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On a Theorem of Burgess and Stephenson

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Abstract. A theorem of Burgess and Stephenson asserts that in an exchange ring with central idempotents, every maximal left ideal is also a right ideal. The proof uses sheaf-theoretic techniques. In this paper, we give a short elementary proof of this important theorem.

A ring R is called *left duo* [3] if every left ideal is an ideal. In 1995, Yu [8] called R *left quasi-duo* if every maximal left ideal is an ideal. In 2005, Lam and Dugas [5, Question 4.1] declared themselves "practically clueless" as to whether every left quasi-duo ring is right quasi-duo. Previously, Yu [8, Proposition 4.1] had shown that the answer is affirmative if the ring is an exchange ring with central idempotents, but his proof depends critically on a 1979 theorem of Burgess and Stephenson [1] that every such exchange ring is actually left quasi-duo. However, their proof is sheaf-theoretic, and in this short note, we give an elementary proof of this important theorem using some basic results about exchange rings from [6].

Throughout the paper we assume that all rings R are associative with unity and all modules are unitary. We denote the Jacobson radical by $J(R)$, and the left annihilator of $X \subseteq R$ by $\mathtt{l}(X)$. The notation $A \triangleleft R$ asserts that A is an ideal of R.

Commutative rings and local rings are all quasi-duo (left and right), and the property is retained by images and direct products. Moreover, if D is a division ring, then the $n \times n$ upper triangular matrix ring $T_n(D)$ is left quasi-duo, but the ring $M_n(D)$ of all $n \times n$ matrices is not left quasi-duo if $n \geq 2$, because $R(1 - e_{nn})$ is a maximal left ideal that is not a right ideal. Hence, being left quasi-duo is not a Morita invariant, and the only semisimple rings that are left quasi-duo are the finite direct products of division rings.

A ring R is called *left primitive* if it has a faithful simple left module. The following useful lemma (and its converse) was proved in 2002 by Huh, Jang, Kim, and Lee [4, Proposition 1]. We include a short proof for completeness.

Lemma 1 *A ring R is left quasi-duo if every left primitive factor ring R/P is a division ring.*

Proof If L is a maximal left ideal of R, then

$$P =: \mathtt{l}(R/L) = \{b \in R \mid bR \subseteq L\}$$

is a left primitive ideal of R. Hence R/P is a division ring by hypothesis, so P is maximal as a left ideal. But $P \subseteq L$, so it follows that $L = P$. In particular, $L \triangleleft R$, as required. ∎

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Following Crawley and Jónsson [2], a module M has the (*finite*) *exchange property* if, for any module X and (finite) index set I, we have

$$X = M' \oplus N = \bigoplus_{i\in I} X_i, \quad M' \cong M, \quad \text{implies} \quad X = M' \oplus (\bigoplus_{i\in I} X_i')$$

for submodules $X_i' \subseteq X_i$.

In 1972, Warfield [7] showed that ${}_RR$ has the finite exchange propery if and only if the same is true of R_R, and called R an exchange ring in this case. An elementary characterization of these exchange rings appears in [6] and enables the proof of the main result of this paper. The following lemma is needed. We call a ring *abelian* if every idempotent is central.

Lemma 2 *If R is an indecomposable abelian exchange ring, then R is local.*

Proof As R is indecomposable, 0 and 1 are the only central idempotents in R, and so the only idempotents as R is abelian. But R is exchange, so every left ideal not contained in $J(R)$ contains a nonzero idempotent [6, Proposition 1.9]. In particular, every maximal left ideal of R is contained in the Jacobson radical. It follows that R is local. ■

Theorem 3 (Burgess and Stephenson) *Every abelian exchange ring is left quasi-duo.*

Proof Let R be an abelian exchange ring. By Lemma 1, it suffices to show that every left primitive image R/P is a division ring. Observe that R/P is exchange by [6, Proposition 1.4] and abelian by [6, Corollary 1.3]. Hence, it is enough to prove the following claim.

Claim *Every abelian, left primitive, exchange ring R is a division ring.*

Proof of Claim If R is such a ring, let ${}_RK$ be simple and faithful and choose $0 \neq k \in K$. Consider any element $a \notin \mathtt{l}(k)$, $a \in R$. Since R is indecomposable (left primitive rings are prime), Lemma 2 shows that R is local. As $\mathtt{l}(k)$ is a maximal left ideal, it follows that $\mathtt{l}(k) = J(R)$. But then $\mathtt{l}(k)K = J(R)K = 0$ as ${}_RK$ is simple, and so $\mathtt{l}(k) = 0$, because ${}_RK$ is faithful. It follows that R is a division ring, proving the claim and hence the theorem. ■

Note that the ring $\mathbb{Z}$ of integers is abelian and quasi-duo, but not exchange, and $\begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ & \mathbb{Z}_2 \end{bmatrix}$ is exchange and quasi-duo but not abelian (here, $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$).

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