On the reconstruction of unknown driving forces from low-mode observations in the 2D Navier–Stokes equations

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This article is concerned with the problem of determining an unknown source of non-potential, external time-dependent perturbations of an incompressible fluid from large-scale observations on the flow field. A relaxation-based approach is proposed for accomplishing this, which makes use of a nonlinear property of the equations of motions to asymptotically enslave small scales to large scales. In particular, an algorithm is introduced that systematically produces approximations of the flow field on the unobserved scales in order to generate an approximation to the unknown force; the process is then repeated to generate an improved approximation of the unobserved scales, and so on. A mathematical proof of convergence of this algorithm is established in the context of the two-dimensional Navier–Stokes equations with periodic boundary conditions under the assumption that the force belongs to the observational subspace of phase space; at each stage in the algorithm, it is shown that the model error, represented as the difference between the approximating and true force, asymptotically decreases to zero in a geometric fashion provided that sufficiently many scales are observed and certain parameters of the algorithm are appropriately tuned.

1. Introduction

In the derivation of any model, parameters arise that capture intrinsic properties of the phenomenon of interest. In the case of modelling a turbulent, incompressible fluid flow via the Navier–Stokes equations (NSE), assuming a constant density, the two relevant parameters are essentially the kinematic viscosity of the fluid and the external force. For instance, in eddy diffusivity models, turbulent viscosity coefficients must be specified [6]. In practice, these parameters must be empirically determined. On the contrary, in proposing any turbulent closure, one inevitably

commits model errors. These errors may themselves then be modelled as a body force, whose exact form in terms of the mean field is unknown.

In the current study, we propose an algorithm for determining all large-scale features of an external driving force in the two-dimensional incompressible NSE (2D NSE) down to the observation scale. In practice, such an algorithm can be used to filter model errors that are represented as external driving forces. In our idealized set-up, the fluid occupies a periodic domain, $\Omega = [0, 2\pi]^2$, the density of the fluid is normalized to unity, the kinematic viscosity of the fluid is known perfectly, but the external driving force is not. We point out that the problem of determining the viscosity based on direct observation of the velocity field has recently been studied in [**9**, **15**, **21**, **22**, **61**], where estimators for the viscosity were proposed, their consistency and asymptotic normality were established [**21**], convergence analyses for viscosity-recovery algorithms were carried out [**61**], and numerical tests were performed [**15**, **22**]. The problem of multi-parameter recovery in chaotic dynamical systems was studied in [**16**, **63**], while a recent study [**17**] analysed the sensitivity of the 2D NSE to changes in the viscosity, as well as its implications for a certain downscaling algorithm for data assimilation.

Generally speaking, one of the objectives of this article is to study the extent to which external forces can be determined based on error-free, but partial observations of flow field through a practically implementable algorithm. In our ideal set-up, we will assume that we have access to a time series of the velocity field, but only through sufficiently small, but nevertheless finitely many, length scales. Ultimately, we introduce an algorithm for reconstructing large-scale features of the unknown force and establish its convergence under the assumption that the force acts only on length scales that are directly observed, that is to say, that the force belongs to the span of the observational field.

To be more precise, we recall that the 2D NSE on $\Omega = [0, 2\pi]^2$ is given by

$$\partial_t u + (u\cdot\nabla)u = \nu\Delta u - \nabla p + f, \quad \nabla\cdot u = 0, \tag{1.1}$$

where the kinematic viscosity, $\nu > 0$, is given and fixed, and the time series, $\{P_N u(t)\}_{t\in[0,T]}$, is known, up to some time $T > 0$, where P_N is the L^2-orthogonal projection onto Fourier wavenumbers corresponding to $|k| \leqslant N$, and $u\cdot\nabla = u^j\partial_j$, where repeated indices indicate summation over those indices. However, the external force, f, is time-independent, but unknown. The scalar pressure field is denoted by p; upon taking the divergence of (1.1), one sees that p is determined entirely by u, f via the Poisson equation $-\Delta p = \partial_i\partial_j(u^i u^j) + \partial_j f^j$. We will assume that u, p, and f are both mean-free and periodic over Ω, and that f is divergence-free. The main result is that for forces satisfying $P_N f = f$, where N is sufficiently large, there exists a sequence of times $\{t_n\}_n$ and sequence of approximating forces, $\{f_n\}_n$, depending only on the observations $\{P_N u(t)\}_{t\geqslant 0}$ such that f_n converges to f. We will in fact address the more general case of time-dependent forces.

The motivating idea for the algorithm we propose is based on the notion of 'asymptotic completeness' for nonlinear dissipative systems. For systems that possess this property, it asserts that having direct (observational) access to a sufficiently rich, but nevertheless finite-dimensional, set of scales, is enough to asymptotically determine the unobserved scales. This property was rigorously shown to hold for

(1.1) by Foiaş and Prodi [**44**] in the case when access to sufficiently many Fourier modes of the velocity field is available; for the case of three dimensions, the reader is referred to [**11**, **20**]. Specifically, given two solutions u_1, u_2 of (1.1) corresponding to external forces f_1, f_2, it was shown that there exists $N \geqslant 1$, depending on ν and f_1, f_2, but only through their size, such that $P_N(u_1(t) - u_2(t)) \to 0$ and $f_1(t) - f_2(t) \to 0$ as $t \to \infty$ together imply $u_1(t) - u_2(t) \to 0$ as $t \to \infty$. In this case, it is then said that the Fourier modes corresponding to wavenumbers $|k| \leqslant N$ are *determining* for system (1.1); the smallest such number, N, is referred to as the number of *determining modes*. In light of this result, one sees that the problem of inferring an unknown force may be possible to solve provided that sufficiently many modes are observed and that one identifies an algorithm that asymptotically reconstructs higher modes that can subsequently be made use of to approximate the force.

To see the main difficulty that must be overcome in doing so, suppose that one is given access to $P_N u(t)$, for $t \geqslant t_0$. Then, a naïve, but reasonable first approximation to the low modes of the force may be given by simply evaluating the nonlinear differential operator determined by (1.1) along $P_N u$. This yields

$$\partial_t P_N u - \nu \Delta P_N u + P_N((P_N u \cdot \nabla) P_N u) + \nabla p_N =: f_0, \qquad (1.2)$$

where p_N is determined by enforcing f_0 to be divergence-free. On the contrary, by applying the low-pass filter P_N to (1.1), one also obtains:

$$\partial_t P_N u - \nu \Delta P_N u + P_N((P_N u \cdot \nabla) P_N u) + \nabla p_N = P_N f + \mathcal{R}_N, \qquad (1.3)$$

where $\mathcal{R}_N$ denotes the Reynolds stress, defined as:

$$\mathcal{R}_N := P_N((P_N u \cdot \nabla) P_N u) - P_N((u \cdot \nabla) u) + \nabla p_N - \nabla P_N p. \qquad (1.4)$$

One therefore has the following relation:

$$\mathcal{R}_N = f_0 - P_N f. \qquad (1.5)$$

In particular, determination of $P_N f$ is equivalent to determination of $\mathcal{R}_N$. The fundamental difficulty of the problem arises from the elementary fact that $\mathcal{R}_N$ depends on both $P_N u$ and $(I - P_N)u$, which is tantamount to the closure problem of turbulence. In comparison, if (1.1) were replaced by the linear heat equation, then observation of $P_N u$ along a trajectory of u provides an exact reconstruction for $P_N f$ along that same trajectory. Thus, any resolution of this difficulty must address a way to reconstruct high-mode information from low-mode data. In § 3, we introduce an algorithm that, under suitable conditions, accomplishes this and systematically decreases the 'Reynolds stress' at each stage in a geometric fashion, yielding a convergent scheme for approximating the force on the low modes. From this point of view, the proposed scheme may be viewed as a 'nonlinear filtering' of large-scale error.

For systems that possess an *inertial manifold*, reconstruction of small scales from large scales would be possible. Indeed, the existence of an inertial manifold implies a strong form of enslavement of scales as it asserts the existence of a map Φ such that $P_N u|_{t=t_0} \mapsto (I - P_N)u|_{t \geqslant t_0}$, that is, $u(t) = P_N u(t) + \Phi(P_N u(t_0))(t)$, for all $t \geqslant t_0$.

In other words, knowledge of low modes at a single time is sufficient to determine the high-mode behaviour at all future times. The inertial manifold would then be determined by the graph of Φ (see [**45**, **47**]). However, the existence of such a map for (1.1) remains an outstanding open problem. One of the main points of this article is that one needs to only rely on the weaker property of asymptotic completeness to reconstruct sufficiently high-mode components of the state variables in order to eventually recover the low modes of the forcing.

There are at least two ways available in the literature for doing, one due to Foias *et al.* in [**39**] and another by Azouani *et al.* in [**4**]. The first construction [**39**] allows one to encode projections of solutions on the global attractor of (1.1) as travelling wave solutions to some infinite-dimensional ordinary differential equation. However, it effectively requires one to solve the evolution equation of (1.1) corresponding to the high modes [**40**]. The second method introduces a downscaling algorithm for data assimilation [**4**], in which large-scale observations are exogenously inserted into (1.1) as a feedback-control term that serves to drive the corresponding solution of this modified system towards the reference solution, but only on large scales. By virtue of the Foias–Prodi property of determining modes, it was then shown that by tuning the strength of the feedback-control system appropriately, the generated approximating signal asymptotically synchronizes to the reference signal to which the large-scale observations correspond. We will make use of the latter approach to systematically reconstruct high-mode information on the reference field.

The remainder of the paper is organized as follows. In § 2, we establish the notation and functional setting in which the result will be proved, as well as classical well-posedness results that we will make use of. A derivation of the algorithm that reconstructs the external force is presented in § 3. We provide formal statements of the convergence results in § 4, then outline their proof in § 5. The main step is to reduce the analysis of the convergence to establish suitable 'sensitivity estimates'. These estimates are proved in § 6. We finally supply rigorous proofs of the main results in § 7. Technical auxiliary results are relegated to Appendix A.

2. Notation and functional setting

Let $\Omega = [0, 2\pi]^2$ and $L_\sigma^p = L_\sigma^p(\Omega)$, for $p \in [1, \infty]$, denote the space of p-integrable (in the sense of Lebesgue), mean-free, solenoidal vector fields over Ω, which are periodic in each direction; its norm is given by

$$\|u\|_{L^p}^p := \int_\Omega |u(x)|^p \, \mathrm{d}x, \quad \text{for } p \in [1, \infty), \quad \|u\|_{L^\infty} := \text{ess sup}_{x \in \Omega} |u(x)|, \quad (2.1)$$

where ess sup denotes the essential supremum. We let $H_\sigma^k = H_\sigma^k(\Omega)$ denote the space of periodic, mean-free, solenoidal vector fields over Ω whose weak derivatives (in the sense of Sobolev) up to order k belong to L_σ^2; its norm is given by

$$\|u\|_{H^k}^2 := \sum_{|\alpha| \leqslant k} \int_\Omega |\partial^\alpha u(x)|^2 \, \mathrm{d}x, \quad (2.2)$$

where $\alpha \in (\mathbb{N} \cup \{0\})^2$ is a multi-index. We will abuse notation and use L^2_σ, H^k_σ to denote the corresponding spaces for scalar functions as well; in this case, $|u|$ is interpreted as absolute value, rather than Euclidean norm.

We will make use of the functional form of the NSE, which is given by

$$\frac{\mathrm{d}}{\mathrm{d}t} u + \nu A u + B(u,u) = Pf, \tag{2.3}$$

where

$$Au = -P\Delta u \quad \text{and} \quad B(u,v) = P(u \cdot \nabla)v,$$

and P denotes the Leray–Helmholtz projection, that is, the orthogonal projection onto divergence-free vector fields; we refer to A as the Stokes operator. Note that by orthogonality, $\|P\|_{L^2 \to L^2} \leqslant 1$. Moreover,

$$\widehat{Pu}(n) := \hat{u}(n) - \frac{n \cdot \hat{u}(n)}{|n|^2} n, \quad n \in \mathbb{Z}^2 \setminus \{(0,0)\}, \tag{2.4}$$

where $\hat{u}(n)$ denotes the Fourier coefficient of u corresponding to wavenumber $n \in \mathbb{Z}^2$. Also, powers of A can be defined spectrally via:

$$\widehat{A^{k/2}u}(n) := |n|^k \hat{u}(n). \tag{2.5}$$

Hence, P commutes with Δ in the setting of periodic boundary conditions. Recall that for $u \in H^1_\sigma$ the Poincaré inequality states:

$$\|u\|_{L^2} \leqslant \|A^{1/2}u\|_{L^2} = \|A^{1/2}u\|_{L^2}. \tag{2.6}$$

It follows that for each integer $k \geqslant 1$, there exists a constant $c_k > 0$ such that

$$c_k^{-2} \|u\|_{H^k}^2 \leqslant \sum_{|\alpha|=k} \int_\Omega |\partial^\alpha u(x)|^2 \, \mathrm{d}x \leqslant c_k^2 \|u\|_{H^k}^2, \tag{2.7}$$

whenever $u \in H^k_\sigma$. In particular, by the Parseval identity, it follows that there also exist constants $c_k > 0$ such that

$$c_k^{-1} \|u\|_{H^k} \leqslant \|A^{k/2}u\|_{L^2} \leqslant c_k \|u\|_{H^k}, \tag{2.8}$$

for all $u \in H^k_\sigma$.

Given $t_0 \geqslant 0$ and $f \in L^\infty(t_0, \infty; L^2_\sigma)$, we define the Grashof-type number by

$$\tilde{G} := \frac{\sup_{t \geqslant t_0} \|f(t)\|_{L^2}}{\kappa_0^2 \nu^2}, \tag{2.9}$$

where $\kappa_0 = (2\pi)/L$, where L is the linear length of the domain; since $\Omega = [0, 2\pi]^2$, we see that $\kappa_0 = 1$. We note that the Grashof number is traditionally denoted by G, that is, undecorated, when the force is *independent of time*. Since we allow for time-dependence in the force, we will distinguish between the two notations by making use of tilde. See, for instance [5], where this distinction is also maintained.

Observe that $Pf = f$, for any $f \in L^2_\sigma$. One has the following classical results regarding the existence theory for (2.3) [26, 41, 65, 66].

THEOREM 2.1. *Let $t_0 \geqslant 0$ and $f \in L^\infty(t_0, \infty; L^2_\sigma)$. For all $u_0 \in H^1_\sigma$, there exists a unique $u \in C([t_0, T]; H^1_\sigma) \cap L^2(t_0, T; H^2)$ satisfying (2.3), for all $T > 0$, such that $(\mathrm{d}/\mathrm{d}t)u \in L^2(t_0, T; L^2_\sigma)$ and*

$$\|A^{1/2}u(t)\|^2_{L^2} \leqslant \|A^{1/2}u(t_0)\|^2_{L^2}\, \mathrm{e}^{-\nu(t-t_0)} + \nu^2 \tilde{G}^2(1 - \mathrm{e}^{-\nu(t-t_0)}), \qquad (2.10)$$

for all $t \geqslant t_0 \geqslant 0$.

From (2.10), we see that for $t_0 = 0$ and all $t > 0$ sufficiently large, depending on $\|A^{1/2}u_0\|_{L^2}$, one has:

$$\|A^{1/2}u(t)\|_{L^2} \leqslant \sqrt{2}\nu\tilde{G} =: \tilde{c}_1\tilde{R}_1, \quad \tilde{c}_1 = \sqrt{2}. \qquad (2.11)$$

Let $\tilde{B}_1$ denote the ball of radius $\tilde{R}_1$, centred at the origin in L^2_σ. Observe that $A^{1/2}u_0 \in \tilde{B}_1$ implies via (2.10) that $u(t; u_0, f) \in \tilde{B}_1$, for all $t \geqslant 0$, where $u(t; u_0, f)$ denotes the unique solution of (2.3) with initial data u_0 and external forcing f. Moreover, there exists a constant $\tilde{c}_2 > 0$ such that if $f \in L^\infty(0, \infty; H^1_\sigma)$ and $Au_0 \in \alpha\tilde{B}_2$, where $\alpha > 0$ is arbitrary, $\tilde{B}_2$ denotes the ball of radius $\tilde{R}_2 = \tilde{c}_2\nu(\tilde{G} + \tilde{\sigma}_1)\tilde{G}$, centred at the origin in L^2_σ, and $\alpha\tilde{B}_2$ the same ball of radius $\alpha\tilde{R}_2$, then

$$\|Au(t)\|^2_{L^2} \leqslant (1 + \alpha^2)\tilde{c}_2^2\nu^2(\tilde{\sigma}_1 + \tilde{G})^2\tilde{G}^2, \qquad (2.12)$$

for all $t \geqslant 0$, where $\tilde{\sigma}_1$ denotes a 'shape factor' defined by

$$\tilde{\sigma}_1 := \frac{\sup_{t \geqslant t_0}\|A^{1/2}f(t)\|_{L^2}}{\sup_{t \geqslant t_0}\|f(t)\|_{L^2}}. \qquad (2.13)$$

In other words, $Au_0 \in \alpha\tilde{B}_2$ implies $Au(t) \in (1 + \alpha^2)^{1/2}\tilde{B}_2$, for all $t \geqslant 0$. Observe that $\tilde{\sigma}_1 \geqslant 1$ by Poincaré's inequality. In particular, if $Au_0 \in \tilde{B}_2$, then

$$\|Au(t)\|_{L^2} \leqslant \sqrt{2}\tilde{c}_2\nu(\tilde{G} + \tilde{\sigma}_1)\tilde{G} = \sqrt{2}\tilde{R}_2. \qquad (2.14)$$

Bounds sharper than (2.14) were established in [**27**] in the setting where f was time-independent. In this particular case, one has that:

$$\|Au(t)\|_{L^2} \leqslant c_2\nu(\sigma_1^{1/2} + G)G =: R_2, \qquad (2.15)$$

holds for all $t \geqslant 0$, for some $c_2 > 0$, provided that $u_0 \in B_2$, the ball of radius R_2 in H^2_σ. Here, G denotes the Grashof number, which is simply given by (2.9) when f is time-independent. Similarly, σ_1 is given by (2.13) when f is time-independent. For the sake of completeness, we supply a short proof of (2.15) in Appendix A.

3. Description of the algorithm

We consider the following feedback control system:

$$\frac{\mathrm{d}}{\mathrm{d}t}v + \nu Av + B(v, v) = h - \mu P_N(v - u), \qquad (3.1)$$

where h, possibly time-dependent, is given. The well-posedness theory and synchronization properties of this model were originally developed in [**4**] for a more general

class of observables, which includes projection onto finitely many Fourier modes as a special case, in [**7**, **12**] in the setting of noisy observations, while the issue of synchronization in higher-order topologies was studied in [**8**, **10**]. In the idealized setting considered in this article, the existence, uniqueness results in [**4**] will suffice for our purposes. This is stated in the following theorem.

THEOREM 3.1. *Let $t_0 \geqslant 0$ and $h \in L^\infty(t_0, \infty; L^2_\sigma)$. Let u denote the unique solution of* (2.3) *corresponding to initial data $u_0 \in H^1_\sigma$ guaranteed by theorem* 2.1. *There exists a constant $\tilde{c} > 0$ such if μ, N satisfy*

$$\mu \leqslant \tilde{c}\nu N^2 \tag{3.2}$$

then given $v_0 \in H^1_\sigma$, there exists a unique solution, v, to the initial value problem corresponding to (3.1) *such that*

$$v \in C([t_0, T]; H^1_\sigma) \cap L^2(t_0, T; H^2_\sigma) \quad and \quad \frac{\mathrm{d}}{\mathrm{d}t}v \in L^2(t_0, T; L^2_\sigma), \tag{3.3}$$

for any $T > 0$.

The feedback control system (3.1) was originally conceived as a way to assimilate observations on the flow field into the equations of motion in order to reconstruct the unobserved scales of motion. There exists a considerable body of research analysing the extent to which this is possible in various situations in hydrodynamics such as Rayleigh–Bénard convection [**3**, **14**, **32**–**35**, **37**], turbulence [**2**, **19**, **23**, **38**, **58**, **67**, **68**], geophysical fluids [**1**, **30**, **36**, **52**, **53**, **64**], dispersive equations [**54**, **55**], as well as various numerical analytical and computational studies [**13**, **18**, **31**, **42**, **48**–**51**, **59**].

Given h, we may thus obtain from (3.1) an approximate reconstruction of the high modes of u via $Q_N v = (I - P_N)v$. We therefore propose the following algorithm: let $h = f_0$ denote the initial guess for the forcing field, defined for all $t \geqslant t_0$, for some fixed initial time $t_0 \geqslant 0$, and let v^0 to be an arbitrary initial state; f_0 is considered to be the approximation at stage $n = 0$ that is prescribed by the user and may, in fact, be chosen arbitrarily. Suppose that $P_N f_0 = f_0$. Then, at stage $n = 1$, we consider:

$$\frac{\mathrm{d}}{\mathrm{d}t}v_1 + \nu A v_1 + B(v_1, v_1) = f_0 - \mu P_N(v_1 - u), \quad v_1(t_0) = v_1^0, \quad t \in I_0 := [t_0, \infty). \tag{3.4}$$

We define the first approximation to the flow field by

$$u_1 = P_N u + Q_N v_1, \quad \text{for } t \in I_0. \tag{3.5}$$

By design, u_1 will relax towards u after a transient period, $\rho_1 := t_1 - t_0$, that is proportional to the relaxation timescale, μ^{-1}, where $t_1 \gg t_0$, but only up to an error of size $O(g_0)$, where $g_0 := f_0 - f$, which represents the 'model error'. Only after this period has transpired we will extract the first approximation to f via the

formula:

$$f_1(t) := \frac{\mathrm{d}}{\mathrm{d}t} P_N u_1 + \nu A P_N u_1 + P_N B(u_1, u_1),$$

$$\text{for all } t \in I_1 := [t_0 + \rho_1, \infty), \text{ for some } \rho_1 \gg 0. \tag{3.6}$$

Observe that $P_N f_1 = f_1$. To obtain new approximations in subsequent stages, we proceed recursively: suppose that at stage $n - 1$, a force, $f_{n-1} = P_N f_{n-1}$ over $t \in I_{n-1} := [t_{n-2} + \rho_{n-1}, \infty)$, has been produced, where the relaxation period satisfies $\rho_{n-1} \gg 0$, and an arbitrary initial state, v_n^0, has been given. Then, consider:

$$\frac{\mathrm{d}}{\mathrm{d}t} v_n + \nu A v_n + B(v_n, v_n) = f_{n-1} - \mu P_N(v_n - u), \quad v_n(t_{n-1}) = v_n^0, \quad t \in I_{n-1}. \tag{3.7}$$

Let $u_n = P_N u + Q_N v_n$, for $t \in I_{n-1}$, and define the approximation to the force at stage n by

$$f_n(t) := \frac{\mathrm{d}}{\mathrm{d}t} P_N u_n + \nu A P_N u_n + P_N B(u_n, u_n),$$

$$\text{for all } t \in I_n := [t_{n-1} + \rho_n, \infty), \text{ for some } \rho_n \gg 0. \tag{3.8}$$

This procedure produces a sequence of forces $f_1|_{I_1}, f_2|_{I_2}, f_3|_{I_3}, \ldots$ that approximates f on time intervals I_n whose left-hand endpoints are increasing with n. In particular, the sequence $\{f_n|_{I_n}\}_n$ *asymptotically* approximates f.

The key step to ensuring convergence of the generated sequence $\{f_n\}_{n \geqslant 1}$ to the true forcing, f, is to control model errors, $g_n := f_n - f$, at each stage, in terms of the synchronization errors, w_n, which, in turn, are controlled by the model error *from the previous stage*; this will be guaranteed to be the case after transient periods of length $\rho_n := t_n - t_{n-1}$, which allows relaxation in (3.4) to take place. However, it will be shown that this convergence can only be guaranteed to occur on the 'observational subspace' $P_N L_\sigma^2$, for N sufficiently large. Indeed, a crucial observation at this point is that if $f = P_N f$, then $f_n = P_N f_n$, for all $N \geqslant 1$. We refer the reader to remarks 4.6 and 5.2 for additional remarks on the basic expectations for recovering force from low-mode data and the underlying limitations of this algorithm. We refer the reader to remark 7.1 for a more detailed discussion on the size of the transient periods, ρ_n.

REMARK 3.2. We remark that the first guess, f_0, need not be arbitrary and can be chosen to be:

$$f_0 := \frac{\mathrm{d}}{\mathrm{d}t} P_N u + \nu A P_N u + P_N B(P_N u, P_N u), \tag{3.9}$$

as suggested in (1.2), where $t_0 > 0$ is chosen sufficiently large so that $u(t_0)$ is contained in an absorbing ball for (2.3). Similarly, the initial states, v_n^0, at each stage need not be arbitrary. Since $\{P_N u(t)\}_{t \geqslant t_0}$ is assumed to be given, one can initialize the system governing v_n at time $t = t_{n-1}$ with $v_n^0 = P_N u(t_{n-1})$ at each stage. These natural choices would presumably aid in the implementation of the proposed algorithm; we refer the reader to remark 7.1 for additional remarks related to this point.

Before outlining the proofs, we rigorously state the main results of the article.

4. Statements of main results

Let $A^{1/2}u_0 \in \tilde{B}_1$ and $Au_0 \in \tilde{B}_2$ and $f \in C([t_0, \infty); L^2_\sigma) \cap L^\infty(t_0, \infty; L^2_\sigma)$, for some $t_0 \geqslant 0$, where $\tilde{B}_1, \tilde{B}_2$ were defined as in § 2. Let u denote the unique strong solution of (2.3) corresponding to forcing f and initial data u_0 guaranteed by theorem 2.1 corresponding to initial velocity u_0 and external forcing f. Given $t_0 \geqslant 0$, let γ_0 denote the initial relative error defined by

$$\gamma_0 := \frac{\sup_{t \geqslant t_0} \|f_0(t) - f(t)\|_{L^2}}{\sup_{t \geqslant t_0} \|f(t)\|_{L^2}}, \tag{4.1}$$

where f_0 is a user-prescribed initial guess for the force which satisfies $f_0 \in C([t_0, \infty); L^2_\sigma) \cap L^\infty(t_0, \infty; L^2_\sigma)$. Recall that the assumptions on the initial data imply [via (2.10), (2.15)] that $A^{1/2}u(t) \in \tilde{B}_1$ and $Au(t) \in 2\tilde{B}_2$, for all $t \geqslant 0$.

THEOREM 4.1. *Suppose that $f = P_N f$, for some $N \geqslant 1$, and that $\{P_N u(t)\}_{t \geqslant t_0}$ is given. There exists a positive constant c, depending on γ_0, such that for any $\beta \in (0, 1)$, if $N \geqslant 1$ satisfies:*

$$\beta^2 N^2 > c(\tilde{\sigma}_1 + \tilde{G})\tilde{G}^2, \tag{4.2}$$

then there exists a choice for the tuning parameter μ and increasing sequence of times $t_{n-1} \leqslant t_n$, for $n \geqslant 2$, such that

$$\sup_{t \geqslant t_n} \|f_n(t) - f(t)\|_{L^2} \leqslant \beta \left(\sup_{t \geqslant t_{n-1}} \|f_{n-1}(t) - f(t)\|_{L^2} \right), \tag{4.3}$$

for all $n \geqslant 1$, where $f_n = P_N f_n$, $f_n \in C([t_0, \infty); L^2_\sigma) \cap L^\infty(t_0, \infty; L^2_\sigma)$, and each f_n is given by (3.8).

REMARK 4.2. It is worth pointing out that (4.2) depends on the unknown forcing f. However, it must be emphasized that this condition only depends on f through its size. From a practical perspective, one must always approach the problem of parameter estimation with prior knowledge in hand. In this light, what condition (4.2) indicates is that if one has access to the Grashof number of the flow, for instance through measurement of the Reynolds number (see remark 4.7), then the only prior knowledge on the force that is needed to achieve exact recovery is knowledge of its shape factor $\tilde{\sigma}_1$. It moreover indicates that having such knowledge in one's possession can quantitatively inform what balance is needed between the number of observations and the algorithmic parameter, μ, to ensure a full reconstruction of the unknown force.

REMARK 4.3. Regarding condition (4.2), we remind the reader that $\tilde{G}$ also depends on viscosity. Thus, the intuition that the number of observations needed should increase as the viscosity decreases or that fewer observations are needed when viscosity is large is reflected in the statement.

Note that when f is time-dependent, theorem 4.1 only asserts recovery of the external force asymptotically in time. However, when the force is time-periodic or time-independent, then theorem 4.1 immediately implies that the external force is eventually recovered; we provide a statement of the time-independent case in the following corollary since the corresponding approximating forces are obtained by evaluating the sequence of approximating forces asserted in theorem 4.1 at certain times.

COROLLARY 4.4. *Suppose that $f = P_N f$, for some $N \geqslant 1$, is time-independent, and that $\{P_N u(t)\}_{t \geqslant t_0}$ is given. There exists a positive constant c_0, depending on γ_0, such that for any $\beta \in (0, 1)$, if $N \geqslant 1$ satisfies:*

$$\beta^2 N^2 > c_0(\tilde{\sigma}_1 + G)G^2, \tag{4.4}$$

then there exists a choice for the tuning parameter μ and an increasing sequence of times $t_{n-1} \leqslant t_n$ such that

$$\|f_n - f\|_{L^2} \leqslant \beta \|f_{n-1} - f\|_{L^2}, \tag{4.5}$$

for all $n \geqslant 1$, where $f_n = P_N f_n$ and each f_n is given by (3.8) evaluated at $t = t_n$.

In the setting of time-independent forcing, one can in fact 'recycle' the data provided that a sufficiently long time series is available. By 'recycle' we mean that once an approximation to the force is proposed by the algorithm at a given stage, we may use this proposed forcing to re-run the algorithm over *the same time window* to produce the subsequent approximation of the force in the following stage, and so on. In this way, the same data set is used over and over in order to generate new approximations to the force.

THEOREM 4.5. *Let $T > 0$. Suppose that $f = P_N f$, for some $N \geqslant 1$, is time-independent, and that $\{P_N u(t)\}_{t \in [0, T)}$ is given, for some $t_0 \geqslant 0$. There exists a positive constant c_0, depending on γ_0, such that for any $\beta \in (0, 1)$, if $N \geqslant 1$ satisfies:*

$$\beta^2 N^2 > c_0(\tilde{\sigma}_1 + G)G^2, \tag{4.6}$$

then for each $n \geqslant 1$, there exists a choice for the tuning parameter μ and a sequence of times, $t_n > 0$, such that if $T > t_0 + t_n$, for all $n \geqslant 1$, then

$$\|f_n - f\|_{L^2} \leqslant \beta \|f_{n-1} - f\|_{L^2}, \tag{4.7}$$

for all $n \geqslant 1$, where $f_n = P_N f_n$, $f_n \in L^2_\sigma$, and each is determined by a procedure similar to that in § 3, except that v_n, u_n, f_n are always derived on the interval $[0, T]$.

The proof of theorem 4.5 follows along the same lines as that of theorem 4.1, except that one requires a few technical modifications of the set-up described in § 3; we supply the relevant details of these modifications in § 7.

REMARK 4.6. A well-known example given by Marchioro [60] exhibits a scenario where observation of low modes below the spectrum of the forcing are sufficient

to determine the forcing. In particular, the study [**60**] identifies a class of low-mode body forces for which the asymptotic behaviour of solutions to (2.3) are characterized by a one-point global attractor for (2.3) whose unique stationary point is supported on a frequency shell strictly smaller than that of the force. Of course, this phenomenon is made possible due to the presence of nonlinearity in (2.3).

The extent to which it is possible to determine features of the forcing beyond the scales which are observed is, in general, not known and in particular, not addressed by theorem 4.5. Indeed, condition (4.6) identifies an upper bound on the number of modes that one should observe in order to uniquely determine the forcing provided that scales beyond those that are directly observed are not forced to begin with. In particular, it would be interesting to (1) study the sharpness of condition (4.6), and (2) whether one may identify classes of forces, beyond the Marchioro class, which inject energy into scales larger than those which are directly observed, preferably much larger, but can nevertheless be reconstructed by these observations. These issues are left to be investigated in a future study.

REMARK 4.7. There are at least three natural directions that warrant further investigation. First and foremost, a systematic computational study that probes the efficacy and limitations of this method for recovering the force should be carried out, especially in the context of noisy observations. For instance, each of the above theorems suggests that a large number of modes are required to achieve convergence. In the context of a turbulent flow, $G \sim \mathrm{Re}^2$, where Re denotes the Reynolds number of the flow [**28**, **29**], which may be intractably large in practice. On the contrary, the analysis performed here inherently takes into account 'worst-case' scenarios that may saturate various inequalities that were used, but which may occur rarely in reality. This direction will be explored in future research.

Secondly, the choice of Fourier modes as the form of observations is chosen due to its conceptual and analytical convenience. In principle, other observations on the velocity, such as nodal values or local spatial averages can also be used. However, the choice of Fourier modes allows one to commute the 'observation operator', P_N with derivatives, which greatly facilitates the analysis. The failure of this commutation introduces further analytical difficulties. Indeed, the original feedback control system was introduced with a general interpolant observable operator, I_h, replacing P_N in (3.1). In connection to this, the reader is referred to the classical studies [**24**, **25**, **46**, **56**, **57**], where the notion of 'finite determining parameters', properly generalizing 'determining modes', was developed.

Thirdly, although the existence of an inertial manifold for the 2D NSE is an open problem, there are several systems, which do possesses inertial manifolds [**45**, **66**], such as the Kuramoto–Sivashinsky equation [**43**]; it would be interesting to explore what can be gained in this particular context. In a similar vein, it would also be interesting to explore the usage of approximate inertial manifolds for the 2D NSE, as it is used, for instance, in post-processing Galerkin methods. Indeed, these ideas have been successfully used in the context of downscaling data assimilation algorithms in [**62**].

5. Outline of the convergence argument

To prove theorem 4.1, the object of interest will be the error in the forcing, which we also refer to as 'model error'. This is denoted by

$$g_n := f_n - f, \tag{5.1}$$

where f_n is generated from the scheme described in § 3. We claim the following: there exists a sequence of times $t_n \geqslant t_{n-1}$, for all $n \geqslant 1$, such that

$$\sup_{t \geqslant t_n} \|A^{1/2} g_n(t)\|_{L^2} \leqslant \beta \sup_{t \geqslant t_{n-1}} \|A^{1/2} g_{n-1}(t)\|_{L^2}, \tag{5.2}$$

for some $\beta \in (0, 1)$. The lengths of the relaxation periods, $\rho_n := t_n - t_{n-1}$, between the moments, t_{n-1}, t_n, at which we choose to reconstitute new forces, f_{n-1}, f_n, respectively, are prescribed to be sufficiently large in order to allow time for system (3.7) to relax. As we will see, the length of these relaxation periods, ρ_n, will essentially be determined by the relaxation parameter μ; the reader is referred to remark 7.1 for a precise relation.

Let us denote the synchronization error by

$$w_n := v_n - u.$$

Observe that the evolution of w_n over the time interval $[t_{n-1}, \infty)$ is governed by

$$\frac{\mathrm{d}}{\mathrm{d}t} w_n + \nu A w_n + B(w_n, w_n) + DB(u)w_n = g_{n-1} - \mu P_N w_n,$$
$$w_n(t_{n-1}) = v_n^0 - u(t_{n-1}), \tag{5.3}$$

where $DB(u)v := B(u, v) + B(v, u)$. Let $B_N = P_N B$ and $DB_N = P_N DB$. Then, using (2.3), the facts that $P_N f = f$ and $u_n = P_N u + Q_N v_N$, and (3.8), we obtain

$$
\begin{aligned}
g_n &= \left(\frac{\mathrm{d}}{\mathrm{d}t} P_N u_n + \nu A P_N u_n + B_N(u_n, u_n) \right) - \left(\frac{\mathrm{d}}{\mathrm{d}t} P_N u + \nu A P_N u + B_N(u, u) \right) \\
&= B_N(P_N u + Q_N v_n, P_N u + Q_N v_n) - B_N(P_N u + Q_N u, P_N u + Q_N u) \\
&= B_N(Q_N w_n, Q_N w_N) + DB_N(u) Q_N w_n, \tag{5.4}
\end{aligned}
$$

which holds over $[t_{n-1}, \infty)$. In analogy to (1.5), we define the 'Reynolds stress' at stage n by

$$\mathcal{R}_N^{(n)} := B_N(Q_N w_n, Q_N w_n) + DB_N(u) Q_N w_n = g_n. \tag{5.5}$$

Ultimately, (5.5) enables us to envision a recursion in the model error at each stage through the dependence of $\mathcal{R}_N^{(n)}$ on $\mathcal{R}_N^{(n-1)}$ via the synchronization error w_n. Hence, in order to prove that $\mathcal{R}_N^{(n)}$ vanishes in the limit as $n \to \infty$, we require *sensitivity-type* estimates, that is, estimates on w_n. The estimates will take on the following

form:

$$\|A^{1/2}w_n(t)\|_{L^2} \leqslant \nu \left(\frac{\nu}{\mu}\right)^{1/2} O\left(\frac{\sup_{t \geqslant t_n}\|g_{n-1}(t)\|_{L^2}}{\nu^2}\right), \tag{5.6}$$

for all $t \geqslant t_n$, for some sufficiently large $t_n \geqslant t_{n-1}$. Before we go on to develop estimates of form (5.6) in § 6, let us first determine the precise manner in which their application will arise.

REMARK 5.1. In what follows and for the remainder of the paper, we make use of the convention that C denotes a generic dimensionless constant, which may change line-to-line and possibly be large, but will always be independent of N, ν, γ_0.

To estimate (5.5), we will invoke the following inequalities for $B(u, v)$, which follows from a direct application of Hölder's inequality:

$$\|B(u, v)\|_{L^2} \leqslant \min\{\|u\|_{L^\infty}\|A^{1/2}v\|_{L^2}, \|u\|_{L^4}\|A^{1/2}v\|_{L^4}\}. \tag{5.7}$$

We will also make use of the fact that P_N, Q_N commute with $A^{m/2}$, for all integers m, and that the following inequalities hold for all $N > 0$, $\ell > 0$, and $m \in \mathbb{Z}$:

$$\|A^{m/2}P_N v\|_{L^2} \leqslant N^m\|P_N v\|_{L^2}, \quad \|A^{m/2}Q_N v\|_{L^2} \leqslant N^{-\ell}\|A^{(m+\ell)/2}Q_N v\|_{L^2}. \tag{5.8}$$

Lastly, we make the following elementary, but important observation for treating the first term appearing in (5.5): for vector fields $u = (u_1, u_2)$ and $v = (v_1, v_2)$, we have:

$$B(u, v)_i = \sum_{j=1}^{2} P(u \cdot \nabla v_i) = P\nabla \cdot (uv_i), \quad i = 1, 2,$$

where $B(u, v)_i$ denotes the i-th component of the vector field $B(u, v)$. Thus, upon applying (5.8) and Hölder's inequality, we obtain:

$$\|B_N(u, v)\|_{L^2}^2 \leqslant 2\sum_{i=1}^{2}\|P_N A^{1/2}(uv_i)\|_{L^2}^2 \leqslant 2N^2\sum_{i=1}^{2}\|uv_i\|_{L^2}^2 \leqslant 2N^2\|u\|_{L^4}^2\|v\|_{L^4}^2. \tag{5.9}$$

From (5.5), we now apply (5.9), (5.7), (5.8), and interpolation to obtain

$$\begin{aligned}
\|\mathcal{R}_N^{(n)}\|_{L^2} &\leqslant \|P_N A^{1/2}(Q_N w_n \otimes Q_N w_n)\|_{L^2} + \|DB_N(u)Q_N w_n\|_{L^2} \\
&\leqslant CN\|Q_N w_n\|_{L^4}^2 + \|u\|_{L^\infty}\|A^{1/2}Q_N w_n\|_{L^2} + \|Q_N w_n\|_{L^4}\|A^{1/2}u\|_{L^4} \\
&\leqslant CN\|A^{1/2}Q_N w_n\|_{L^2}\|Q_N w_n\|_{L^2} + C\|Au\|_{L^2}^{1/2}\|u\|_{L^2}^{1/2}\|A^{1/2}Q_N w_n\|_{L^2} \\
&\quad + \|A^{1/2}Q_N w_n\|_{L^2}^{1/2}\|Q_N w_n\|_{L^2}^{1/2}\|Au\|_{L^2}^{1/2}\|A^{1/2}u\|_{L^2}^{1/2} \\
&\leqslant C\|A^{1/2}Q_N w_n\|_{L^2}^2 + C\|Au\|_{L^2}^{1/2}\|u\|_{L^2}^{1/2}\|A^{1/2}Q_N w_n\|_{L^2} \\
&\quad + \frac{C}{N^{1/2}}\|A^{1/2}Q_N w_n\|_{L^2}\|Au\|_{L^2}^{1/2}\|A^{1/2}u\|_{L^2}^{1/2} \\
&\leqslant C_0\nu(\tilde{\sigma}_1 + \tilde{G})^{1/2}\tilde{G}\left(1 + \frac{\|A^{1/2}Q_N w_n\|_{L^2}}{\nu}\right)\|A^{1/2}Q_N w_n\|_{L^2}, \tag{5.10}
\end{aligned}$$

for some universal constant $C_0 > 0$, independent of n, for all $t \geqslant t_{n-1}$. Note that we also invoked the assumption that u belongs to the absorbing ball in H_σ^1 and H_σ^2 [see (2.11), (2.14), respectively] in obtaining the final inequality. It is at this point that one applies (5.6) in order to properly close the recursive estimate. In order to rigorously carry out this argument, let us therefore prove that (5.6) indeed holds.

REMARK 5.2. In the case when the force contains modes beyond those that are observed, one can identify an obstruction that precludes a proof in the manner described above. Suppose that $f \neq P_N f$. Then, modify the ansatz (3.8) for the force at each stage by removing the projection onto low modes. In particular, re-define f_n so that:

$$f_n = \frac{\mathrm{d}}{\mathrm{d}t} u_n + \nu A u_n + B(u_n, u_n).$$

Then, the model error becomes:

$$g_n = \frac{\mathrm{d}}{\mathrm{d}t} Q_N w_n + \nu A Q_N w_n + B(Q_N w_n, Q_N w_n) + DB(u)Q_N w_n. \tag{5.11}$$

Upon applying the complementary projection, Q_N, to (5.3), and combining the result with (5.11), we see that:

$$Q_N g_n = -B^N(P_N w_n, P_N w_n) - B^N(P_N w_n, Q_N w_n) - B^N(Q_N w_n, P_N w_n)$$
$$- DB^N(u)P_N w_n + Q_N g_{n-1},$$

where $B^N = Q_N B$ and $DB^N = Q_N DB$. Due to the presence of $Q_N g_{n-1}$ on the right-hand side, one cannot expect to obtain a convergent recursive relation of the form $\|Q_N g_n\|_{L^2} \leqslant \beta \|Q_N g_{n-1}\|_{L^2}$, for some $\beta \in (0, 1)$. Although one can establish an estimate of $Q_N g_n$ from this relation that is of the form $\|Q_N g_n - Q_N g_{n-1}\|_{L^2} \leqslant O_N(\|A^{1/2}Q_N w_n\|_{L^2})$, where the suppressed constant has a favourable dependence on N, a subsequent analysis will nevertheless be unable to establish a suitable recursion relation since we will only ever have access to an estimate of the form (5.6).

6. Sensitivity analysis

We establish a more precise form of the crucial estimates (5.6), which form the bridge to the desired recursion for the model error at each stage of the approximation. For this, we recall the notation introduced in § 5. In particular, we prove the following.

PROPOSITION 6.1. *There exist universal constants $\bar{c}_0, \underline{c}_0 \geqslant 1$ such that if μ, N satisfy*

$$\underline{c}_0(\tilde{\sigma}_1 + \tilde{G})\tilde{G}^2 \leqslant \frac{\mu}{\nu} \leqslant \bar{c}_0 N^2, \tag{6.1}$$

then for each $n \geqslant 1$, there exist relaxation periods, $\rho_n = t_n - t_{n-1}$, for some $t_n > t_{n-1}$, such that

$$\sup_{t \in I_n} \left(\frac{\|A^{1/2} w_n(t)\|_{L^2}}{\nu} \right) \leqslant \left(\frac{2C_1 \nu}{\mu} \right)^{1/2} \left(\frac{\sup_{t \in I_{n-1}} \|g_{n-1}(t)\|_{L^2}}{\nu^2} \right), \tag{6.2}$$

where $I_n := [t_{n-1} + \rho_n, \infty)$, *for some universal constant* $C_1 \geqslant 1$, *independent of n. Moreover,* ρ_n *satisfies* (6.5).

Proof. Fix $n \geqslant 1$. The enstrophy-balance for w_n is obtained by taking the L^2-inner product of (5.3) by Aw_n, which yields:

$$\frac{1}{2}\frac{d}{dt}\|A^{1/2}w_n\|_{L^2}^2 + \nu\|Aw_n\|_{L^2}^2 = -\langle B(u, w_n), Aw_n\rangle - \langle B(w_n, u), Aw_n\rangle$$
$$+ \langle g_{n-1}, Aw_n\rangle - \mu\|A^{1/2}P_N w_n\|_{L^2}^2$$
$$= I + II + III + IV. \tag{6.3}$$

Observe that by interpolation, the Cauchy–Schwarz inequality, Poincaré's inequality, and (2.11), (2.14), we may estimate:

$$|I| \leqslant C\|u\|_{L^\infty}\|A^{1/2}w_n\|_{L^2}\|Aw_n\|_{L^2}$$
$$\leqslant C\|Au\|_{L^2}^{1/2}\|u\|_{L^2}^{1/2}\|A^{1/2}w_n\|_{L^2}\|Aw_n\|_{L^2}$$
$$\leqslant C\mu\left(\frac{\nu}{\mu}\right)(\tilde{\sigma}_1 + \tilde{G})\tilde{G}^2\|A^{1/2}w_n\|_{L^2}^2 + \frac{\nu}{100}\|Aw_n\|_{L^2}^2.$$

Similarly, we obtain:

$$|II| \leqslant C\|A^{1/2}w_n\|_{L^2}^{1/2}\|w_n\|_{L^2}^{1/2}\|Au\|_{L^2}^{1/2}\|A^{1/2}u\|_{L^2}^{1/2}\|Aw_n\|_{L^2}$$
$$\leqslant C\mu\left(\frac{\nu}{\mu}\right)(\tilde{\sigma}_1 + \tilde{G})\tilde{G}^2\|A^{1/2}w_n\|_{L^2}^2 + \frac{\nu}{100}\|Aw_n\|_{L^2}^2.$$

For the third term, we apply the Cauchy–Schwarz inequality to obtain:

$$|III| \leqslant C\frac{\|g_{n-1}\|_{L^2}^2}{\nu} + \frac{\nu}{100}\|Aw_n\|_{L^2}^2.$$

Lastly, we have:

$$IV = -\mu\|A^{1/2}w_n\|_{L^2}^2 + \mu\|A^{1/2}Q_N w_n\|_{L^2}^2$$
$$\leqslant -\mu\|A^{1/2}w_n\|_{L^2}^2 + \frac{\mu}{N^2}\|Aw_n\|_{L^2}^2.$$

Upon combining the estimates for I–IV and invoking (6.1), where $\underline{c}_0, \overline{c}_0 \geqslant 1$ are chosen appropriately relative to the constants C appearing above, we arrive at:

$$\frac{d}{dt}\|A^{1/2}w_n\|_{L^2}^2 + \frac{3}{2}\nu\|Aw_n\|_{L^2}^2 + \frac{3}{2}\mu\|A^{1/2}w_n\|_{L^2}^2 \leqslant C\nu^3\left(\frac{\|g_{n-1}\|_{L^2}}{\nu^2}\right)^2.$$

An application of Gronwall's inequality yields:

$$\left(\frac{\|A^{1/2}w_n(t)\|_{L^2}}{\nu}\right)^2 \leqslant e^{-\mu(t-t_n)}e^{-\mu\rho_n}\left(\frac{\|A^{1/2}w_n(t_{n-1})\|_{L^2}}{\nu}\right)^2$$
$$+ C_1\left(\frac{\nu}{\mu}\right)\left(\frac{\sup_{t\geqslant t_{n-1}}\|g_{n-1}(t)\|_{L^2}}{\nu^2}\right)^2, \tag{6.4}$$

for all $t \geqslant t_n$, for some $C_1 \geqslant 1$ independent of n. Now, recall that $w_n(t_{n-1}) = v_n^0 - u(t_{n-1})$. We choose $\rho_n > 0$ such that:

$$\rho_n \geqslant \frac{1}{\mu} \ln \left[\left(\frac{\mu}{C_1 \nu} \right) \left(\frac{\nu \|A^{1/2} w_n(t_{n-1})\|_{L^2}}{\sup_{t \in I_{n-1}} \|g_{n-1}(t)\|_{L^2}} \right)^2 \right]. \tag{6.5}$$

Returning to (6.4), it follows that

$$\sup_{t \in I_n} \left(\frac{\|A^{1/2} w_n(t)\|_{L^2}}{\nu} \right) \leqslant \left(\frac{2C_1 \nu}{\mu} \right)^{1/2} \left(\frac{\sup_{t \in I_{n-1}} \|g_{n-1}(t)\|_{L^2}}{\nu^2} \right),$$

as desired.   □

7. Proofs of convergence

Let us assume that $f = P_N f$. At the initializing stage, $n = 0$, we consider any $f_0 \in C([0,\infty); L_\sigma^2) \cap L^\infty([0,\infty;); L_\sigma^2)$. Recall that in each subsequent stage, we will produce a new approximation, f_n, for the force via the ansatz (3.8). We proceed by induction.

Proof of theorem 4.1. Fix $\beta \in (0,1)$. We choose c_0 to satisfy:

$$c_0 \geqslant \max \left\{ \underline{c}_0, 2C_1 \left(\frac{C_0}{\beta} \right)^2 \left[1 + \left(\frac{2C_1}{\underline{c}_0 (\tilde{\sigma}_1 + \tilde{G})} \right)^{1/2} \gamma_0 \right]^2 \right\},$$

where $C_0, C_1, \underline{c}_0$ are the universal constants appearing in (5.10), (6.1), (6.2), respectively, and γ_0 denotes the initial relative error defined by (4.1). Fix any $c_1 \leqslant \tilde{c}$, where $\tilde{c}$ is the constant appearing in (3.2). Then, assume that μ, N satisfies (4.2) with $c = c_0 c_1^{-1}$. Then, choose μ such that:

$$c_0 (\tilde{\sigma}_1 + \tilde{G}) \tilde{G}^2 < \beta^2 \frac{\mu}{\nu} \leqslant c_1 \beta^2 N^2. \tag{7.1}$$

Observe that N satisfies (4.2) with $c = c_0 c_1^{-1}$. Since $c_0 \geqslant \underline{c}_0$ by choice, it immediately follows that (6.1) also holds.

Let $n = 1$. Denote $\mathcal{R}_N^{(0)} := g_0 = f_0 - f$. Observe that from (2.9), (4.1), we have:

$$\sup_{t \geqslant t_0} \|\mathcal{R}_N^{(0)}(t)\|_{L^2} = \gamma_0 \nu^2 \tilde{G}. \tag{7.2}$$

Combining proposition 6.1, (5.10) for $n = 1$, and (7.2) ensures that there exists a relaxation period $\rho_1 > 0$, for some $t_1 \geqslant t_0$, such that

$$\sup_{t \geqslant I_1} \|\mathcal{R}_N^{(1)}(t)\|_{L^2} \leqslant (2C_1)^{1/2} C_0 (\tilde{\sigma}_1 + \tilde{G})^{1/2} \tilde{G} \left(1 + \left(\frac{2C_1 \nu}{\mu} \right)^{1/2} \gamma_0 \tilde{G} \right) \left(\frac{\nu}{\mu} \right)^{1/2}$$

$$\sup_{t \geqslant I_0} \|\mathcal{R}_N^{(0)}(t)\|_{L^2}, \tag{7.3}$$

where $I_1 = [t_0 + \rho_1, \infty)$.

Further assume that μ satisfies:

$$\beta^2 \frac{\mu}{\nu} \geqslant 2C_1 C_0^2 \left((\tilde{\sigma}_1 + \tilde{G})^{1/2} + \left(\frac{2C_1}{\underline{c}_0} \right)^{1/2} \gamma_0 \right)^2 \tilde{G}, \tag{7.4}$$

where C_1 is the same constant appearing in (7.3). Then, (7.3), (6.1), and (7.4) imply:

$$\sup_{t \geqslant I_1} \| \mathcal{R}_N^{(1)}(t) \|_{L^2} \leqslant \beta \left(\sup_{t \geqslant I_0} \| \mathcal{R}_N^{(0)}(t) \|_{L^2} \right). \tag{7.5}$$

This establishes the base case.

Now, suppose that for all $k = 1, \ldots, n$, there exist relaxation periods ρ_k, such that:

$$\sup_{t \geqslant I_k} \| \mathcal{R}_N^{(k)}(t) \|_{L^2} \leqslant \beta \sup_{t \geqslant I_{k-1}} \| \mathcal{R}_N^{(k-1)}(t) \|_{L^2}. \tag{7.6}$$

With (7.2), it follows that:

$$\sup_{t \geqslant I_k} \| \mathcal{R}_N^{(k)}(t) \|_{L^2} \leqslant \beta^k \gamma_0 \tilde{G} \leqslant \gamma_0 \tilde{G}, \tag{7.7}$$

for all $k = 1, \ldots, n$. We may thus deduce from proposition 6.1, (5.10), and (7.7) that

$$\sup_{t \geqslant I_{n+1}} \| \mathcal{R}_N^{(n+1)}(t) \|_{L^2} \leqslant (2C_1)^{1/2} C_0 (\tilde{\sigma}_1 + \tilde{G})^{1/2} \tilde{G} \left(1 + \left(\frac{2C_1 \nu}{\mu} \right)^{1/2} \gamma_0 \tilde{G} \right) \left(\frac{\nu}{\mu} \right)^{1/2}$$

$$\sup_{t \geqslant I_n} \| \mathcal{R}_N^{(n)}(t) \|_{L^2}. \tag{7.8}$$

An application of (7.4) and (6.1) then implies

$$\sup_{t \geqslant I_{n+1}} \| \mathcal{R}_N^{(n+1)}(t) \|_{L^2} \leqslant \beta \left(\sup_{t \geqslant I_n} \| \mathcal{R}_N^{(n)}(t) \|_{L^2} \right),$$

as desired. This completes the induction. $\qquad \square$

REMARK 7.1. From the proof of theorem 4.1, we can obtain more informative estimates on the relaxation periods, ρ_n. Indeed, by (6.5) and (7.2), we see that the first relaxation period satisfies:

$$\rho_1 \geqslant \frac{1}{\mu} \ln \left[\left(\frac{\mu}{C_1 \nu} \right) \frac{1}{\gamma_0^2 \tilde{G}^2} \left(\frac{\| A^{1/2}(v_1^0 - u(t_0)) \|_{L^2}}{\nu} \right)^2 \right]. \tag{7.9}$$

In subsequent stages, $n \geqslant 1$, by (6.5) and (6.2) applied at the preceding stage, we may deduce that the relaxation periods satisfy:

$$\rho_n \geqslant \frac{1}{\mu} \ln \left(\frac{\sup_{t \in I_{n-2}} \| \mathcal{R}_N^{(n-2)}(t) \|_{L^2}}{\sup_{t \in I_{n-1}} \| \mathcal{R}_N^{(n-1)}(t) \|_{L^2}} \right) \geqslant -\frac{1}{\mu} \ln \beta. \tag{7.10}$$

One may then observe that (7.9) and (7.10) follow a consistent pattern if we allow ourselves to make special, but nevertheless natural choices for initial force ansatz,

f_0, and the initial data, v_1^0, of the first nudged system. Indeed, suppose that the solution, u, of (2.3) possesses additional regularity, for instance, $Au(t_0) \in L_\sigma^2$. Now, initialize the nudged system (3.4) with $v_1^0 = P_N u(t_0)$ and let the initial guess f_0 be given by (1.2). Then, the first relaxation period satisfies:

$$\rho_1 \geqslant \frac{1}{\mu} \ln \left[\left(\frac{c_1}{C_1} \right) \left(\frac{\nu^2}{\sup_{t \in I_0} \|\mathcal{R}_N^{(0)}(t)\|_{L^2}} \right)^2 \right], \tag{7.11}$$

where we applied the Poincaré inequality, the upper bound in (7.1), and (2.9).

Now, we prove a variation on the time-independent case, which allows one to recycle the existing data.

Proof sketch of theorem 4.5. We modify the algorithm in §3 as follows: let $J_0 = I_0 = [t_0, \infty)$. For convenience, we assume that $t_0 = 0$. Suppose that $\{P_N u(t)\}_{t \in J_0}$ is known. At stage $n = 1$, we suppose initial data v_1^0 is given and solve (3.4) to produce the solution v_1 over J_0. We then define (3.5) over J_0 and immediately generate $f_1(t)$ via (3.6) over J_0. To define the stage 1 approximation to f, we evaluate $f_1(t)$ after a transient period of length $\rho_1 > 0$ to obtain $f_1 := f_1(\rho_1)$.

In subsequent stages $n \geqslant 1$, we generate v_n over J_0 via (3.7), where $f_{n-1} := f_{n-1}(\rho_{n-1})$ was generated from the preceding stage, $n - 1$. We then define the stage-n approximation to the true state over J_0 by $u_n = P_N u + Q_N v_n$. We generate a new force, $f_n(t)$, via the ansatz (3.8). The stage-n approximation of f is then given by $f_n(\rho_n)$, for some $\rho_n > 0$.

To assess the error, we once again form the synchronization error, $w_n = v_n - u$, and the model error, $g_n(t) = f_n(t) - f$. For each $n \geqslant 1$, the identity (5.5) still holds. Consequently, (5.10) holds as well. The remaining ingredient to establish convergence is a time-independent force analogue of proposition 6.1.

Supposing that (6.1) holds with $\tilde{G} \mapsto G$, the proof of proposition 6.1 proceeds the same way, except that the analysis is carried out entirely over the interval J_0, to arrive at the analogue of (6.4):

$$\left(\frac{\|A^{1/2} w_n(\rho_n)\|_{L^2}}{\nu} \right)^2 \leqslant \mathrm{e}^{-\mu \rho_n} \left(\frac{\|A^{1/2} w_n(0)\|_{L^2}}{\nu} \right)^2 + C_1 \left(\frac{\nu}{\mu} \right) \left(\frac{\|g_{n-1}\|_{L^2}}{\nu^2} \right)^2. \tag{7.12}$$

One chooses $\rho_n > 0$ according to:

$$\rho_n \geqslant \frac{1}{\mu} \left[\frac{\mu}{C_1 \nu} \left(\frac{\nu \|A^{1/2} w_n(0)\|_{L^2}}{\|g_{n-1}\|_{L^2}} \right)^2 \right],$$

so that (7.12) yields:

$$\left(\frac{\|A^{1/2} w_n(\rho_n)\|_{L^2}}{\nu} \right)^2 \leqslant 2 C_1 \left(\frac{\nu}{\mu} \right) \left(\frac{\|g_{n-1}\|_{L^2}}{\nu^2} \right)^2.$$

It is at this time $t = \rho_n$, that we evaluate (3.8).

After these adjustments, it is now clear that the proof of theorem 4.5 follows an analogous manner to the proof theorem 4.1, mutatis mutandis. $\qquad\square$

Appendix A. Uniform-in-time estimates for palenstrophy

We now provide uniform-in-time estimates in H^2 for the reference flow field when the external force field is time-dependent. We begin by establishing an alternative form of the standard enstrophy balance of (2.3) as it is presented in theorem 2.1. Indeed, upon taking the L^2-inner product of (2.3) with Au, we obtain:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|A^{1/2}u\|_{L^2}^2 + \nu\|Au\|_{L^2}^2 = \langle f, Au\rangle.$$

By the Cauchy–Schwarz inequality and (2.9), we see that:

$$|\langle f, Au\rangle| \leqslant \nu^3\tilde{G}^2 + \frac{\nu}{4}\|Au\|_{L^2}^2.$$

By the Poincaré inequality, it follows that:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{\nu t}\|A^{1/2}u\|_{L^2}^2\right) + \frac{\nu}{2}\|Au\|_{L^2}^2 \leqslant 2\nu^3\tilde{G}^2\,\mathrm{e}^{\nu t}.$$

Integrating over $t \geqslant t_0$ yields:

$$\|A^{1/2}u(t)\|_{L^2}^2 + \frac{\nu}{2}\int_{t_0}^t \mathrm{e}^{-\nu(t-s)}\|Au(s)\|_{L^2}^2$$
$$\leqslant \|A^{1/2}u(t_0)\|_{L^2}^2\,\mathrm{e}^{-\nu(t-t_0)} + 2\nu^2\tilde{G}^2(1 - \mathrm{e}^{-\nu(t-t_0)}),$$

which is (2.10), as desired. Thus, for $u_0 \in \tilde{B}_1$, we may deduce:

$$\|A^{1/2}u(t)\|_{L^2}^2 + \frac{\nu}{2}\int_{t_0}^t \mathrm{e}^{-\nu(t-s)}\|Au(s)\|_{L^2}^2 \leqslant 2\nu^2\tilde{G}^2, \tag{A.1}$$

for all $t \geqslant t_0 \geqslant 0$.

Proof of (2.12). Upon taking the L^2-inner product of (2.3) with A^2u, we obtain:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|Au\|_{L^2}^2 + \nu\|A^{3/2}u\|_{L^2}^2 = -\langle B(u,u), A^2u\rangle + \langle f, A^2u\rangle = I + II.$$

Observe that integration by parts multiple times yields:

$$I = -\langle B(Au, u), Au\rangle - 2\sum_{\ell=1,2}\langle B(\partial_\ell u, \partial_\ell u), Au\rangle$$

where we applied the identity $\langle B(u,v), v\rangle = 0$. It then follows from Hölder's inequality, interpolation, Young's inequality, and (2.10) that:

$$|I| \leqslant 3\|A^{1/2}u\|_{L^2}\|Au\|_{L^4}^2 \leqslant 3c_L\|A^{1/2}u\|_{L^2}\|Au\|_{L^2}\|A^{3/2}u\|_{L^2}$$
$$\leqslant \frac{\nu}{8}\|A^{3/2}u\|_{L^2}^2 + 36c_L^2\nu\tilde{G}^2\|Au\|_{L^2}^2,$$

where $c_L \geqslant 1$ is the associated constant of interpolation, i.e. $\|Au\|_{L^4}^2 \leqslant c_L\|Au\|_{L^2}\|A^{3/2}u\|_{L^2}$. We treat II with integration by parts and the Cauchy–Schwarz inequality to obtain:

$$|II| \leqslant |\langle A^{1/2}f, A^{3/2}u\rangle| \leqslant 2\nu^3\left(\frac{\|A^{1/2}f\|_{L_t^\infty L_x^2}}{\nu^2}\right)^2 + \frac{\nu}{8}\|A^{3/2}u\|_{L^2}^2$$
$$\leqslant 2\nu^3\tilde{\sigma}_1^2\tilde{G}^2 + \frac{\nu}{8}\|A^{3/2}u\|_{L^2}^2.$$

We now combine the estimates I, II, and apply Poincaré's inequality to arrive at:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|Au\|_{L^2}^2 + \nu\|Au\|_{L^2}^2 + \frac{\nu}{2}\|A^{3/2}u\|_{L^2}^2 \leqslant 36c_L^2\nu\tilde{G}^2\|Au\|_{L^2}^2 + 2\nu^3\tilde{\sigma}_1^2\tilde{G}^2.$$

An application of Gronwall's inequality yields:

$$\|Au(t)\|_{L^2}^2 + \frac{\nu}{2}\int_{t_0}^t \mathrm{e}^{-\nu(t-s)}\|A^{3/2}u(s)\|_{L^2}^2\,\mathrm{d}s$$
$$\leqslant \|Au(t_0)\|_{L^2}^2\,\mathrm{e}^{-\nu(t-t_0)} + 36c_L^2\nu\tilde{G}^2\int_{t_0}^t \mathrm{e}^{-\nu(t-s)}\|Au(s)\|_{L^2}^2\,\mathrm{d}s$$
$$+ 2\nu^2\tilde{\sigma}_1^2\tilde{G}^2(1 - \mathrm{e}^{-\nu(t-t_0)}),$$

for all $t \geqslant t_0 \geqslant 0$. We now apply (A.1) to bound:

$$\|Au(t)\|_{L^2}^2 + \frac{\nu}{2}\int_{t_0}^t \mathrm{e}^{-\nu(t-s)}\|A^{3/2}u(s)\|_{L^2}^2\,\mathrm{d}s$$
$$\leqslant \|Au(t_0)\|_{L^2}^2\,\mathrm{e}^{-\nu(t-t_0)} + \tilde{c}_2^2\nu^2(\tilde{\sigma}_1 + \tilde{G})^2\tilde{G}^2,$$

where $\tilde{c}_2 = 12c_L$, which holds for all $t \geqslant t_0 \geqslant 0$.

Therefore, if $\|Au_0\|_{L^2} \leqslant \alpha\tilde{c}_2\nu(\tilde{\sigma}_1 + \tilde{G})\tilde{G}$, for any $\alpha > 0$, then

$$\|Au(t)\|_{L^2}^2 \leqslant \tilde{c}_2^2(1 + \alpha^2)\nu^2(\tilde{\sigma}_1 + \tilde{G})^2\tilde{G}^2,$$

for all $t \geqslant 0$. This completes the proof. $\qquad\square$

References

1   D. A. F. Albanez and M. J. Benvenutti. Continuous data assimilation algorithm for simplified bardina model. *Evol. Equ. Control Theory* **7** (2018), 33–52.

2   D. A. F. Albanez, H. J. Nussenzveig Lopes and E. S. Titi. Continuous data assimilation for the three-dimensional Navier–Stokes-α model. *Asymptotic Anal.* **97** (2016), 139–164.

3   M. U. Altaf, E. S. Titi, T. Gebrael, O. M. Knio, L. Zhao and M. F. McCabe. Downscaling the 2D Bénard convection equations using continuous data assimilation. *Comput. Geosci.* **21** (2017), 393–410.

4   A. Azouani, E. J. Olson and E. S. Titi. Continuous data assimilation using general interpolant observables. *J. Nonlinear Sci.* **24** (2014), 277–304.

5   N. Balci, C. Foias and M. S. Jolly. 2-D turbulence for forcing in all scales. *J. Math. Pures Appl.* **94** (2010), 1–32.

6   L. C. Berselli, T. Iliescu and W. J. Layton. *Mathematics of large eddy simulation of turbulent flows*. Scientific Computation (Berlin: Springer-Verlag, 2006).

7   H. Bessaih, E. J. Olson and E. S. Titi. Continuous data assimilation with stochastically noisy data. *J. Nonlinear Sci.* **28** (2015), 729–753.

8   A. Biswas, K. R. Brown and V. R. Martinez. Higher-order synchronization and a refined paradigm for global interpolant observables. *Ann. Appl. Math.* **38** (2022), 1–60.

9   A. Biswas and J. Hudson. Determining the viscosity of the Navier–Stokes equations from observations of finitely many modes. *Inverse Probl.* **39** (2023), 125012.

10   A. Biswas and V. R. Martinez. Higher-order synchronization for a data assimilation algorithm for the 2D Navier–Stokes equations. *Nonlinear Anal. Real World Appl.* **35** (2017), 132–157.

11   A. Biswas and R. Price. Continuous data assimilation for the three-dimensional Navier–Stokes equations. *SIAM J. Math. Anal.* **53** (2021), 6697–6723.

12   D. Blömker, K. Law, A. M. Stuart and K. C. Zygalakis. Accuracy and stability of the continuous-time 3DVAR filter for the Navier–Stokes equation. *Nonlinearity* **26** (2013), 2193–2219.

13   J. Blocher, V. R. Martinez and E. Olson. Data assimilation using noisy time-averaged measurements. *Physica D* **376-377** (2018), 49–59. Special Issue: Nonlinear Partial Differential Equations in Mathematical Fluid Dynamics.

14   Y. Cao, M. S. Jolly, E. S. Titi and J. P. Whitehead. Algebraic bounds on the Rayleigh–Bénard attractor. *Nonlinearity* **34** (2021), 509–531.

15   E. Carlson, J. Hudson and A. Larios. Parameter recovery for the 2 dimensional Navier–Stokes equations via continuous data assimilation. *SIAM J. Sci. Comput.* **42** (2020), A250–A270.

16   E. Carlson, J. Hudson, A. Larios, V. R. Martinez, E. Ng and J. P. Whitehead. Dynamically learning the parameters of a chaotic system using partial observations. *Discrete Contin. Dyn. Syst.* **42** (2022), 3809–3839.

17   E. Carlson and A. Larios. Sensitivity analysis for the 2D Navier–Stokes equations with applications to continuous data assimilation. *J. Nonlinear Sci.* **31** (2021), 84.

18   E. Celik, E. Olson and E. S. Titi. Spectral filtering of interpolant observables for a discrete-in-time downscaling data assimilation algorithm. *SIAM J. Appl. Dyn. Syst.* **18** (2019), 1118–1142.

19   N. Chen, Y. Li and E. Lunasin. An efficient continuous data assimilation algorithm for the Sabra shell model of turbulence. *Chaos* **31** (2021), 103123.

20   A. Cheskidov and M. Dai. Kolmogorov's dissipation number and the number of degrees of freedom for the 3D Navier–Stokes equations. *Proc. R. Soc. Edinburgh* **149** (2019), 429–446.

21   I. Cialenco and N. Glatt-Holtz. Parameter estimation for the stochastically perturbed Navier–Stokes equations. *Stochastic Process. Appl.* **121** (2011), 701–724.

22   P. Clark Di Leoni, A. Mazzino and L. Biferale. Inferring flow parameters and turbulent configuration with physics-informed data assimilation and spectral nudging. *Phys. Rev. Fluids* **3** (2018), 104604.

23   P. Clark Di Leoni, A. Mazzino and L. Biferale. Synchronization to big data: nudging the Navier–Stokes equations for data assimilation of turbulent flows. *Phys. Rev. X* **10** (2020), 011023.

24   B. Cockburn, D. A. Jones and E. S. Titi. Determining degrees of freedom for nonlinear dissipative equations. *C. R. Acad. Sci. Paris Sér. I Math.* **321** (1995), 563–568.

25   B. Cockburn, D. A. Jones and E. S. Titi. Estimating the number of asymptotic degrees of freedom for nonlinear dissipative systems. *Math. Comput.* **66** (1997), 1073–1087.

26   P. Constantin and C. Foias. *Navier–Stokes equations.* Chicago Lectures in Mathematics (Chicago, IL: University of Chicago Press, 1988).

27   R. Dascaliuc, C. Foias and M. S. Jolly. Relations between energy and enstrophy on the global attractor of the 2-D Navier–Stokes equations. *J. Dyn. Differ. Equ.* **17** (2005), 643–736.

28   R. Dascaliuc, C. Foias and M. S. Jolly. Some specific mathematical constraints on 2D turbulence. *Physica D* **237** (2008), 3020–3029.

29   R. Dascaliuc, C. Foias and M. S. Jolly. On the asymptotic behavior of average energy and enstrophy in 3D turbulent flows. *Physica D* **238** (2009), 725–736.

30   S. Desamsetti, H. P. Dasari, S. Langodan, E. S. Titi, O. Knio and I. Hoteit. Efficient dynamical downscaling of general circulation models using continuous data assimilation. *Q. J. R. Meteorol. Soc.* **145** (2019), 3175–3194.

31   A. E. Diegel and L. G. Rebholz. Continuous data assimilation and long-time accuracy in a C^0 interior penalty method for the Cahn–Hilliard equation. *Appl. Math. Comput.* **424** (2022), 127042.

32   A. Farhat, N. E. Glatt-Holtz, V. R. Martinez, S. A. McQuarrie and J. P. Whitehead. Data assimilation in large Prandtl Rayleigh–Bénard convection from thermal measurements. *SIAM J. Appl. Dyn. Syst.* **19** (2020), 510–540.

33   A. Farhat, H. Johnston, M. S. Jolly and E. S. Titi. Assimilation of nearly turbulent Rayleigh–Bénard flow through vorticity or local circulation measurements: a computational study. *J. Sci. Comput.* **77** (2018), 1–15.

34   A. Farhat, M. S. Jolly and E. S. Titi. Continuous data assimilation for the 2D Bénard convection through velocity measurements alone. *Physica D* **303** (2015), 59–66.

35   A. Farhat, E. Lunasin and E. S. Titi. Data assimilation algorithm for 3D Bénard convection in porous media employing only temperature measurements. *J. Math. Anal. Appl.* **438** (2016), 492–506.

36   A. Farhat, E. Lunasin and E. S. Titi. On the Charney conjecture of data assimilation employing temperature measurements alone: the paradigm of 3D planetary geostrophic model. *Math. Clim. Weather Forecast.* **2** (2016), 61–74.

37   A. Farhat, E. Lunasin and E. S. Titi. Continuous data assimilation for a 2D Bénard convection system through horizontal velocity measurements alone. *J. Nonlinear Sci.* **27** (2017), 1065–1087.

38   A. Farhat, E. Lunasin and E. S. Titi. *A data assimilation algorithm: the paradigm of the 3D Leray-α model of turbulence.* In *Nonlinear Partial Differential Equations Arising from Geometry and Physics.* London Mathematical Society. Lecture Notes Series (Singapore: Cambridge University Press, 2017).

39   C. Foias, M. S. Jolly, R. Kravchenko and E. S. Titi. A determining form for the two-dimensional Navier–Stokes equations: the Fourier modes case. *J. Math. Phys.* **53** (2012), 115623.

40   C. Foias, M. S. Jolly, D. Lithio and E. S. Titi. One-dimensional parametric determining form for the two-dimensional Navier–Stokes equations. *J. Nonlinear Sci.* **27** (2017), 1513–1529.

41   C. Foias, O. Manley, R. Rosa and R. Temam. *Navier–Stokes equations and turbulence.* Encyclopedia of Mathematics and its Applications, vol. 83 (Cambridge: Cambridge University Press, 2001).

42   C. Foias, C. Mondaini and E. S. Titi. A discrete data assimilation scheme for the solutions of the 2D Navier–Stokes equations and their statistics. *SIAM J. Appl. Dyn. Syst.* **15** (2016), 2019–2142.

43   C. Foias, B. Nicolaenko, G. R. Sell and R. Temam. Inertial manifolds for the Kuramoto–Sivashinsky equation and an estimate of their lowest dimension. *J. Math. Pures Appl.* **67** (1988), 197–226.

44   C. Foiaş and G. Prodi. Sur le comportement global des solutions non-stationnaires des équations de Navier–Stokes en dimension 2. *Rend. Sem. Mat. Univ. Padova* **39** (1967), 1–34.

45   C. Foias, G. R. Sell and T. Temam. Inertial manifolds for nonlinear evolutionary equations. *J. Differ. Equ.* **73** (1988), 309–353.

46   C. Foias and R. Temam. Determination of the solutions of the Navier–Stokes equations by a set of nodal values. *Math. Comput.* **43** (1984), 117–133.

47   C. Foias and R. Temam. *The connection between the Navier–Stokes equations, dynamical systems, and turbulence theory.* In *Directions in Partial Differential Equations (Madison, WI, 1985).* Publ. Math. Res. Center Univ. Wisconsin, vol. 54, pp. 55–73 (Boston, MA: Academic Press, 1987).

48   B. García-Archilla and J. Novo. Error analysis of fully discrete mixed finite element data assimilation schemes for the Navier–Stokes equations. *Adv. Comput. Math.* **46** (2020), 61.

49   B. García-Archilla, J. Novo and E. S. Titi. Uniform in time error estimates for a finite element method applied to a downscaling data assimilation algorithm for the Navier–Stokes equations. *SIAM J. Numer. Anal.* **58** (2020), 410–429.

50   M. Gesho, E. J. Olson and E. S. Titi. A computational study of a data assimilation algorithm for the two-dimensional Navier–Stokes equations. *Commun. Comput. Phys.* **19** (2016), 1094–1110.

51   H. A. Ibdah, C. F. Mondaini and E. S. Titi. Fully discrete numerical schemes of a data assimilation algorithm: uniform-in-time error estimates. *IMA J. Numer. Anal.* **40** (2019), 2584–2625.

52   M. S. Jolly, V. R. Martinez, E. J. Olson and E. S. Titi. Continuous data assimilation with blurred-in-time measurements of the surface quasi-geostrophic equation. *Chin. Ann. Math. Ser. B* **40** (2019), 721–764.

53   M. S. Jolly, V. R. Martinez and E. S. Titi. A data assimilation algorithm for the 2D subcritical surface quasi-geostrophic equation. *Adv. Nonlinear Stud.* **35** (2017), 167–192.

54   M. S. Jolly, T. Sadigov and E. S. Titi. A determining form for the damped driven nonlinear Schrödinger equation–Fourier modes case. *J. Differ. Equ.* **258** (2015), 2711–2744.

55   M. S. Jolly, T. Sadigov and E. S. Titi. Determining form and data assimilation algorithm for weakly damped and driven Korteweg–De Vries equation – Fourier modes case. *Nonlinear Anal. Real World Appl.* **36** (2017), 287–317.

56   D. A. Jones and E. S. Titi. Determining finite volume elements for the 2D Navier–Stokes equations. *Physica D* **60** (1992), 165–174.

57   D. A. Jones and E. S. Titi. On the number of determining nodes for the 2D Navier–Stokes equations. *J. Math. Anal.* **168** (1992), 72–88.

58   A. Larios and Y. Pei. Approximate continuous data assimilation of the 2D Navier–Stokes equations via the Voigt-regularization with observable data. *Evol. Equ. Control Theory* **9** (2020), 733–751.

59   A. Larios, L. G. Rebholz and C. Zerfas. Global in time stability and accuracy of IMEX-FEM data assimilation schemes for Navier–Stokes equations. *Comput. Methods Appl. Mech. Eng.* **345** (2019), 1077–1093.

60   C. Marchioro. An example of absence of turbulence for any Reynolds number. I. *Commun. Math. Phys.* **105** (1986), 99–106.

61   V. R. Martinez. Convergence analysis of a viscosity parameter recovery algorithm for the 2D Navier–Stokes equations. *Nonlinearity* **35** (2022), 2241–2287.

62   C. F. Mondaini and E. S. Titi. Uniform-in-time error estimates for the postprocessing Galerkin method applied to a data assimilation algorithm. *SIAM J. Numer. Anal.* **56** (2018), 78–110.

63   B. Pachev, J. P. Whitehead and S. A. McQuarrie. *Concurrent multi-parameter learning demonstrated on the Kuramoto–Sivashinsky equation* (2021).

64   Y. Pei. Continuous data assimilation for the 3D primitive equations of the ocean. *Commun. Pure Appl. Anal.* **18** (2019), 643–661.

65   R. Temam. *Navier–Stokes Equations: Theory and Numerical Analysis*, Reprint of the 1984 edn (Providence, RI: AMS Chelsea Publishing, 2001).

66   R. Temam. *Infinite-dimensional dynamical systems in mechanics and physics*, 2nd edn. Applied Mathematical Sciences, vol. 68 (New York: Springer-Verlag, 1997).

67   C. Yu, A. Giorgini, M. S. Jolly and A. Pakzad. Continuous data assimilation for the 3D
      Ladyzhenskaya model: analysis and computations. *Nonlinear Anal.: Real World Appl.* **68**
      (2022), 103659.
68   M. Zauner, V. Mons, O. Marquet and B. Leclaire. Nudging-based data assimilation of the
      turbulent flow around a square cylinder. *J. Fluid Mech.* **937** (2022), A38.