - xL_1)dx + \epsilon_k$$

where ϵ_k is a small constant depending on k such that $\epsilon_k \to 0$ as $k \to \infty$. Since

$$D - xL_1 \sim_{\mathbb{Q}} (1-x)L_1 + L_2$$

and $L_2^2 < 0$, we have $\mathrm{vol}(D - xL_1) = 0$ for $x \geqslant 1$. It implies that $\tau(L_1) = 1$. Meanwhile, the equalities

$$(D - xL_1) \cdot L_2 = ((1-x)L_1 + L_2) \cdot L_2 = \frac{2}{2n-1} - \frac{2n}{2n-1}x$$

imply that $(D - xL_1)$ is nef whenever $\frac{1}{n} \geqslant x$. Thus

$$\mathrm{vol}(D - xL_1) = (D - xL_1)^2 = \frac{4}{2n-1} - \frac{4}{2n-1}x - \frac{2n-2}{2n-1}x^2$$

for $\frac{1}{n} \geqslant x$. We next consider the volume of $D - xL_1$ for $1 \geqslant x \geqslant \frac{1}{n}$. Let

$$P = (1-x)D + (1-x)\frac{1}{n-1}L_2$$

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be the nef divisor for $1 \geqslant x \geqslant \frac{1}{n}$. Then we write

$$D - xL_1 = P + \left(\frac{n}{n-1}x - \frac{1}{n-1}\right) L_2.$$

Since $P \cdot L_2 = 0$, the right-hand side of the above equation is the Zariski decomposition of $D - xL_1$. Thus

$$\mathrm{vol}(D - xL_1) = P^2 = \frac{2}{n-1}(1-x)^2$$

for $1 \geqslant x \geqslant \frac{1}{n}$. Then we have

$$\begin{aligned}
&\frac{1}{D^2}\int_0^{\tau(L_1)} \mathrm{vol}(D - xL_1)dx \\
&\quad = \frac{2n-1}{4}\left(\int_0^{\frac{1}{n}} \frac{4}{2n-1} - \frac{4}{2n-1}x - \frac{2n-2}{2n-1}x^2 dx + \int_{\frac{1}{n}}^1 \frac{2}{n-1}(1-x)^2 dx\right) \\
&\quad = \frac{2n-1}{4}\left(\frac{12n^2 - 8n + 2}{3(2n-1)n^3} + \frac{2(n-1)^2}{3n^3}\right) = \frac{2n+1}{6n}.
\end{aligned}$$

Thus we obtain

$$b \leqslant \frac{2n+1}{6n} + \epsilon_k.$$

It implies that $\lambda b \leqslant 1$. By the inversion of adjunction formula we have

$$\frac{1}{\lambda} < (D - bL_1) \cdot L_1 = \frac{2}{2n-1} + \frac{2n-2}{2n-1}b.$$

It implies that

$$\frac{(2n-3)(4n+1)}{6n(2n-2)} = \left(\frac{1}{\lambda} - \frac{2}{2n-1}\right)\frac{2n-1}{2n-2} < b.$$

This is impossible. Therefore the log pair $(S, \lambda D)$ is log canonical along $S \setminus \{\mathsf{p}_w\}$. □

5. On the singular point of family No. 40

In this section we prove the following theorem.

THEOREM 5.1. *Let $S_n \subset \mathbb{P}(1, 1, n, n, 2n-1)$ be a quasi-smooth member of family No. 40 where n is a positive integer. Then $\delta(S_n) > \frac{6n}{4n+3}$. Moreover, S_n admits an orbifold Kähler–Einstein metric.*

We divide the proof of the above theorem into a sequence of lemmas.

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5.1. Basis

Let $\mathcal{L} = H^0(S_n, \mathcal{O}_{S_n}(k))$ be the vector space where k is a positive integer. In this subsection, we find a monomial basis of $\mathcal{L}$. We define a subset of $\mathcal{L}$ as follows:

$$\mathcal{B} = \left\{ f \in \mathbb{C}[x,y,z,t,w]_k \;\middle|\; \begin{array}{l} f \text{ is a monomial whose form is one of the following:} \\ w^e, z^c t^d w^e, x^a y^b t^d, x^a y^b z t^d \text{ or } x^a z^c t^d. \end{array} \right\}$$

where $\mathbb{C}[x, y, z, t, w]_k$ is the set of quasi-homogeneous polynomials of degree k with weights $\mathrm{wt}(x) = \mathrm{wt}(y) = 1$, $\mathrm{wt}(z) = \mathrm{wt}(t) = n$ and $\mathrm{wt}(w) = 2n - 1$. The equations

$$-wx = z^2 + zf_n(x,y) + t\hat{f}_n(x,y) + f_{2n}(x,y) \tag{5.1}$$

and

$$-wy = t^2 + zg_n(x,y) + t\hat{g}_n(x,y) + g_{2n}(x,y) \tag{5.2}$$

hold in S_n. From the equations (5.1) and (5.2), we can obtain

$$yz^2 = xt^2 + zh_{n+1}(x,y) + t\hat{h}_{n+1}(x,y) + h_{2n+1}(x,y). \tag{5.3}$$

From the equations (5.1), (5.2) and (5.3) we can see that $\mathcal{L}$ is generated by $\mathcal{B}$ on S_n.

Claim. The set $\mathcal{B}$ is the basis of $\mathcal{L}$.

In a neighbourhood U of S_n at p_w, we may regard z and t are local coordinates with weights $\mathrm{wt}(z) = 1$ and $\mathrm{wt}(t) = 1$. Then U is isomorphic to the quotient of $\mathbb{C}^2$ by the action $\zeta \cdot (z, t) \mapsto (\zeta z, \zeta t)$ where ζ is a primitive $(2n-1)$-th root of unity. We have the isomorphism $\sigma\colon \mathbb{C}/\mathbb{Z}_{2n-1} \to U$ given by $(z, t) \mapsto (z^2 + f_{>2n}, t^2 + g_{>2n}, z, t)$ where $f_{>2n}$ and $g_{>2n}$ are power series such that the orders are greater than $2n$. Then for a section $s(x, y, z, t, w) \in \mathcal{L}$ the local equation in U is given by $\sigma^*(s(x, y, z, t, 1))$. We consider the following set:

$$\mathcal{T} = \left\{ g \in \mathbb{C}[z,t] \;\middle|\; \begin{array}{l} \text{There is a monomial } \mathbf{x} \text{ in } \mathcal{B} \text{ such that} \\ \text{the Zariski tangent term of } \sigma^*(\mathbf{x}) \text{ is } g. \end{array} \right\}.$$

Let $\mathbf{x} = x^a y^b z^c t^d w^e$ be a monomial in $\mathcal{L}$. Then $\sigma^*(\mathbf{x})$ is

$$(z^2 + f_{>2n})^a (t^2 + g_{>2n})^b z^c t^d = z^{2a+c} t^{2b+d} + h(z,t)$$

where $h(z, t)$ is the power series such that the order of $h(z, t)$ is greater than $2a + 2b + c + d$. Thus the Zariski tangent term of $\sigma^*(\mathbf{x})$ is $z^{2a+c}t^{2b+d}$. It implies that every element of $\mathcal{T}$ is a monomial in $\mathbb{C}[z, t]$.

Lemma 5.2. *The number of elements of the set $\mathcal{T}$ is equal to the number of elements of the set $\mathcal{B}$.*

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Proof. Let $\mathbf{x_1} = x^{a_1}y^{b_1}z^{c_1}t^{d_1}$ and $\mathbf{x_2} = x^{a_2}y^{b_2}z^{c_2}t^{d_2}$ be monomials in the set $\mathcal{B}$ such that the Zariski tangent terms of $\sigma^*(\mathbf{x_1})$ and $\sigma^*(\mathbf{x_2})$ are equal. Then we have

$$c_1 + 2a_1 = c_2 + 2a_2, \qquad d_1 + 2b_1 = d_2 + 2b_2.$$

Since the two monomials $\mathbf{x_1}$ and $\mathbf{x_2}$ have same degree, we have

$$a_1 + b_1 + n(c_1 + d_1) = a_2 + b_2 + n(c_2 + d_2).$$

From the above equations, we obtain the equations

$$a_1 + b_1 = a_2 + b_2, \qquad c_1 + d_1 = c_2 + d_2.$$

If $a_1 = a_2$ then we have $b_1 = b_2$, $c_1 = c_2$ and $d_1 = d_2$. Thus we can assume that $a_1 > a_2$. Then we have $b_1 < b_2$, $c_1 < c_2$ and $d_1 > d_2$. We can write the two monomials $\mathbf{x_1}$ and $\mathbf{x_2}$ as

$$x^{a_2}y^{b_1}z^{c_1}t^{d_2}x^{a_1-a_2}t^{d_1-d_2}, \qquad x^{a_2}y^{b_1}z^{c_1}t^{d_2}y^{b_2-b_1}z^{c_2-c_1}.$$

They imply that $2(a_1 - a_2) = c_2 - c_1$ and $2(b_2 - b_1) = d_1 - d_2$. We also have $a_1 - a_2 = b_2 - b_1$ and $c_2 - c_1 = d_1 - d_2$. Thus the two monomials $\mathbf{x_1}$ and $\mathbf{x_2}$ are

$$x^{a_2}y^{b_1}z^{c_1}t^{d_2}(xt^2)^{a_1-a_2}, \qquad x^{a_2}y^{b_1}z^{c_1}t^{d_2}(yz^2)^{a_1-a_2}.$$

However monomials of the form $(yz^2)^{\xi}x^ay^bz^ct^d$ are not contained in the set $\mathcal{B}$ where ξ is a positive integer. Therefore the two monomials $\mathbf{x_1}$ and $\mathbf{x_2}$ are equal. □

By lemma 5.2, we obtain the following.

Corollary 5.3. *The set $\mathcal{B}$ is the basis of $\mathcal{L}$.*

Proof. We consider the following set:

$$\mathcal{Z} = \left\{ g \in \mathbb{C}[z,t] \;\middle|\; \begin{array}{l}\text{There is a section } s \text{ in } \mathcal{L} \text{ such that}\\ \text{the Zariski tangent term of } \sigma^*(s) \text{ is } g.\end{array} \right\}.$$

It is obvious that $\dim_{\mathbb{C}} \mathcal{Z} \leqslant \dim_{\mathbb{C}} \mathcal{L}$. Since $\mathcal{T} \subset \mathcal{Z}$, we have $|\mathcal{T}| \leqslant \dim_{\mathbb{C}} \mathcal{Z}$. We also have $\dim_{\mathbb{C}} \mathcal{L} \leqslant |\mathcal{B}|$. By lemma 5.2 we have $\dim_{\mathbb{C}} \mathcal{L} = |\mathcal{B}|$. Consequently, $\mathcal{B}$ is the basis of $\mathcal{L}$. □

5.2. Monomial

We consider the ring $\mathbb{C}[z, t]$. The order of monomials in the ring $\mathbb{C}[z, t]$ is the graded lexicographic order with $z < t$. We set $l = h^0(S_n, \mathcal{O}_{S_n}(k))$. All elements of the basis $\mathcal{B}$ can be written

$$x^{a_1}y^{b_1}z^{c_1}t^{d_1}w^{e_1}, \ldots, x^{a_l}y^{b_l}z^{c_l}t^{d_l}w^{e_l}$$

in the order of their Zariski tangent terms. we set $a = \sum_{i=1}^{l} a_i$, $b = \sum_{i=1}^{l} b_i$, $c = \sum_{i=1}^{l} c_i$, $d = \sum_{i=1}^{l} d_i$ and $e = \sum_{i=1}^{l} e_i$.

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LEMMA 5.4. *For every basis $\{s_1, \ldots s_l\}$ of $\mathcal{L}$, the Newton polygon of the power series by applying the coordinate change $z \mapsto z - \sum_{j>0} \alpha_j t^j$ and $t \mapsto t$ to the power series $\prod_{i=1}^{l} \sigma^*(s_i(x, y, z, t, 1))$ contains the point corresponding to the monomial $z^{c+2a}t^{d+2b}$.*

Proof. We set $\xi_i = \sigma^*(x^{a_i} y^{b_i} z^{c_i} t^{d_i} w^{e_i})$ for each i. Then the Zariski tangent term of ξ_i is the monomial $z^{c_i+2a_i} t^{d_i+2b_i}$ for each i. Let ζ_i be the power series by applying the coordinate change $z \mapsto z - \sum_{j>0} \alpha_j t^j$ and $t \mapsto t$ to ξ_i for each i. And let T be the $l \times l$ matrix whose entry in row i and column j is the coefficient of the monomial $z^{c_j+2a_j} t^{d_j+2b_j}$ of ζ_i. Since the Zariski tangent terms of ζ_i are $(z - \alpha_1 t)^{c_i+2a_i} t^{d_i+2b_i}$, all monomials less than $z^{c_i+2a_i} t^{d_i+2b_i}$ in the monomial ordering are not contained in ζ_i for each i. Thus the matrix T is the upper triangular matrix whose every diagonal entry is 1.

For any $l \times l$ invertible matrix M there is a permutation matrix P such that PMT is the upper triangular matrix. Then the power series η_i with $i = 1, \ldots l$ given by

$$\begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_l \end{bmatrix} = PM \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \vdots \\ \zeta_l \end{bmatrix}$$

contain the monomial $z^{c_i+2a_i} t^{d_i+2b_i}$. Thus the Newton polygon of $\prod_{i=1}^{l} \eta_i$ contains the point corresponding to the monomial $z^{c+2a} t^{d+2b}$. □

LEMMA 5.5. *The inequalities $\frac{1}{kl}(c + 2a) \leqslant \frac{1}{3n} + \frac{2}{3} + \epsilon_k$ and $\frac{1}{kl}(d + 2b) \leqslant \frac{1}{3n} + \frac{2}{3} + \epsilon_k$ hold where ϵ_k is a small constant depending on k such that $\epsilon_k \to 0$ as $k \to \infty$.*

Proof. We consider the monomials

$$x^{a_1} y^{b_1} z^{c_1} t^{d_1} w^{e_1}, \ldots, x^{a_l} y^{b_l} z^{c_l} t^{d_l} w^{e_l}$$

of the basis $\mathcal{B}$. Let B_i be the effective Cartier divisor given by $x^{a_i} y^{b_i} z^{c_i} t^{d_i} w^{e_i} = 0$ for each i. Then

$$B := \frac{B_1 + \cdots + B_l}{kl}$$

is the anti-canonical $\mathbb{Q}$-divisor of k-basis type. Moreover klB is given by $x^a y^b z^c t^d w^e = 0$ where $a = \sum_{i=1}^{l} a_i$, $b = \sum_{i=1}^{l} b_i$, $c = \sum_{i=1}^{l} c_i$, $d = \sum_{i=1}^{l} d_i$ and $e = \sum_{i=1}^{l} e_i$. By corollary 2.8 we have the following inequalities:

$$\frac{a}{kl} \leqslant \frac{1}{3} + \epsilon_k, \quad \frac{b}{kl} \leqslant \frac{1}{3} + \epsilon_k, \quad \frac{c}{kl} \leqslant \frac{1}{3n} + \epsilon_k, \quad \frac{d}{kl} \leqslant \frac{1}{3n} + \epsilon_k$$

where ϵ_k is a small constant depending on k such that $\epsilon_k \to 0$ as $k \to \infty$. Thus we have the inequalities $\frac{1}{kl}(c + 2a) \leqslant \frac{1}{3n} + \frac{2}{3} + \epsilon_k$ and $\frac{1}{kl}(d + 2b) \leqslant \frac{1}{3n} + \frac{2}{3} + \epsilon_k$. □

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5.3. The proof of the theorem 5.1

By using lemmas 4.1 and 5.6 we prove that the log pair $(S_n, \lambda D)$ is log canonical, that is, $\delta(S_n) \geqslant \frac{1}{\lambda} > 1$.

LEMMA 5.6. *Let D be an anti-canonical $\mathbb{Q}$-divisor of k-basis type on S_n with $k \gg 0$. The log pair $(S_n, \lambda D)$ is log canonical at the point p_w.*

Proof. Let D be an anti-canonical $\mathbb{Q}$-divisor of k-basis type on S_n with $k \gg 0$. Then there is a basis $\{s_1, \ldots, s_l\}$ of the space $H^0(S_n, \mathcal{O}_{S_n}(k))$ such that

$$D = \frac{D_1 + \cdots + D_l}{kl}$$

where D_i is the effective divisor of the section s_i for each i. In the open set U, the effective divisor $\sum_{i=1}^{l} D_i$ is given by the equation $s := \prod_{i=1}^{l} s_i(x, y, z, t, 1) = 0$. We consider the Newton polygon N of $\sigma^*(s)$ in the coordinates (u, v) of $\mathbb{R}^2$. Let Λ be the edge of the Newton polygon N that intersects the diagonal line given by $u = v$. If the edge Λ is either vertical or horizontal then the log canonical threshold of the log pair $(S_n, \sum_{i=1}^{l} D_i)$ at p_w is determined by the edge Λ (see [**14**, step A]). By lemma 5.4 the point corresponding to the monomial $z^{c+2a}t^{d+2b}$ is contained in the Newton polygon N. Thus we have

$$\mathrm{lct}_0(\mathbb{C}^2, (\sigma^*(s)) \geqslant \min\left\{\frac{1}{c+2a}, \frac{1}{d+2b}\right\}.$$

By lemma 5.5 we then have

$$\mathrm{lct}_0(\mathbb{C}^2, \sigma^*(s)) \geqslant \frac{\lambda}{kl}.$$

Thus the log pair $(S_n, \lambda D)$ is log canonical at the point p_w.

Suppose that the edge Λ is neither vertical nor horizontal. By [**14**, step C], we can obtain a power series η applying a change of coordinates $z \mapsto z - \sum_{j>0} \alpha_j t^j$ and $t \mapsto t$ to $\sigma^*(s)$ such that the edge Λ' of the Newton polygon N' of the power series η that intersects the diagonal line given by $u = v$ determine the log canonical threshold of the log pair $(S_n, \sum_{i=1}^{l} D_i)$ at p_w. By lemma 5.4 the point corresponding to the monomial $z^{c+2a}t^{d+2b}$ is contained in the Newton polygons N' of the power series η, we have

$$\mathrm{lct}_0(\mathbb{C}^2, \eta) \geqslant \min\left\{\frac{1}{c+2a}, \frac{1}{d+2b}\right\}.$$

By lemma 5.5 we then have

$$\mathrm{lct}_0(\mathbb{C}^2, \eta) \geqslant \frac{\lambda}{kl}.$$

Therefore the log pair $(S_n, \lambda D)$ is log canonical at the point p_w. □

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