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ABSTRACT HARMONIC ANALYSIS OF RELATIVE CONVOLUTIONS OVER CANONICAL HOMOGENEOUS SPACES OF SEMIDIRECT PRODUCT GROUPS

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Abstract

This paper presents a structured study for abstract harmonic analysis of relative convolutions over canonical homogeneous spaces of semidirect product groups. Let H, K be locally compact groups and $\theta : H \to \mathrm{Aut}(K)$ be a continuous homomorphism. Let $G_\theta = H \ltimes_\theta K$ be the semidirect product of H and K with respect to θ and G_θ/H be the canonical homogeneous space (left coset space) of G_θ/H. We present a unified approach to the harmonic analysis of relative convolutions over the canonical homogeneous space G_θ/H.

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1. Introduction

The notion of relative-convolution operators is a theoretical generalization of other classical operators in mathematical analysis and functional analysis, such as two-sided convolutions and Toeplitz operators; see [14] and references therein. The abstract theory of relative convolutions for homogeneous spaces of locally compact groups was introduced in [15] and studied in depth in [18].

The class of locally compact semidirect product groups, as a large class of non-Abelian groups, has significant roles in theories connecting mathematical physics and mathematical theory of coherent states analysis and covariant transforms; see [1, 2, 4, 7–9, 16, 17] and references therein. This research work consists of theoretical aspects of the nature of abstract relative convolutions over canonical homogeneous spaces of locally compact semidirect product groups. This article aims to further develop harmonic analysis aspects of a unified approach to the abstract notion of convolution and involution over canonical homogeneous spaces of semidirect product groups, which is equivalent with relative convolutions in the operator theory framework.

This paper is organized as follows. Section 2 is devoted to fixing notation and a summary of classical harmonic analysis on locally compact homogeneous spaces and locally compact semidirect product groups. In Section 3, we assume that H and K are locally compact groups and $\theta : H \to \mathrm{Aut}(K)$ is a continuous homomorphism. Further, it is assumed that $G_\theta = H \ltimes_\theta K$ is the semidirect product of H and K with respect to θ. We present abstract properties of canonical homogeneous spaces of semidirect groups. Then we study a unified approach to the theoretical aspects of the notion of relative convolution operators over the canonical homogeneous space (left coset space) G_θ/H.

2. Preliminaries and notation

Let X be a locally compact Hausdorff space. By $C_c(X)$, we mean the space of all continuous complex-valued functions on X with compact supports. If μ is a positive Radon measure on X, for each $1 \leq p < \infty$ the Banach space of equivalence classes of μ-measurable complex-valued functions $f : X \to \mathbb{C}$ such that

$$\|f\|_{L^p(X,\mu)} = \left(\int_X |f(x)|^p \, d\mu(x)\right)^{1/p} < \infty$$

is denoted by $L^p(X,\mu)$, which contains $C_c(X)$ as a $\|.\|_{L^p(X,\mu)}$-dense subspace. Let G be a locally compact group with the left Haar measure m_G and the modular function Δ_G. For $p \geq 1$, the notation $L^p(G)$ stands for the Banach function space $L^p(G, m_G)$. If $p = 1$, then the standard convolution for $f, g \in L^1(G)$ is defined via

$$f * g(x) = \int_G f(y)g(y^{-1}x)\, dm_G(y) \quad (x \in G).$$

That is, in the operator terms

$$f * g(\cdot) = \int_G f(y)L_y g(\cdot)\, dm_G(y).$$

The involution for $f \in L^1(G)$ is defined by $f^*(x) = \Delta_G(x^{-1})\overline{f(x)}$ for $x \in G$. Then the Banach function space $L^1(G)$ equipped with the above convolution and involution is a Banach $*$-algebra. The Banach $*$-algebra $L^1(G)$ is commutative if and only if G is Abelian; see [5, 11, 19] and references therein.

Let H and K be locally compact groups with identity elements e_H and e_K respectively and left Haar measures m_H and m_K respectively. Let $\theta : H \to \mathrm{Aut}(K)$ be a homomorphism such that the map $(h, k) \mapsto \theta_h(k)$ is continuous from $H \times K$ onto K. There is a natural topology, sometimes called the Braconnier topology, turning $\mathrm{Aut}(K)$ into a Hausdorff topological group (not necessarily locally compact), which is defined by the sub-base of identity neighbourhoods

$$\mathcal{B}(F, U) = \{\alpha \in \mathrm{Aut}(K) : \alpha(k), \alpha^{-1}(k) \in Uk, \forall k \in F\},$$

where $F \subseteq K$ is compact and $U \subseteq K$ is an identity neighbourhood. Continuity of a homomorphism $\theta : H \to \mathrm{Aut}(K)$ is equivalent with the continuity of the map

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$(h, k) \mapsto \theta_h(k)$ from $H \times K$ onto K; see [11, 13]. Then the semidirect product $G_\theta = H \ltimes_\theta K$ is the locally compact topological group with the underlying set $H \times K$ which is equipped by the product topology and the group operation is defined by

$$(h, k) \ltimes_\theta (h', k') = (hh', k\theta_h(k')) \quad \text{and} \quad (h, k)^{-1} = (h^{-1}, \theta_{h^{-1}}(k^{-1})).$$

The left Haar measure of the locally compact group G_θ is

$$dm_{G_\theta}(h, k) = \delta^\theta_{H,K}(h)\, dm_H(h)\, dm_K(k)$$

and the modular function Δ_{G_θ} is

$$\Delta_{G_\theta}(h, k) = \delta^\theta_{H,K}(h)\Delta_H(h)\Delta_K(k) \quad \forall (h, k) \in G_\theta,$$

where the positive continuous homomorphism $\delta^\theta_{H,K} : H \to (0, \infty)$ is given by

$$dm_K(k) = \delta^\theta_{H,K}(h)\, dm_K(\theta_h(k)) \quad \forall h \in H. \tag{2.1}$$

The homomorphism $\theta : H \to \mathrm{Aut}(K)$ is called trivial if $\theta_h = I_K$ for all $h \in H$, where I_K is the identity automorphism. If $\theta : H \to \mathrm{Aut}(K)$ is trivial, then the semidirect product of H and K with respect to θ is precisely the direct product of H and K. If $\theta : H \to \mathrm{Aut}(K)$ is nontrivial, then $\widetilde{H} := \{(h, e_K) : h \in H\}$ (respectively $\widetilde{K} = \{(e_H, k) : k \in K\}$) is a closed nonnormal (respectively normal) subgroup of G_θ. From now on we may use H instead of $\widetilde{H}$.

Let H be a closed subgroup of a locally compact group G with the left Haar measures m_H and m_G, respectively. The left coset space $G/H = \{xH : x \in G\}$ is considered as a locally compact homogeneous space that G acts on from the left. The locally compact left coset space G/H is called a locally compact pure homogeneous space if the closed subgroup H is not normal in G. The function space $C_c(G/H)$ consists of all $P_H(f)$ functions, where $f \in C_c(G)$ and

$$P_H(f)(xH) = \int_H f(xh)\, dm_H(h).$$

The mapping $P_H : C_c(G) \to C_c(G/H)$ defined by $f \mapsto P_H(f)$ is a surjective bounded linear operator; see [5, 11, 12, 19]. Let μ be a Radon measure on G/H and $x \in G$. The translation μ^x of μ is defined by $\mu^x(E) = \mu(xE)$ for each Borel subset E of G/H. The measure μ is called G-invariant if $\mu^x = \mu$ for all $x \in G$. The measure μ is called strongly quasi-invariant if some continuous function $\lambda : G \times G/H \to (0, \infty)$ satisfies $d\mu^x(yH) = \lambda(x, yH)\, d\mu(yH)$ for $x, y \in G$. If the function $\lambda(x, .)$ reduces to a constant, μ is called relatively invariant under G. A rho-function for the pair (G, H) is a continuous function $\rho : G \to (0, \infty)$ which satisfies $\rho(xh) = \Delta_H(h)\Delta_G(h)^{-1}\rho(x)$ for each $x \in G$ and $h \in H$. If H is a closed subgroup of G, the pair (G, H) admits a rho-function and, for each rho-function ρ on G, there is a strongly quasi-invariant measure μ on G/H such that each $f \in C_c(G)$ satisfies

$$\int_{G/H} P_H(f)(xH)\, d\mu(xH) = \int_G f(x)\rho(x)\, dm_G(x)$$

and $d\mu^x(yH) = \rho(xy)\rho(y)^{-1}\, d\mu(yH)$ for $x, y \in G$. Moreover, all strongly quasi-invariant measures in G/H arise from a rho-function in this manner and all these measures are strongly equivalent. The homogeneous space G/H has a G-invariant measure if and only if the constant function $\rho = 1$ is a rho-function for the pair (G, H) or, equivalently, $\Delta_G|_H = \Delta_H$. If μ is the strongly quasi-invariant measure on G/H arising from the rho-function ρ, the mapping $T_H : L^1(G) \to L^1(G/H, \mu)$, defined by

$$T_H(f)(xH) = \int_H \frac{f(xh)}{\rho(xh)}\, dm_H(h),$$

is a surjective bounded linear operator with $\|T_H\| \le 1$ and satisfies the following Weil's formula [5, 6, 11, 19]:

$$\int_{G/H} T_H(f)(xH)\, d\mu(xH) = \int_G f(x)\, dm_G(x).$$

Let μ be a relatively invariant measure on the homogeneous space G/H which arises from the rho-function ρ. Then $\mathfrak{L}_{G,H}$, that is, the left regular representation of G on the Hilbert space $L^2(G/H, \mu)$ is the continuous unitary representation $\mathfrak{L}_{G,H} : G \to \mathcal{U}(L^2(G/H, \mu))$ defined by $g \mapsto \mathfrak{L}_{G,H}(g)$, where the unitary operator $\mathfrak{L}_{G,H}(g) : L^2(G/H, \mu) \to L^2(G/H, \mu)$ is given by [5, 11]

$$[\mathfrak{L}_{G,H}(g)\varphi](xH) = \sqrt{\frac{\rho(e)}{\rho(g)}}\varphi(g^{-1}xH) \tag{2.2}$$

for $g \in G$, $xH \in G/H$ and $\varphi \in L^2(G/H, \mu)$.

Let $q : G \to G/H$ be the canonical projection, that is, $x \mapsto q(x) = xH$ for all $x \in G$. Let $\mathbf{s} : G/H \to G$ be a continuous section, that is, a continuous map which is a right inverse to the canonical projection q, that is, $q \circ \mathbf{s} = I_{G/H}$. Then we can define the map $\widetilde{\mathbf{s}} : G \to H$ given by $x \mapsto \widetilde{\mathbf{s}}(x) := \mathbf{s}(xH)^{-1}x$ and also the (left) action of G on G/H can be rewritten as follows [15, 18]:

$$gxH = g\mathbf{s}(xH)H \quad (g, x \in G).$$

Then a Radon measure μ on G/H is called $\mathbf{s}$-invariant under G if

$$d\mu(gxH) = \frac{\Delta_H(\mathbf{s}(gxH)^{-1}g\mathbf{s}(xH))}{\Delta_G(\mathbf{s}(gxH)^{-1}g\mathbf{s}(xH))}\, d\mu(xH)$$

for all $g \in G$.

Let $\alpha : H \to \mathbb{C}$ be a unitary character of H, that is, a one-dimensional continuous irreducible unitary representation of H. Let $(\mathrm{ind}_H^G(\alpha), \mathcal{H}_{\mathrm{ind}_H^G(\alpha)})$ be the induced representation by α. For $\varphi \in L^1(G/H, \mu)$, the relative convolution operator S_φ^α is given by [15, 18]

$$S_\varphi^\alpha := \int_{G/H} \varphi(yH)\, \mathrm{ind}_H^G(\alpha)(\mathbf{s}(yH))\, d\mu(yH). \tag{2.3}$$

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3. Canonical homogeneous spaces of semidirect product groups

Throughout this paper, we assume that H, K are locally compact groups with given left Haar measures m_H and m_K, respectively, and $\theta : H \to \mathrm{Aut}(K)$ is a continuous homomorphism. Let $G_\theta = H \ltimes_\theta K$ be the semidirect product of H and K with respect to θ. Then the left coset space

$$G_\theta/H = \{(h, k)H : (h, k) \in G_\theta\}$$

is a locally compact homogeneous space. The locally compact homogeneous space (left coset space) G_θ/H is called a *canonical homogeneous space* of the locally compact semidirect product $G_\theta = H \ltimes_\theta K$. From now on, for $k \in K$ the notation kH stands for the left coset $(e_H, k)H$.

The following theorem states some properties of the canonical homogeneous space G_θ/H.

Theorem 3.1. *Let H, K be locally compact groups and $\theta : H \to \mathrm{Aut}(K)$ be a continuous homomorphism. Let $G_\theta = H \ltimes_\theta K$ be the semidirect product of H and K with respect to θ. Then:*

(1) *the closed subgroup H is normal in G_θ if and only if θ is the trivial homomorphism;*
(2) *for $k, k' \in K$, $kH = k'H$ if and only if $k = k'$;*
(3) *the canonical left coset space G_θ/H is precisely the set $\{kH : k \in K\}$.*

Proof. (1) It is straightforward.

(2) Let $k, k' \in K$. The left cosets kH and $k'H$ are equal if and only if $k^{-1}k' \in H$, which implies that $k = k'$.

(3) Let $(h, k) \in G_\theta$. Due to the semidirect product law, we can write

$$(h, k) = (e_H, k) \ltimes_\theta (h, e_K).$$

Therefore, in the terms of left cosets,

$$(h, k)H = (e_H, k) \ltimes_\theta (h, e_K)H = (e_H, k)H = kH.$$

Thus, the canonical left coset space G_θ/H is exactly $\{kH : k \in K\}$. □

Remark 3.2. Theorem 3.1 shows that if θ is not the trivial homomorphism, then the canonical left coset space G_θ/H is not a locally compact group. Thus, in this case traditional harmonic analysis tools and notions such as convolution and involution are not well defined for the canonical pure homogeneous space G_θ/H.

As an immediate consequence of the preceding results, we can conclude the next useful corollary.

Corollary 3.3. *Let H, K be locally compact groups, $\theta : H \to \mathrm{Aut}(K)$ be a continuous homomorphism and $G_\theta = H \ltimes_\theta K$. Let ρ be a rho-function for the pair (G_θ, H). Then:*

(1) *the linear map* $P_H : C_c(G_\theta) \to C_c(G_\theta/H)$ *is given by*

$$P_H(f)(kH) = \int_H f(h,k)dm_H(h) \quad \forall f \in C_c(G_\theta);$$

(2) *the linear map* $T_H : C_c(G_\theta) \to C_c(G_\theta/H)$ *is given by*

$$T_H(f)(kH) = \int_H \frac{f(h,k)}{\rho(h,k)}\, dm_H(h) \quad \forall f \in C_c(G_\theta). \tag{3.1}$$

Remark 3.4. The linear maps P_H and T_H have significant roles in classical approaches of the abstract harmonic analysis over locally compact homogeneous spaces; see [5, 11, 19]. Corollary 3.3 asserts connections of partial integrations on H with the linear maps P_H and T_H.

The function $\rho_\theta : G_\theta \to (0,\infty)$ given by

$$\rho_\theta(h,k) := \Delta_H(h)\Delta_{G_\theta}(h)^{-1} = \delta^\theta_{H,K}(h)^{-1} \quad \forall (h,k) \in G_\theta = H \ltimes_\theta K \tag{3.2}$$

is a rho-function for the pair (G_θ, H), which is called a *canonical rho-function* for the pair (G_θ, H). For more details concerning analytic and algebraic aspects of the canonical rho-function, we refer the reader to [10] and the comprehensive list of references therein.

The canonical rho-function satisfies

$$\int_{G_\theta} f(h,k)\rho_\theta(h,k)\, dm_{G_\theta}(h,k) = \int_H \int_K f(h,k)\, dm_H(h)\, dm_K(k)$$

for all $f \in C_c(G_\theta)$.

Proposition 3.5. *The induced strongly quasi-invariant measure* μ_θ *via the canonical rho-function* ρ_θ *defined in (3.2) is a relatively invariant measure on the canonical homogeneous space* G_θ/H.

Proof. Let μ_θ be the induced strongly quasi-invariant measure on the canonical rho-function ρ_θ defined in (3.2) and $(h,k),(h',k') \in G_\theta$. Then we can write

$$\frac{\rho((h,k) \ltimes_\theta (h',k'))}{\rho(h',k')} = \frac{\delta^\theta_{H,K}(h')}{\delta^\theta_{H,K}(hh')} = \frac{1}{\delta^\theta_{H,K}(h)}.$$

Thus,

$$\begin{aligned} d\mu_\theta^{(h,k)}(k'H) &= \frac{\rho((h,k) \ltimes_\theta (h',k'))}{\rho(h',k')}\, d\mu_\theta(k'H) \\ &= \delta^\theta_{H,K}(h)^{-1}\, d\mu_\theta(k'H) \\ &= \rho_\theta(h,k)\, d\mu_\theta(k'H), \end{aligned}$$

which implies that μ_θ is relatively invariant. □

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4. Function spaces over canonical homogeneous spaces of semidirect product groups

Throughout this section, we present a unified approach to the abstract harmonic analysis of L^p-function spaces over canonical homogeneous spaces of semidirect product groups. It is still assumed that H, K are locally compact groups with given left Haar measures m_H and m_K, respectively, $\theta : H \to \mathrm{Aut}(K)$ is a continuous homomorphism and $G_\theta = H \ltimes_\theta K$ is the semidirect product of H and K with respect to θ.

The following theorem states basic properties of the relatively invariant measure on the canonical left coset space G_θ/H which arises from the canonical rho-function ρ_θ defined in (3.2).

Theorem 4.1. *Let μ_θ be the relatively invariant measure on the canonical left coset space G_θ/H which arises from the canonical rho-function ρ_θ defined in (3.2). Then*

$$\int_{G_\theta/H} \psi(kH)\, d\mu_\theta(kH) = \int_K \psi(kH)\, dm_K(k) \quad \forall \psi \in L^1(G_\theta/H, \mu_\theta), \tag{4.1}$$

$$\int_{G_\theta/H} v(kH)\, d\mu_\theta(kH) = \int_K v(k)\, dm_K(k) \quad \forall v \in L^1(K). \tag{4.2}$$

Proof. Let $\psi \in L^1(G_\theta/H, \mu_\theta)$ and $f \in L^1(G_\theta)$ with $T_H(f) = \psi$. Using Weil's formula, we can write

$$\begin{aligned}\int_{G_\theta/H} \psi(kH)\, d\mu_\theta(kH) &= \int_{G_\theta/H} T_H(f)(kH)\, d\mu_\theta(kH)\\ &= \int_{G_\theta} f(h,k)\, dm_{G_\theta}(h,k).\end{aligned}$$

By (3.1),

$$\begin{aligned}\int_{G_\theta} f(h,k)\, dm_{G_\theta}(h,k) &= \int_H \int_K f(h,k)\delta^\theta_{H,K}(h)\, dm_H(h)\, dm_K(k)\\ &= \int_H \int_K \frac{f(h,k)}{\rho(h,k)}\, dm_H(h)\, dm_K(k)\\ &= \int_K \Big(\int_H \frac{f(h,k)}{\rho(h,k)}\, dm_H(h)\Big)\, dm_K(k).\end{aligned}$$

Thus,

$$\begin{aligned}\int_K \Big(\int_H \frac{f(h,k)}{\rho(h,k)}\, dm_H(h)\Big)\, dm_K(k) &= \int_K T_H(f)(kH)\, dm_K(k)\\ &= \int_K \psi(kH)\, dm_K(k),\end{aligned}$$

which implies (4.1). The same argument implies (4.2). $\square$

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The mapping $\Gamma = \Gamma_\theta : C_c(K) \to C_c(G_\theta/H)$ given by $v \mapsto \Gamma_\theta(v)$, where $\Gamma_\theta(v)$ is defined by $\Gamma_\theta(v)(sH) = v(s)$ for $s \in K$, is well defined, surjective and injective.

The next result is a straightforward consequence of Theorem 4.1, which shows that the linear map Γ_θ is useful for analysing functions on the canonical homogeneous space G_θ/H.

Corollary 4.2. *Let μ_θ be the relatively invariant measure on the canonical left coset space G_θ/H which arises from the canonical rho-function ρ_θ defined in (3.2) and $p \geq 1$. Then:*

(1) *the linear map $\Gamma_\theta : C_c(K) \to C_c(G_\theta/H)$ satisfies*

$$\|\Gamma_\theta(v)\|_{L^p(G_\theta/H,\mu_\theta)} = \|v\|_{L^p(K)} \quad \forall v \in C_c(K); \tag{4.3}$$

(2) *the linear map $\Gamma_\theta : C_c(K) \to C_c(G_\theta/H)$ has a unique extension to the linear map $\Gamma_\theta : L^p(K) \to L^p(G_\theta/H, \mu_\theta)$, which satisfies*

$$\|\Gamma_\theta(v)\|_{L^p(G_\theta/H,\mu_\theta)} = \|v\|_{L^p(K)} \quad \forall v \in L^p(K).$$

Proof. (1) Let $p \geq 1$ and $v \in C_c(K)$. Then $\Gamma_\theta(v) \in C_c(G_\theta/H)$ and hence we have $\psi := |\Gamma_\theta(v)|^p \in C_c(G_\theta/H)$. Using Theorem 4.1,

$$\begin{aligned}\int_{G_\theta/H} |\Gamma_\theta(v)(kH)|^p \, d\mu_\theta(kH) &= \int_K |\Gamma_\theta(v)(kH)|^p \, dm_K(k)\\ &= \int_K |v(k)|^p \, dm_K(k),\end{aligned}$$

which implies (4.3).

(2) It is straightforward. □

For $\varphi, \varphi' \in C_c(G_\theta/H)$, define the *$\theta$-convolution* of φ and φ' by

$$\varphi *_\theta \varphi'(sH) := \int_{G_\theta/H} \varphi(kH)\varphi'(k^{-1}sH)\, d\mu_\theta(kH) \quad \forall sH \in G_\theta/H. \tag{4.4}$$

Then the integral given in (4.4) converges and the mapping $(\varphi, \varphi') \mapsto \varphi *_\theta \varphi'$ is bilinear. Let $v, v' \in C_c(K)$ with $\varphi = \Gamma_\theta(v)$ and $\varphi' = \Gamma_\theta(v')$. Then

$$\begin{aligned}\varphi *_\theta \varphi'(sH) &= \int_{G_\theta/H} \varphi(kH)\varphi'(k^{-1}sH)\, d\mu_\theta(kH)\\ &= \int_{G_\theta/H} v(k)v'(k^{-1}s)\, d\mu_\theta(kH)\end{aligned}$$

for all $sH \in G_\theta/H$.

The following proposition states the relation of the θ-convolution with the standard convolution on K.

Proposition 4.3. *Let $\varphi, \varphi' \in C_c(G_\theta/H)$ and $v, v' \in C_c(K)$ with $\varphi = \Gamma_\theta(v)$ and $\varphi' = \Gamma_\theta(v')$. Then*

$$\varphi *_\theta \varphi' = \Gamma_\theta(v * v'). \tag{4.5}$$

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PROOF. Let $\varphi, \varphi' \in C_c(G_\theta/H)$. Let $v, v' \in C_c(K)$ with $\varphi = \Gamma_\theta(v)$ and $\varphi' = \Gamma_\theta(v')$. For $s \in K$, the function defined by

$$\psi_s(kH) := \varphi(kH)\varphi'(k^{-1}sH) = v(k)v'(k^{-1}s) \quad \forall kH \in G_\theta/H,$$

belongs to $C_c(G_\theta/H)$. Invoking Theorem 4.1,

$$\begin{aligned}\int_{G_\theta/H} \varphi(kH)\varphi'(k^{-1}sH)\, d\mu_\theta(kH) &= \int_{G_\theta/H} \psi_s(kH)\, d\mu_\theta(kH) \\ &= \int_K \psi_s(kH)\, dm_K(k).\end{aligned}$$

Then

$$\begin{aligned}\varphi *_\theta \varphi'(sH) &= \int_{G_\theta/H} \varphi(kH)\varphi'(k^{-1}sH)\, d\mu_\theta(kH) \\ &= \int_K \psi_s(kH)\, dm_K(k) \\ &= \int_K v(k)v'(k^{-1}s)\, dm_K(k) \\ &= v * v'(s),\end{aligned}$$

which implies (4.5). □

Similarly, one can define the *θ-involution* of $\varphi \in C_c(G_\theta/H)$ by

$$\varphi^{*_\theta}(sH) := \Delta_K(s^{-1})\overline{\varphi(s^{-1}H)} \quad \forall sH \in G_\theta/H. \tag{4.6}$$

It is evident that

$$\varphi^{*_\theta} = \Gamma_\theta(v^*),$$

where $v \in C_c(K)$ with $\varphi = \Gamma_\theta(v)$.

The next theorem guarantees that the θ-convolution and the θ-involution defined by (4.4) and (4.6) on $C_c(G_\theta/H)$ have unique extensions to the Banach function space $L^1(G_\theta/H, \mu_\theta)$, where μ_θ is the relatively invariant measure on G_θ/H which arises from the canonical rho-function given in (3.2).

THEOREM 4.4. *Let μ_θ be the relatively invariant measure on G_θ/H which arises from the canonical rho-function ρ_θ given by (3.2). The θ-convolution given in (4.4) and the θ-involution given in (4.6) have unique extensions to the Banach function space $L^1(G_\theta/H, \mu_\theta)$, in which the Banach function space $L^1(G_\theta/H, \mu_\theta)$ equipped with the extended θ-convolution and the extended θ-involution is a Banach $*$-algebra.*

PROOF. Let $\varphi, \varphi' \in C_c(G_\theta/H)$ and $v, v' \in C_c(K)$ such that $\Gamma_\theta(v) = \varphi$ and $\Gamma_\theta(v') = \varphi'$. Then Proposition 4.3 implies that $\varphi *_\theta \varphi' = \Gamma_\theta(v * v')$. Thus,

$$(\varphi *_\theta \varphi')^{*_\theta} = \varphi'^{*_\theta} *_\theta \varphi^{*_\theta}. \tag{4.7}$$

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By (4.3), we can write

$$\begin{aligned}\|\varphi *_\theta \varphi'\|_{L^1(G_\theta/H,\mu_\theta)} &= \|\Gamma_\theta(\nu * \nu')\|_{L^1(G_\theta/H,\mu_\theta)}\\ &= \|\nu * \nu'\|_{L^1(K)}\\ &\leq \|\nu\|_{L^1(K)}\|\nu'\|_{L^1(K)}\\ &= \|\Gamma_\theta(\nu)\|_{L^1(G_\theta/H,\mu_\theta)}\|\Gamma_\theta(\nu')\|_{L^1(G_\theta/H,\mu_\theta)}\\ &= \|\varphi\|_{L^1(G_\theta/H,\mu_\theta)}\|\varphi'\|_{L^1(G_\theta/H,\mu_\theta)}.\end{aligned}$$

Similarly,

$$\begin{aligned}\|\varphi^{*_\theta}\|_{L^1(G_\theta/H,\mu_\theta)} &= \|\Gamma_\theta(\nu^*)\|_{L^1(G_\theta/H,\mu_\theta)}\\ &= \|\nu^*\|_{L^1(K)}\\ &= \|\nu\|_{L^1(K)}\\ &= \|\varphi\|_{L^1(G_\theta/H,\mu_\theta)}.\end{aligned}$$

Thus, for $\varphi, \varphi' \in C_c(G_\theta/H)$,

$$\|\varphi *_\theta \varphi\|_{L^1(G_\theta/H,\mu_\theta)} \leq \|\varphi\|_{L^1(G_\theta/H,\mu_\theta)}\|\varphi'\|_{L^1(G_\theta/H,\mu_\theta)}, \tag{4.8}$$

$$\|\varphi^{*_\theta}\|_{L^1(G_\theta/H,\mu_\theta)} = \|\varphi\|_{L^1(G_\theta/H,\mu_\theta)}. \tag{4.9}$$

For $\varphi, \varphi' \in L^1(G_\theta/H, \mu_\theta)$, define the extended θ-convolution and θ-involution respectively by

$$\varphi *_\theta \varphi' := \lim_n \varphi_n *_\theta \varphi'_n, \tag{4.10}$$

$$\varphi^{*_\theta} := \lim_n \varphi_n^{*_\theta} \tag{4.11}$$

with $\{\varphi_n\}, \{\varphi'_n\} \subset C_c(G_\theta/H)$, $\varphi = \lim_n \varphi_n$ and $\varphi' = \lim_n \varphi'_n$, where the limits are considered in the topology induced by the norm $\|.\|_{L^1(G_\theta/H,\mu_\theta)}$. The extended θ-convolution and θ-involution defined by (4.10) and (4.11) are well defined and satisfy (4.7), (4.8) and (4.9). Thus, the extended θ-convolution and the extended θ-involution make the Banach function space $L^1(G_\theta/H, \mu_\theta)$ into a Banach function $*$-algebra. □

From now on, we may use the notations $*_\theta$ for the extended θ-convolution and also ${}^{*_\theta}$ for the extended θ-involution on $L^1(G_\theta/H, \mu_\theta)$.

Corollary 4.5. *Let $\varphi, \varphi' \in L^1(G_\theta/H, \mu_\theta)$. Then*

$$\varphi *_\theta \varphi'(sH) = \int_{G_\theta/H} \varphi(kH)\varphi'(k^{-1}sH)\, d\mu_\theta(kH) \quad \forall sH \in G_\theta/H$$

and

$$\varphi^{*_\theta}(sH) = \Delta_K(s^{-1})\overline{\varphi(s^{-1}H)} \quad \forall sH \in G_\theta/H.$$

Corollary 4.6. *Let μ_θ be the relatively invariant measure on G_θ/H which arises from the canonical rho-function ρ_θ given by (3.2). Then:*

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(1) *the mapping* $\Gamma_\theta : L^1(K) \to L^1(G_\theta/H, \mu_\theta)$ *is an isometric* $*$*-isomorphism of Banach* $*$*-algebras;*

(2) *the* θ*-convolution is commutative if and only if* K *is Abelian.*

We deduce the following results concerning the structure of the Banach $*$-algebra $L^1(G_\theta/H, \mu_\theta)$.

Corollary 4.7. *Let* μ_θ *be the relatively invariant measure on* G_θ/H *which arises from the canonical rho-function* ρ_θ *given by (3.2). Then:*

(1) *the Banach* $*$*-algebra* $L^1(G_\theta/H, \mu_\theta)$ *admits an approximation identity;*

(2) *the Banach* $*$*-algebra* $L^1(G_\theta/H, \mu_\theta)$ *is unital if and only if* K *is discrete.*

Then we can characterize left ideals of the Banach $*$-algebra $L^1(G_\theta/H, \mu_\theta)$ as follows.

Proposition 4.8. *Let* μ_θ *be the relatively invariant measure on* G_θ/H *which arises from the canonical rho-function* ρ_θ *given by (3.2) and* $\mathcal{J}$ *be a closed subspace of* $L^1(G_\theta/H, \mu_\theta)$*. Then* $\mathcal{J}$ *is a left* θ*-ideal if and only if* $\mathcal{J}$ *is left translation invariant under* K*.*

The next theorem can be considered as a generalization of Theorem 4.4 for L^p-function spaces of the canonical homogeneous space G_θ/H.

Theorem 4.9. *Let* μ_θ *be the relatively invariant measure on* G_θ/H *which arises from the canonical rho-function* ρ_θ *given by (3.2) and* $1 < p < \infty$*. The* θ*-convolution* $*_\theta : C_c(G_\theta/H) \times C_c(G_\theta/H) \to C_c(G_\theta/H)$ *has a unique extension to a bounded bilinear map from* $L^1(G_\theta/H, \mu_\theta) \times L^p(G_\theta/H, \mu_\theta)$ *into* $L^p(G_\theta/H, \mu_\theta)$*, which makes the Banach function space* $L^p(G_\theta/H, \mu_\theta)$ *into a Banach* $L^1(G_\theta/H, \mu_\theta)$*-module.*

Proof. Let $\psi, \varphi \in C_c(G_\theta/H)$ and $1 < p < \infty$. Let $u, v \in C_c(K)$ with $\Gamma_\theta(u) = \psi$ and $\Gamma_\theta(v) = \varphi$. Then

$$\begin{aligned}
\|\psi *_\theta \varphi\|_{L^p(G_\theta/H,\mu_\theta)} &= \|\Gamma_\theta(u * v)\|_{L^p(G_\theta/H,\mu_\theta)} \\
&= \|u * v\|_{L^p(K)} \\
&\le \|u\|_{L^1(K)}\|v\|_{L^p(K)} \\
&\le \|\Gamma_\theta(u)\|_{L^1(G_\theta/H,\mu_\theta)}\|\Gamma_\theta(v)\|_{L^p(G_\theta/H,\mu_\theta)} \\
&= \|\psi\|_{L^1(G_\theta/H,\mu_\theta)}\|\varphi\|_{L^p(G_\theta/H,\mu_\theta)}.
\end{aligned}$$

Thus, the θ-convolution $*_\theta : C_c(G_\theta/H) \times C_c(G_\theta/H) \to C_c(G_\theta/H)$ given by $(\psi, \varphi) \mapsto \psi *_\theta \varphi$ is continuous as a bilinear map. Since $C_c(G_\theta/H)$ is $\|.\|_{L^p(G_\theta/H,\mu_\theta)}$-dense in $L^p(G_\theta/H, \mu_\theta)$, we can extend it to a bilinear map

$$*_\theta : L^1(G_\theta/H, \mu_\theta) \times L^p(G_\theta/H, \mu_\theta) \to L^p(G_\theta/H, \mu_\theta)$$

which satisfies

$$\|\psi *_\theta \varphi\|_{L^p(G_\theta/H,\mu_\theta)} \le \|\psi\|_{L^1(G_\theta/H,\mu_\theta)}\|\varphi\|_{L^p(G_\theta/H,\mu_\theta)}$$

for all $\psi \in L^1(G_\theta/H, \mu_\theta)$ and $\varphi \in L^p(G_\theta/H, \mu_\theta)$. □

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For $h \in H$ and $p \geq 1$, define the operator

$$D_h : L^p(K) \to L^p(K)$$

via $v \to D_h v$, where $D_h v : K \to \mathbb{C}$ is given by

$$D_h v(k) := \delta^{\theta}_{H,K}(h)^{1/p} v(\theta_{h^{-1}}(k)) \quad \forall k \in K. \tag{4.12}$$

The next theorem states basic properties of the operators defined via (4.12).

First we need the following result concerning the action of H on the modular function of K.

Proposition 4.10. *Let H, K be locally compact groups and $\theta : H \to \mathrm{Aut}(K)$ be a continuous homomorphism. Then*

$$\Delta_K(\theta_h(k)) = \Delta_K(k) \quad \forall h \in H \ \forall k \in K. \tag{4.13}$$

Proof. Let $h \in H$ and $k \in K$. Let $v \in L^1(K)$ be a nonzero positive function. Then we can write

$$\Delta_K(\theta_h(k))^{-1} \int_K v(s)\, dm_K(s) = \int_K v(s\theta_h(k))\, dm_K(s).$$

Using (2.1),

$$\begin{aligned}
\int_K v(s\theta_h(k))\, dm_K(s) &= \int_K v(\theta_h(s)\theta_h(k))\, dm_K(\theta_h(s)) \\
&= \delta^{\theta}_{H,K}(h)^{-1} \int_K v(\theta_h(s)\theta_h(k))\, dm_K(s) \\
&= \delta^{\theta}_{H,K}(h)^{-1} \int_K v(\theta_h(sk))\, dm_K(s) \\
&= \delta^{\theta}_{H,K}(h)^{-1} \Delta_K(k)^{-1} \int_K v(\theta_h(s))\, dm_K(s) \\
&= \delta^{\theta}_{H,K}(h)^{-1} \Delta_K(k)^{-1} \int_K v(s)\, dm_K(\theta_{h^{-1}}(s)) \\
&= \Delta_K(k)^{-1} \int_K v(s)\, dm_K(s).
\end{aligned}$$

Thus,

$$\Delta_K(\theta_h(k))^{-1} \int_K v(s)\, dm_K(s) = \Delta_K(k)^{-1} \int_K v(s)\, dm_K(s),$$

which implies (4.13). □

Then we can prove the following theorem.

Theorem 4.11. *Let H, K be locally compact groups and $\theta : H \to \mathrm{Aut}(K)$ be a continuous homomorphism. Let $h \in H$ and $p \geq 1$. Then:*

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(1) $D_h : L^p(K) \to L^p(K)$ *is an isometric isomorphism of Banach spaces;*
(2) $D_h : L^2(K) \to L^2(K)$ *is a unitary operator satisfying* $(D_h)^* = D_{h^{-1}}$*;*
(3) $D_h : L^1(K) \to L^1(K)$ *is an* $*$*-isometric isomorphism of Banach* $*$*-algebras.*

PROOF. (1) For $v \in L^1(K)$,

$$\begin{aligned}\|D_h v\|^p_{L^p(K)} &= \delta^\theta_{H,K}(h)\int_K |v(\theta_{h^{-1}}(k))|^p\,dm_K(k)\\ &= \delta^\theta_{H,K}(h)\int_K |v(k)|^p\,dm_K(\theta_h(k))\\ &= \int_K |v(k)|^p\,dm_K(k)\\ &= \|v\|^p_{L^p(K)},\end{aligned}$$

which implies that the linear map $D_h : L^p(K) \to L^p(K)$ is an isometric operator. Thus, it is injective. Evidently the operator D_h maps $L^p(K)$ onto $L^p(K)$ as well. Hence, the operator D_h is an isometric isomorphism of Banach spaces.

(2) It is straightforward.

(3) Let $v, v' \in L^1(K)$. Then, for $s \in K$,

$$\begin{aligned}D_h(v * v')(s) &= \delta^\theta_{H,K}(h)v * v'(\theta_{h^{-1}}(s))\\ &= \delta^\theta_{H,K}(h)\int_K v(k)v'(k^{-1}\theta_{h^{-1}}(s))\,dm_K(k)\\ &= \delta^\theta_{H,K}(h)\int_K v(\theta_{h^{-1}}(k))v'(\theta_{h^{-1}}(k)^{-1}\theta_{h^{-1}}(s))\,dm_K(\theta_{h^{-1}}(k))\\ &= \delta^\theta_{H,K}(h)^2\int_K v(\theta_{h^{-1}}(k))v'(\theta_{h^{-1}}(k^{-1})\theta_{h^{-1}}(s))\,dm_K(k)\\ &= \delta^\theta_{H,K}(h)^2\int_K v(\theta_{h^{-1}}(k))v'(\theta_{h^{-1}}(k^{-1}s))\,dm_K(k)\\ &= \int_K D_h(v)(k)D_h(v')(k^{-1}s)\,dm_K(k)\\ &= D_h(v) * D_h(v')(s).\end{aligned}$$

Invoking (4.13), we can write

$$\begin{aligned}D_h(v^*)(s) &= \delta^\theta_{H,K}(h)v^*(\theta_{h^{-1}}(s))\\ &= \delta^\theta_{H,K}(h)\Delta_K(\theta_{h^{-1}}(s)^{-1})\overline{v(\theta_{h^{-1}}(s)^{-1})}\\ &= \delta^\theta_{H,K}(h)\Delta_K(s^{-1})\overline{v(\theta_{h^{-1}}(s)^{-1})}\\ &= D_h(v)^*(s).\end{aligned}$$

□

For $h \in H$ and $p \geq 1$, define the operator

$$\mathcal{D}_h : L^p(G_\theta/H, \mu_\theta) \to L^p(G_\theta/H, \mu_\theta)$$

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via $\varphi \to \mathcal{D}_h\varphi$, where $\mathcal{D}_h\varphi : G_\theta/H \to \mathbb{C}$ is given by

$$\mathcal{D}_h\varphi(kH) := \delta_{H,K}(h)^{1/p}\varphi(\theta_{h^{-1}}(k)H) \quad \forall k \in K.$$

Let $v \in L^p(K)$ with $\varphi = \Gamma_\theta(v)$. Then, for $k \in K$,

$$\begin{aligned}\mathcal{D}_h\varphi(kH) &= \delta^\theta_{H,K}(h)^{1/p}\varphi(\theta_{h^{-1}}(k)H)\\ &= \delta^\theta_{H,K}(h)^{1/p}v(\theta_{h^{-1}}(k))\\ &= D_hv(k).\end{aligned}$$

The next proposition states some properties of the operators $\mathcal{D}_h$ with $h \in H$.

Proposition 4.12. *Let H, K be locally compact groups, $\theta : H \to \mathrm{Aut}(K)$ be a continuous homomorphism and μ_θ be the relatively invariant measure on G_θ/H which arises from the canonical rho-function given by (3.2). Let $h \in H$ and $p \geq 1$. Then:*

(1) *$\mathcal{D}_h : L^p(G_\theta/H, \mu_\theta) \to L^p(G_\theta/H, \mu_\theta)$ is an isometric isomorphism of Banach spaces;*
(2) *$\mathcal{D}_h : L^p(G_\theta/H, \mu_\theta) \to L^p(G_\theta/H, \mu_\theta)$ is a unitary operator satisfying $(D_h)^* = D_{h^{-1}}$;*
(3) *$\mathcal{D}_h : L^1(G_\theta/H, \mu_\theta) \to L^1(G_\theta/H, \mu_\theta)$ is an $*$-isometric isomorphism of Banach $*$-algebras.*

5. Relative convolutions and θ-convolutions

In this section, we present connections of the θ-convolution and relative convolutions. For a complete picture of the mathematical theory of relative convolutions, see [15, 18]. Throughout this section, it is still assumed that H, K are locally compact groups, $\theta : H \to \mathrm{Aut}(K)$ is a continuous homomorphism and $G_\theta = H \ltimes_\theta K$ is the semidirect product of H and K with respect to θ.

First we state without proof some well-known results concerning the connection of the left regular representation and induced representations.

Proposition 5.1. *Let μ_θ be the relatively invariant measure on G_θ/H which arises from the canonical rho-function ρ_θ given by (3.2). Let $(\iota, \mathbb{C})$ be the trivial representation of H on $\mathbb{C}$. Then $\mathfrak{L}_{G,H}$ is precisely $\mathrm{ind}_H^{G_\theta}(\iota)$.*

Proof. See [5, 11]. □

Proposition 5.2. *The map $\pi_\theta : G_\theta \to \mathcal{U}(L^2(K))$ given by $(h, k) \mapsto \pi_\theta(h, k) := L_kD_h$ is a continuous unitary representation of G_θ on the Hilbert space $L^2(K)$.*

Proof. See [3] and references therein. □

Then we can prove the following useful result.

Theorem 5.3. *Let μ_θ be the relatively invariant measure on G_θ/H which arises from the canonical rho-function ρ_θ given by (3.2). The continuous unitary representations $(\pi_\theta, L^2(K))$ and $(\mathfrak{L}_{G_\theta,H}, L^2(G_\theta/H, \mu_\theta))$ are unitarily equivalent.*

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PROOF. Let $v \in L^2(K)$ and $(h,k) \in G_\theta$. Then, using (2.2), for $s \in K$, we can write

$$\begin{aligned}
\Gamma_\theta[\pi_\theta(h,k)v](sH) &= [\pi_\theta(h,k)v](s) \\
&= [L_k D_h v](s) \\
&= \delta_{H,K}(h)^{1/2} v(\theta_{h^{-1}}(k^{-1}s)) \\
&= \delta_{H,K}(h)^{1/2} \Gamma_\theta(v)(\theta_{h^{-1}}(k^{-1}s)H) \\
&= \delta_{H,K}(h)^{1/2} \Gamma_\theta(v)((h^{-1}, \theta_{h^{-1}}(k^{-1}s))H) \\
&= \sqrt{\frac{\rho_\theta(e_H, e_K)}{\rho_\theta(h,k)}} \Gamma_\theta(v)((h^{-1}, \theta_{h^{-1}}(k^{-1}s))H) \\
&= [\mathfrak{L}_{G_\theta,H}(h,k)v](sH).
\end{aligned}$$

Thus, the map $\Gamma_\theta : L^2(K) \to L^2(G_\theta/H, \mu_\theta)$ is a unitary intertwining linear operator, which guarantees that the continuous unitary representations $(\pi_\theta, L^2(K))$ and $(\mathfrak{L}_{G_\theta,H}, L^2(G_\theta/H, \mu_\theta))$ are unitarily equivalent. □

Let $\mathbf{s}_\theta : G_\theta/H \to G_\theta$ be given by $\mathbf{s}_\theta(kH) := k$ for all $k \in k$. Then $\mathbf{s}_\theta$ is a continuous section, called a canonical section of the homogeneous space G_θ/H.

THEOREM 5.4. *Let μ_θ be the relatively invariant measure over the canonical homogeneous space G_θ/H which arises from the canonical rho-function ρ_θ. Then μ_θ is $\mathbf{s}_\theta$-invariant under G_θ.*

PROOF. Let $g = (h,k) \in G_\theta$. Then we can write

$$\begin{aligned}
\mathbf{s}_\theta((h,k) \ltimes k'H)^{-1} \ltimes (h,k) \ltimes \mathbf{s}_\theta(k'H) &= \mathbf{s}_\theta((h, k\theta_h(k')H)^{-1} \ltimes (h,k) \ltimes k' \\
&= \mathbf{s}_\theta((h, k\theta_h(k')H)^{-1} \ltimes (h, k\theta_h(k')) \\
&= \mathbf{s}_\theta(k\theta_h(k')H)^{-1} \ltimes (h, k\theta_h(k')) \\
&= \theta_h(k')^{-1}k^{-1} \ltimes (h, k\theta_h(k')) \\
&= (h, \theta_h(k')^{-1}k^{-1}k\theta_h(k')) \\
&= (h, e_K),
\end{aligned}$$

for all $k' \in K$. Thus,

$$\begin{aligned}
\frac{\Delta_H(\mathbf{s}_\theta((h,k) \ltimes k'H)^{-1} \ltimes (h,k) \ltimes \mathbf{s}_\theta(k'H))}{\Delta_{G_\theta}(\mathbf{s}_\theta((h,k) \ltimes k'H)^{-1} \ltimes (h,k) \ltimes \mathbf{s}_\theta(k'H))} d\mu_\theta(k'H) &= \frac{\Delta_H(h)}{\Delta_{G_\theta}(h)}\, d\mu_\theta(k'H) \\
&= \delta_{H,K}(h)^{-1}\, d\mu_\theta(k'H) \\
&= \rho_\theta(h,k)\, d\mu_\theta(k'H) \\
&= d\mu_\theta^{(h,k)}(k'H).
\end{aligned}$$ □

Then we deduce the following interesting result, concerning connections of θ-convolutions with relative convolutions.

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THEOREM 5.5. *Let H, K be locally compact groups, $\theta : H \to \mathrm{Aut}(K)$ be a continuous homomorphism and $G_\theta = H \ltimes_\theta K$. The θ-convolution operator over the canonical homogeneous spaces G_θ/H is precisely the relative convolution operator associated to the canonical section $\mathbf{s}_\theta$ and the trivial unitary character $\iota : H \to \mathbb{C}$.*

PROOF. Let μ_θ be the relatively invariant measure over the canonical homogeneous space G_θ/H which arises from the canonical rho-function ρ_θ. Theorem 5.4 guarantees that μ_θ is $\mathbf{s}_\theta$-invariant under G_θ. Let $(\iota, \mathbb{C})$ be the trivial representation of H on $\mathbb{C}$. Also, let $\varphi \in L^1(G_\theta/H, \mu_\theta)$ and $s \in K$. Then, using Proposition 5.1 and (2.3),

$$\begin{aligned}
[S^\iota_\varphi \varphi'](sH) &= \int_{G_\theta/H} \varphi(kH)[\mathrm{ind}_H^G(\iota)(\mathbf{s}_\theta(kH))\varphi'](sH)\, d\mu_\theta(kH) \\
&= \int_{G_\theta/H} \varphi(kH)[\mathrm{ind}_H^G(\iota)(k)\varphi'](sH)\, d\mu_\theta(kH) \\
&= \int_{G_\theta/H} \varphi(kH)\varphi'(k^{-1}sH)\, d\mu_\theta(kH) \\
&= \varphi *_\theta \varphi'(sH)
\end{aligned}$$

for all $\varphi' \in C_c(G_\theta/H)$. This shows that $S^\iota_\varphi \varphi' = \varphi *_\theta \varphi'$ and hence we deduce that the θ-convolution operator over the canonical homogeneous spaces G_θ/H is precisely the relative convolution operator associated to the canonical section $\mathbf{s}_\theta$ and the trivial unitary character $\iota : H \to \mathbb{C}$. □

REMARK 5.6. Theorem 5.5 can be regarded as an explicit construction for relative convolution (convolution-type) operators over canonical homogeneous spaces of semidirect product groups with respect to the canonical section.

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