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CYCLIC ELEMENT THEORY IN CONNECTED AND LOCALLY CONNECTED HAUSDORFF SPACES

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1.0. Introduction. G. T. Whyburn, in 1926, began the development of cyclic element theory for Peano continua. This theory proved fruitful in the study of Peano spaces and a comprehensive development of the theory for metric spaces was presented in [**6**]. An excellent history of the theory is to be found in [**4**]. In [**7**] and [**5**] the generalization of cyclic element theory to more general spaces was begun. However, in each of these papers only basic definitions were set forth and fundamental results obtained. In this paper, we concern ourselves primarily with connected and locally connected Hausdorff spaces, developing the cyclic element theory initiated in [**7**] and demonstrating that the theory has many of the applications to connected and locally connected Hausdorff spaces that the classical theory has to Peano spaces. We note that we generalize many of the results from Chapter IV of [**6**] and that therefore the organization of this paper is closely related to the organization of that chapter. In fact, we might say that the purpose of this paper is to "rewrite" for connected and locally connected Hausdorff spaces the cyclic element theory presented in [**6**].

1.1. Preliminaries. Several of the following definitions are found in [**7**], and all are generalizations of definitions stated in [**6**]. Let M be a connected topological space. A point p of M is a *cut point* of M if and only if $M - p$ is not connected. A point p of M is an *end point* of M if and only if p has a local base of open sets with singleton boundaries. Two points a and b of M are said to be *conjugate in* M ("a is conjugate to b in M") if and only if no point of M separates a and b in M. For $a, b \in M$, $E(a, b)$ denotes the collection of all points of M which separate a and b in M. (It follows that a and b are conjugate if and only if $E(a, b) = \emptyset$). There is a natural (linear) order "$<$" on $E(a, b) \cup \{a, b\}$ defined by $a < x$, $x < b$ for all $x \in E(a, b)$; $a < b$, and if $x, y \in E(a, b)$ then $x < y$ if and only if $x \in E(a, y)$. The order $<$ on $E(a, b) \cup \{a, b\}$ is called the *cut point order* on $E(a, b) \cup \{a, b\}$. A subset E of M is an E_0-set of M if and only if E is nondegenerate, connected, has no cut point of itself, and is maximal with respect to these properties. A *cyclic element* of M is a subset of M which either consists of a single cut point or end point of M or is an E_0-set of M. An A-set of M is a closed subset of M such that

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$M - A$ is the union of a collection of open sets each bounded by a single point of A. If a, b are points of M, $C(a, b)$ denotes the intersection of all A-sets of M which contain both a and b, and the set $C(a, b)$ is called the *cyclic chain in M from a to b.*

Note. For the rest of this paper, unless stated otherwise, "M" denotes a non-degenerate connected and locally connected Hausdorff space.

The following results have been established in [**7**].

a) If $a, b \in M$, then $E(a, b) \cup \{a, b\}$ is closed and compact.

b) If $\mathscr{A}$ is a collection of A-sets of M, then $\bigcap \mathscr{A} = \emptyset$ or is an A-set of M.

c) A nonempty closed set A is an A-set of M if and only if each component of $M - A$ has exactly one boundary point.

d) If A is an A-set of M, and if Z is a connected subset of M, then $A \cap Z$ is connected (possibly empty); thus every A-set of M is connected and locally connected.

e) If a and b are distinct conjugate points of M, then $C(a, b) = \{p \in M : p$ is conjugate to both a and $b\}$, and in this case, $C(a, b)$ is an E_0-set of M. Further, if C is an E_0-set of M and a, b are distinct points of C, then a and b are conjugate in M and $C = C(a, b)$.

f) Any two E_0-sets of M have at most one common point.

g) For any two points a, b of M, $C(a, b) = E(a, b) \cup \{a, b\} \cup C$, where C is the union of all E_0-sets of M which meet $E(a, b) \cup \{a, b\}$ in exactly two points.

Further, the next two results were established in [**5**].

h) If E_1 and E_2 are distinct E_0-sets of M and intersect, their intersection is a cut point of M and $E_1 \cap E_2$ separates $E_1 - E_2$ and $E_2 - E_1$ in M.

i) If $a, b \in M$ and E is an E_0-set of M, then E meets $E(a, b) \cup \{a, b\}$ in at most two points.

1.2. THEOREM. *If $a, b \in M$, then the subspace topology on $E(a, b) \cup \{a, b\}$ is the order topology relative to the cut point order.*

Proof. If $E(a, b) = \emptyset$, then $E(a, b) \cup \{a, b\}$ is discrete with either topology. Assume then, that $E(a, b) \neq \emptyset$. It is well known that in general the order topology on $E(a, b) \cup \{a, b\}$ is a Hausdorff topology that is weaker than the subspace topology. (See, for instance [**9**, p. 206]). It then follows from 1.1-a that both topologies are compact Hausdorff so are identical.

2.0. E_0-sets and the conjugacy relation. Since we shall have occasion to refer to several of the results in this section and since they were not stated in either [**5**] or [**7**] we state them here. However, in many cases we omit the easy proofs. We note also that most of these results are generalizations of theorems that are known for metric spaces (see [**6**, Chapter IV]).

2.1. LEMMA. *If x is an end point of M, S a nondegenerate connected subset of M, and $x \in S$, then x is an end point of S.*

2.2. LEMMA. *If M has no cut point, then M has no end point.*

2.3. LEMMA. *If Z is a connected subset of M and p and q are conjugate in Z, then p and q are conjugate in M.*

2.4. LEMMA. *No E_0-set of M contains an end point of M.*

2.5. LEMMA. *If A is an A-set of M and C is a component of $M - A$, then $\bar{C}$ is an A-set of M.*

Proof. If D is a component of $M - \bar{C}$, then $\partial(D) = \partial(C)$.

2.6. THEOREM. *Of the following statements, if a connected subset A of M satisfies* a), *then A satisfies* b):

a) *If E is a cyclic element of M and $A \cap E$ is non-degenerate, then $E \subset A$.*

b) *If $x, y \in A$ and $N \subset M$ is an irreducible continuum from x to y, then $N \subset A$.*

Proof. Suppose A is connected and satisfies a). Suppose further that $x, y \in A$, N is an irreducible continuum from x to y, and $t \in N - A$. Since $x, y \in A$ and A is connected, $E(x, y) \cup \{x, y\} \subset A$. Thus if an E_0-set E meets $E(x, y) \cup \{x, y\}$ in two points, then $E \cap A$ is nondegenerate and so by assumption $E \subset A$. It follows that $C(x, y) \subset A$. In $M - C(x, y)$, let C_t be the component which contains t, and let $z = \partial(C_t)$. Then $z \in N$. If $z \notin \{x, y\}$, then x, y lie in components C_x, C_y respectively, of $M - (C_t \cup z)$. But then $C_x \cup z$ and $C_y \cup z$ are A-sets, so x and y belong to $(N \cap (C_x \cup z)) \cup (N \cap (C_y \cup z))$, which is a proper subcontinuum of N. Thus $z \in \{x, y\}$ and we may assume $z = x$. But now in $M - (C_t \cup z)$ if D_y is the component containing y, then $N \cap (D_y \cup z)$ is a proper subcontinuum of N and contains x and y. It follows that $N \subset A$.

2.7. COROLLARY. *If E is an E_0-set of M and $a, b \in E$, then E contains every continuum $N \subset M$ such that N is an irreducible continuum from a to b.*

2.8. COROLLARY. *If a and b are conjugate in M and $N \subset M$ is an irreducible continuum from a to b, then every point of N is conjugate to both a and b in M.*

2.9. LEMMA. *If E_1, E_2 are distinct E_0-sets and N is a connected set which meets E_1 and E_2, then $E_1 \cap E_2 \subset N$.*

2.10. LEMMA. *If A is an A-set of M and R is a component of $M - A$, then R meets at most one E_0-set E which meets A.*

Proof. Suppose E_1, E_2 are distinct E_0-sets which meet both R and A. Let $\{b\} = \partial(R)$. Then $b \in E_1 \cap E_2$. But by the above lemma, $E_1 \cap E_2 \subset R$. This is a contradiction since $b \in A$.

2.11. LEMMA. *Let E be an E_0-set of M and C a component of $M - E$. If $b \in M$ such that either $b \in E - \bar{C}$ or $b \notin E$ and $\partial(C_b) \neq \partial(C)$, where C_b is the component of $M - E$ containing b, then $E \subset C(a, b)$ for all a in C.*

Proof. If $b \in E - \bar{C}$ and $a \in C$, let $\{t\} = \partial(C)$. Then $t \neq b$ and $t \in E(a, b)$. Thus by 1.1-i, $E \cap (E(a, b) \cup \{a, b\}) = \{t, b\}$ and $E \subset C(a, b)$. Suppose, then, that $b \notin E$ and $\partial(C_b) \neq \partial(C)$. Let $z = \partial(C_b)$, and $a \in C$. Then C is a component of $M - t$ and is disjoint from $(E - t) \cup C_b$, which is connected. Thus $t \in E(a, b)$. Similarly, $z \in E(a, b)$. Thus $E \cap (E(a, b) \cup \{a, b\}) = \{t, z\}$, so $E \subset C(a, b)$.

2.12. THEOREM. *If M is locally compact and $p \in M$ such that p is neither a cut point nor an end point of M, then there is a point q in M distinct from p and conjugate to p in M.*

Proof. Suppose p is not a cut point and is not conjugate to any other point of M. Let O be any open set such that $p \in O$ and $O \neq M$. Let V be a connected open set such that $p \in V$, $\bar{V}$ is compact, and $\bar{V} \subset O$. For each $x \in \partial(V)$, let G_x be a connected open set containing x such that $p \notin \bar{G}_x$. $\{G_x : x \in \partial(V)\}$ covers the compact set $\partial(V)$, so there is a finite subcover $G_{x_1}, \ldots, G_{x_n}$. Since $M - p$ is connected and locally compact, for each $i = 1, \ldots, n - 1$, there is a continuum N_i in $M - p$ such that x_i, $x_{i+1} \in N_i$. Let $N = (M - V) \cup \bigcup_{i=1}^{n-1} N_i \cup \bigcup_{i=1}^{n} \bar{G}_{x_i}$. Then N is closed and $p \notin N$. Since $M - V \subset N$, $M - N \subset V$ and $p \in M - N$. Let C be the component of $M - N$ such that $p \in C$. Now C has a boundary point q in N. By assumption, p and q are not conjugate in M, so there is a point x of M and a separation (U, W) of $M - x$ such that $p \in U$, $q \in W$. Since $C \cup q$ is connected and contains both p and q, $x \in C$; thus $x \notin N$. Since N is connected and $q \in N$, $N \subset W$. Thus $U \subset M - N \subset V \subset O$ and $\partial(U) = \{x\}$. It follows that p is an end point of M.

The next theorem follows immediately from 2.12 and 1.1-h.

2.13. THEOREM. *If M is locally compact, then every point p of M belongs to a cyclic element of M, and if p is neither a cut point nor an end point of M, then p belongs to a unique cyclic element of M that is an E_0-set of M.*

3.0. H-sets. In [**6**, p. 72] Whyburn defined an H-set in a metric semi-locally connected continuum M to be a connected subset of M which satisfies the following condition:

(*) If $p \in H$, then there is a cyclic element E of M such that $p \in E$ and $E \subset H$.

H-sets were shown to have many of the properties of A-sets, and the closure of

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an H-set was shown to be an A-set. However, in the nonmetric setting, it may be that for some connected set H, every point is contained in a cyclic element E of M such that $E \subset H$, $\bar{H}$ is not an A-set and H fails to have several of the properties that H-sets were shown in [**6**] to possess. It is readily seen that in case M is a metric semi-locally connected continuum, the following definition is equivalent to that given in [**6**]. Again we note that many of the results in this section are generalizations of results established in [**6**].

3.1. *Definition.* A connected subset H of M is an H-set of M if and only if H satisfies one of the following conditions:

a) $H = \{p\}$ for p a cut point or an end point of M.

b) H is nondegenerate and if $a, b \in H$, then $C(a, b) \subset H$.

Remark. Since for any two points a and b of an A-set A of M, $C(a, b)$ is the intersection of all A-sets of M which contain a and b, $C(a, b) \subset A$. Thus every nondegenerate A-set is an H-set, as is any A-set which consists of a single cut point or end point of M. It follows that every cyclic element of M is an H-set of M.

3.2. THEOREM. *If H is an H-set of M and E is an E_0-set of M such that $H \cap E$ is nondegenerate or contains a non-cut point of M, then $E \subset H$ and is an E_0-set of H.*

Proof. If $H \cap E$ is nondegenerate, let s, t be distinct points of $H \cap E$. Then $E = C(s, t) \subset H$. Suppose now that $H \cap E$ contains a point p that is a non-cut point of M. Then by 2.4, p is not an end point of M, so H is nondegenerate. Let $x \in H$ such that $x \neq p$. If $x \in E$, then $E = C(p, x) \subset H$. If $x \notin E$, then there is a point $t \in E$ such that $t \in E(p, x)$ and $E \cap (E(p, x) \cup \{p, x\}) = \{p, t\}$. Thus $E \subset C(p, x) \subset H$. In either case, $E \subset H$. Further, since E is maximal in M with respect to the properties of being nondegenerate, connected, and having no cut point of itself, E is maximal in H with respect to these properties; thus E is an E_0-set of H.

3.3. COROLLARY. *Every E_0-set of an H-set of M is an E_0-set of M.*

3.4. COROLLARY. *A nondegenerate, connected subset H of M is an H-set of M if and only if whenever E is an E_0-set of M such that $H \cap E$ is nondegenerate, then $E \subset H$.*

Proof. Necessity is immediate from 3.2. Suppose then that if $E \cap H$ is nondegenerate for an E_0-set E of M, then $E \subset H$. If $a, b \in H$, since H is connected, $E(a, b) \cup \{a, b\} \subset H$. Thus if an E_0-set E meets $E(a, b) \cup \{a, b\}$ in two points, $H \cap E$ is nondegenerate, so by assumption, $E \subset H$. It follows from 1.1-g that $C(a, b) \subset H$, so H is an H-set of M.

3.5. COROLLARY. *If M is locally compact, H an H-set of M, and $p \in H$, then there is a cyclic element E of M such that $p \in E$ and $E \subset H$.*

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3.6. COROLLARY. *If H is an H-set of M, $x, y \in H$, and N is an irreducible continuum from x to y, then $N \subset H$.*

Proof. We may assume $x \neq y$. By 3.2, H satisfies a) of 2.6, so $N \subset H$.

3.7. THEOREM. *If H is an H-set of M and $H \subset H_0 \subset \bar{H}$, then H_0 is an H-set of M. Further, if M is locally compact, then every point of $\bar{H} - H$ is either a cut point or an end point of M.*

Proof. If H is degenerate, the result is immediate. Suppose then that H is nondegenerate and $x, y \in H_0$. Since $H \cup \{x, y\}$ is connected, $E(x, y) \cup \{x, y\}$ is contained in $H \cup \{x, y\}$. If an E_0-set E of M meets $E(x, y) \cup \{x, y\}$ in two points, then E must meet H in more than one point since $E \cap (H \cup \{x, y\})$ is connected. It follows from 3.2 that $E \subset H \cup \{x, y\}$, so $C(x, y) \subset H \cup \{x, y\} \subset H_0$. Thus H_0 is an H-set of M.

The proof of the second part of the theorem is similar to the proof of 6.8, p. 73 of [**6**].

3.8. THEOREM. *If H is an H-set of M and Z is a connected subset of M, then $H \cap Z$ is connected.*

Proof. Suppose $H \cap Z \neq \emptyset$ and (Z_1, Z_2) is a separation of $H \cap Z$. Let $z_i \in Z_i$, $i = 1, 2$. Then $C(z_1, z_2) \subset H$, so $(C(z_1, z_2) \cap Z_1, C(z_1, z_2) \cap Z_2)$ is a separation of the connected set $C(z_1, z_2) \cap Z$. Thus $H \cap Z$ is connected.

3.9. COROLLARY. *Every H-set in a connected and locally connected Hausdorff space is a connected and locally connected Hausdorff space.*

3.10. COROLLARY. *If H is an H-set of M and Z is a locally connected (semi-locally connected) subset of M, then $H \cap Z$ is locally connected (semi-locally connected).*

3.11. THEOREM. *If H is an H-set of M, then $\bar{H}$ is an A-set of M.*

Proof. Let C be a component of $M - \bar{H}$, and suppose p, q are distinct points of $\partial(C)$. Then $p, q \in \bar{H}$, so $H_0 = H \cup \{p, q\}$ is an H-set of M. Since $C \cup \{p, q\}$ is connected, $H_0 \cap (C \cup \{p, q\}) = \{p, q\}$ is connected. Since this is a contradiction and $\partial(C) \neq \emptyset$, $\partial(C)$ is a singleton.

The proof of the next result is similar to that of 3.11.

3.12. COROLLARY. *If H is an H-set of M and C is a component of $M - H$, then $\bar{C} \cap H$ is a singleton.*

3.13. COROLLARY. *If H is an H-set of M, C a component of $M - H$, and $b = \bar{C} \cap \bar{H}$, then $\bar{C} = C \cup b$ and $\bar{C}$ is an A-set of M.*

Proof. If C is degenerate, then $C = \{b\}$ and the result follows. If not, then $\text{Int}(C) = C - \bar{H} = C - b$, and it follows that $\partial(C) = \{b\}$. If R is a component of $M - \bar{C} = M - (C \cup b)$, then $\emptyset \neq \partial(R) \subset \partial(C) = \{b\}$.

It was proved in [**6**] that if $\mathscr{H}$ is a family of H-sets of a semi-locally connected metric continuum M, and $\bigcup\mathscr{H}$ is connected, then $\bigcup\mathscr{H}$ is an H-set of M. It is not difficult to see that this result does not hold in general. We have, however the following results.

3.14. THEOREM. *If H_1, H_2 are H-sets of M and $H_1 \cap H_2 \neq \emptyset$, then $H_1 \cup H_2$ is an H-set of M.*

Proof. Since $H_1 \cap H_2 \neq \emptyset$, $H_1 \cup H_2$ is connected. If $H_1 \cup H_2$ is degenerate, then $H_1 \cup H_2 = H_1 = H_2$ so is an H-set of M. Suppose, then, that $H_1 \cup H_2$ is nondegenerate and E is an E_0-set of M such that $E \cap (H_1 \cup H_2)$ is nondegenerate. Since $E \cap (H_1 \cup H_2)$ is connected, either $E \cap H_1$ or $E \cap H_2$ is nondegenerate, so $E \subset H_1$ or $E \subset H_2$. The theorem now follows from 3.4.

3.15. COROLLARY. *The union of two intersecting A-sets of M is an A-set of M.*

3.16. THEOREM. *If $\mathscr{H}$ is a family of H-sets of M such that for every two members A, B of $\mathscr{H}$, there is a finite collection $A = H_0, H_1, \ldots, H_n = B$ such that $H_i \cap H_{i+1} \neq \emptyset$, $i = 0, \ldots, n-1$, then $\bigcup\mathscr{H}$ is an H-set of M.*

Proof. It is well known that under the hypotheses of the theorem $\bigcup\mathscr{H}$ is connected, (see, for instance, [**2**, p. 60]). If $\bigcup\mathscr{H}$ is degenerate, the result is immediate, so assume that $\bigcup\mathscr{H}$ is nondegenerate. If $x, y \in \bigcup\mathscr{H}$, and $A, B \in \mathscr{H}$ such that $x \in A$, $y \in B$, let $H_0, H_1, \ldots, H_n$ be members of $\mathscr{H}$ such that $A = H_0$, $B = H_n$, and $H_i \cap H_{i+1} \neq \emptyset$, $i = 0, 1, \ldots, n-1$. It follows from 3.14 that $\bigcup_{i=1}^n H_i$ is an H-set of M, so $C(x, y) \subset \bigcup_{i=1}^n H_i \subset \bigcup\mathscr{H}$.

3.17. THEOREM. *If $\mathscr{H}$ is a family of H-sets of M and $\bigcap\mathscr{H}$ is nondegenerate or consists of a single cut point or end point of M, then $\bigcap\mathscr{H}$ is an H-set of M. If M is locally compact, then every intersection of H-sets of M is an H-set of M.*

Proof. If $\bigcap\mathscr{H}$ is a cut point or an end point of M then $\bigcap\mathscr{H}$ is an H-set of M. Suppose that $\bigcap\mathscr{H}$ is nondegenerate. Let $a \in \bigcap\mathscr{H}$, and let $x \in \bigcap\mathscr{H}$ such that $a \neq x$. Then for each H in $\mathscr{H}$, $C(a, x) \subset H$, so $C(a, x) \subset \bigcap\mathscr{H}$. It follows that $\bigcap\mathscr{H}$ is connected. Similarly, if $x, y \in \bigcap\mathscr{H}$, then $C(x, y) \subset \bigcap\mathscr{H}$, so $\bigcap\mathscr{H}$ is an H-set of M.

Now if M is locally compact and $p \in \bigcap\mathscr{H}$ such that p is neither a cut point nor an end point of M, then p belongs to an E_0-set E of M and $p \in E \cap H$ for all H in $\mathscr{H}$. Thus for each H in $\mathscr{H}$, $E \subset H$, so $E \subset \bigcap\mathscr{H}$ and the result now follows from the first part.

3.18. THEOREM. *If H is an H-set of M, $x, y \in H$ and $X \subset M$ such that x and y lie in distinct components of $H - X$, then x and y lie in distinct components of $M - X$.*

Proof. If not, then x and y lie in a component C of $M - X$. Then $C \cap H$ is connected, contains x and y and is contained in $H - X$; so x and y lie in the same component of $H - X$. This is a contradiction.

3.19. COROLLARY. *Every cut point of an H-set of M is a cut point of M.*

3.20. COROLLARY. *If H is an H-set of M, $A, B \subset H$, and X a closed subset of M such that $X \cap H$ separates A and B in H, then X separates A and B in M.*

3.21. COROLLARY. *If M is locally compact, H an H-set of M, and E a cyclic element of H, then E is a cyclic element of M.*

Proof. If E is an E_0-set or a singleton cut point of H, then the result follows from 3.3 or 3.19. Suppose, then, $E = \{p\}$, p an end point of H. If p is neither a cut point nor an end point of M, then p belongs to an E_0-set E^* of M. By 3.2, E^* is an E_0-set of H. But then p is an end point of H belonging to an E_0-set of H and this is a contradiction. Thus p is either a cut point or an end point of M so E is a cyclic element of M.

3.22. COROLLARY. *Let H be an H-set of M. Then every nondegenerate H-set H^* of H is an H-set of M. If M is locally compact, then every H-set of H is an H-set of M.*

Proof. If H^* is nondegenerate, then so is H. If E is an E_0-set of M such that $E \cap H^*$ is nondegenerate, then $E \cap H$ is nondegenerate, so $E \subset H$ and E is an E_0-set of H. Thus by 3.9 and 3.4, $E \subset H^*$. It follows that H^* is an H-set of M. Now if M is locally compact and H^* is degenerate, then by 3.21, H^* is a cut point or an end point of M.

3.23. COROLLARY. *If A is an A-set of M and B is an A-set of A, then B is an A-set of M.*

3.24. COROLLARY. *If $a, b \in M$, then $C(a, b)$ contains no proper A-set of itself which contains both a and b; i.e., the cyclic chain in $C(a, b)$ from a to b is $C(a, b)$. Further, if t is a cut point of $C(a, b)$, then $t \in E(a, b)$.*

3.25. THEOREM. *If H is an H-set of M, and Z is any connected and locally connected subset of M such that $H \cap Z$ is nondegenerate, then $H \cap Z$ is an H-set of Z.*

Proof. Let E be an E_0-set of Z such that $E \cap H \cap Z$ is nondegenerate. Then $E \subset E^*$ for some E_0-set E^* of M. Then $E^* \subset H$, so $E \subset E^* \cap Z \subset H \cap Z$. Thus $H \cap Z$ is an H-set of Z.

3.26. COROLLARY. *If A is an A-set of M and Z is a connected and locally connected subset of M such that $A \cap Z \neq \emptyset$, then $A \cap Z$ is an A-set of Z.*

3.27. THEOREM. *If $a, b \in M$, then a and b are non-cut points of $C(a, b)$, and if a and b are not conjugate in M, then $C(a, b) - a - b$ is connected.*

Proof. Let D be the component of $C(a, b) - a$ such that $b \in D$. Then $\bar{D} = D \cup a$ is an A-set of $C(a, b)$ and contains both a and b. Thus $D \cup a = C(a, b)$, so $C(a, b) - a = D$, which is connected. Similarly, b is a non-cut point of $C(a, b)$.

Suppose now that a and b are not conjugate and that (U, V) is a separation of $C(a, b) - a - b$. Since $C(a, b) - a$ is connected, $\{a, b\}$ is an irreducible closed cutting of $C(a, b)$ and therefore $U \cup \{a, b\}$ is connected. Thus $E(a, b) \cup$

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$\{a, b\} \subset U \cup \{a, b\}$. Now no E_0-set contains both a and b, so if some E_0-set E meets $E(a, b) \cup \{a, b\}$ in two points, it meets U. Since $E - \{a, b\}$ is connected, $E \subset U \cup \{a, b\}$. It follows that $C(a, b) \subset U \cup \{a, b\}$, so $V = \emptyset$. Thus $C(a, b) - a - b$ is connected.

3.28. THEOREM. *Let A be a closed, connected subset of M. Then among the following statements,* a)–c) *are equivalent and* c) *implies* d); *if M is locally compact, then* a)–d) *are equivalent and* a) *implies* e).

a) *A is an A-set of M.*

b) *If C is a component of $M - A$, then $\bar{C} \cap A$ is a singleton.*

c) *If E is a cyclic element of M and $A \cap E$ is nondegenerate, then $E \subset A$.*

d) *If $a, b \in A$, and N is an irreducible continuum from a to b, then $N \subset A$.*

e) *If $p \in A$, then either $p = A$ or there is a cyclic element E of M such that $p \in E \subset A$.*

Proof. That a) and b) are equivalent was stated in [**7**], and that a) implies c) follows from 3.2 since every nondegenerate A-set of M is an H-set of M. Suppose that A satisfies c). If A is degenerate, then A is an A-set of M. If not, then by 3.4, A is an H-set and therefore an A-set since A is closed. Thus c) implies a). That c) implies d) is 2.6.

Now assume that M is locally compact. We show that d) implies b). Let C be a component of $M - A$ and suppose $\partial(C)$ contains two points p and q. Since A is closed, C is a connected, locally connected, and locally compact Hausdorff space. Let R_p, R_q be disjoint open sets containing p and q, respectively, such that $\bar{R}_p$ and $\bar{R}_q$ are disjoint continua, and let x, y be points of $R_p \cap C$ and $R_q \cap C$ respectively. Let $N_{p,x}$ be an irreducible continuum in $\bar{R}_p$ from p to x; $N_{q,y}$ be an irreducible continuum in $\bar{R}_q$ from q to y, and $N_{x,y}$ an irreducible continuum in C from x to y. Then $N_{p,x} \cup N_{x,y} \cup N_{q,y}$ is a continuum containing p and q so contains an irreducible continuum N from p to q. By d), $N \subset A$. But this is impossible since then $N \subset N_{p,x} \cup N_{q,y}$, and these are disjoint closed sets each of which meets N. Thus $\bar{C} \cap A$ contains at most one point, and since M is connected, $\bar{C} \cap A$ is a singleton.

It remains to show that if M is locally compact, a) implies e). If A is nondegenerate, and p is not a cyclic element, then p belongs to an E_0-set E and the result follows from 3.2.

Remark. It was shown in [**6**] that if M is a locally connected metric continuum, and A is a subcontinuum of M, then all five statements a)–e) are equivalent. We note that this is not true in general.

4. Nodal sets, nodes and cyclic chains.

4.1. *Definition.* A closed subset N of a space S is called a *nodal set of S* if and only if $\partial(N)$ is at most a singleton.

The next result follows easily from Definition 4.1.

4.2. LEMMA. *Let S be a T_1 topological space and $N \subset S$. Then*

a) *every singleton is a nodal set of* S *as are* $\emptyset$ *and* S;

b) *if* N *is a nodal set of* S, *then* $\overline{S - N}$ *is a nodal set of* S;

c) *if* S *is connected and locally connected, and* N *is a nonempty nodal set of* S, *then* N *is connected;*

d) *if* S *is connected,* $p \in S$, *and* (S_1, S_2) *is a separation of* $S - p$, *then* $S_1 \cup p$ *and* $S_2 \cup p$ *are nodal sets of* S;

e) *if* S *is connected and locally connected, then if* N *is a nodal set of* S, N *is an* A*-set of* S;

f) *if* $A \subset S$, *and* N *is a nodal subset of* S, *then* $N \cap A$ *is a nodal set of* A.

4.3. *Definition.* A subset N of a connected space S is called a *node* of S if and only if either $N = \{p\}$ for some end point p of S or N is an E_0-set of S such that N is a nodal set of S.

Remark. It is immediate from the definition that if a connected T_1-space S has no cut point, then S is a node of itself and the only nodal subsets of S are $\emptyset$, S, and the singletons of S.

The proofs of the next two results are easy.

4.4. THEOREM. *Let* N *be a nondegenerate subset of* M. *If* $N = M$, *then* N *is a node of* M *if and only if* M *has no cut point. If* $N \neq M$, *then* N *is a node of* M *if and only if* N *is an* E_0*-set of* M *and* N *contains exactly one cut point of* M.

4.5. COROLLARY. *Every node* N *of* M *contains a non-cut point of* M, *and if* N *is nondegenerate, then every point of* N *distinct from the one boundary point of* N *is neither a cut point of* M *nor an end point of* M.

4.6. LEMMA. *If* N *is a nondegenerate nodal subset of* M, *then either* N *contains a cut point of itself or* N *is an* E_0*-set and therefore a node of* M.

Proof. Suppose N contains no cut point of itself. If $N = M$, then M contains no cut point, so N is a node of M. If $N \neq M$, there is a point p of M such that $\partial(N) = \{p\}$, and $(\operatorname{Int} N, \operatorname{Ext} N)$ is a separation of $M - p$. It follows that if $t \in M - N$, then p separates t and $N - p$; thus N is maximal with respect to being nondegenerate, connected, and having no cut point of itself. Thus N is a node of M.

4.7. THEOREM. *If* N_1, N_2 *are distinct, intersecting nodes of* M, *then neither is degenerate and their intersection is a cut point of* M.

Proof. If $N_1 = \{p\}$, then p is an end point of M and $p \in N_2$. But this implies that $N_2 = \{p\} = N_1$. It follows that neither N_1 nor N_2 is degenerate and each is an E_0-set of M. By 1.1-h, $N_1 \cap N_2$ is a cut point of M.

4.8. THEOREM. *If* x *is a non-cut point of* M *belonging to a node* N *of* M, *then* N *is a node of every* H*-set of* M *containing* x.

Proof. Let H be an H-set of M containing x. If $N = \{x\}$, then x is an end

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point of M. If $H = \{x\}$, then $N = H$ and N is a node of H. If H is nondegenerate, then by 2.1, x is an end point of H, so again N is a node of H. If N is nondegenerate, then N is an E_0-set of M and the theorem follows from 3.2 and 4.2.

4.9. THEOREM. *If H is an H-set of M and C is a component of $M - H$, then $\bar{C}$ is a nodal A-set.*

Proof. By 3.12 and 3.13, $\bar{C}$ is an A-set and $\partial(\bar{C}) = \partial(C) = \bar{C} \cap \bar{H}$ is a singleton.

4.10. THEOREM. *Let N be a node of M and $C(x, y)$ a cyclic chain in M. If $N \cap C(x, y)$ contains a non-cut point of M, then one of x and y is a non-cut point of M that belongs to N.*

Proof. By 4.8, $N \subset C(x, y)$. If $N = \{p\}$, then p is an end point of M, so $p \notin E(x, y)$ and p belongs to no E_0-set of M. It follows from 1.1-g that $p \in \{x, y\}$. Suppose N is nondegenerate and $p \notin \{x, y\}$. Then N meets $E(x, y) \cup \{x, y\}$ in two points. But N contains at most one cut point of M, so $N \cap \{x, y\}$ contains at least one non-cut point of M.

4.11. THEOREM. *If a, b are non-cut points of M which belong to distinct nodes of M, then $C(a, b)$ is a maximal cyclic chain of M; that is, if $C(a, b) \subset C(x, y)$, then $C(a, b) = C(x, y)$.*

Proof. Suppose $C(a, b) \subset C(x, y)$ and N_a, N_b are distinct nodes of M containing a and b respectively. By 4.8, $N_a \cup N_b \subset C(a, b)$. By 4.10, $x, y \in N_a \cup N_b$ and it follows that $C(x, y) \subset C(a, b)$.

The next result follows from the proof of Theorem 4.11.

4.12. COROLLARY. *If a and b are non-cut points of M which belong to distinct nodes N_a and N_b, respectively, of M and $C(a, b) = C(x, y)$, then x and y are non-cut points of M and each of N_a, N_b contains one of the points x, y and not both.*

4.13. THEOREM. *If $C(a, b)$ is a cyclic chain in M and N is a node of $C(a, b)$, then $a \in N$ or $b \in N$.*

Proof. It follows from 3.24 that in $C(a, b)$, if $C^*(a, b)$ is the cyclic chain from a to b, then $C^*(a, b) = C(a, b)$. Now N is a node of $C(a, b)$ such that $N \cap C^*(a, b)$ contains a non-cut point of $C(a, b)$, so by 4.10, $a \in N$ or $b \in N$.

4.14. THEOREM. *If $C(a, b)$ is a cyclic chain in M, then $C(a, b)$ contains at most two nodes of itself. Also, if $C(a, b)$ has two nodes, then $E(a, b) \neq \emptyset$; and if M is locally compact and $E(a, b) \neq \emptyset$, then $C(a, b)$ has two nodes.*

Proof. Suppose that $C(a, b)$ has three nodes, N_1, N_2, and N_3. By 3.27, a and b are non-cut points of $C(a, b)$, and by 4.13, either a or b must lie in two of the sets N_1, N_2, N_3; but this contradicts 4.7.

Now suppose that $C(a, b)$ has two nodes, N_1 and N_2. Then by 4.13, we may assume that $a \in N_1$. Since a is not a cut point of $C(a, b)$, $a \notin N_2$, so $b \in N_2$. Then either $N_2 = \{b\}$, (so b is an end point of $C(a, b)$), or in $C(a, b)$, $b \in \text{Int}(N_2)$. In either case $E(a, b) \neq \emptyset$.

If M is locally compact, then $C(a, b)$ is locally compact. Thus if neither a nor b belongs to an E_0-set of $C(a, b)$ then by 2.14, each is an end point of $C(a, b)$, so $\{a\}$ and $\{b\}$ are distinct nodes of $C(a, b)$. If a belongs to an E_0-set E_1 of $C(a, b)$, then E_1 is an E_0-set of M which meets $E(a, b) \cup \{a, b\}$ in exactly two points, one of which is a. Thus if $E(a, b) \neq \emptyset$, $b \notin E_1$, so E_1 meets $E(a, b)$ in exactly one point. By 3.24, the set of cut points of $C(a, b)$ is identical with $E(a, b)$ and it follows that E_1 is an E_0-set of $C(a, b)$ containing exactly one cut point of $C(a, b)$. Thus by 4.4, E_1 is a node of $C(a, b)$. Now if b is an end point of $C(a, b)$, then $N_2 = \{b\}$ is a node of $C(a, b)$ distinct from E_1. If not, then as in the case for a, b belongs to an E_0-set E_2 of $C(a, b)$ and E_2 is a node of $C(a, b)$ distinct from E_1.

Thus far we have not demonstrated the existence of nodes in a connected and locally connected Hausdorff space M. The next theorem assures us of the existence of nodes in the case that M is a locally connected Hausdorff continuum.

4.15. THEOREM. *If M is compact, then every nondegenerate nodal subset of M contains a node of M.*

Proof. Let N be a nondegenerate nodal subset of M. If M has no cut point, then $N = M$, and N is a node. Assume that M has a cut point. Then M is not a node and we may assume that $N \neq M$. Assume, further, that N contains no nondegenerate node of M. We show that in this case N contains an end point of M.

Let $p \in M$ such that $\partial(N) = \{p\}$. Let $\mathscr{P} = \{(x, C) : C$ is a component of $M - x$ and $\bar{C} \subset \text{Int}(N)\}$. Now, $(\text{Ext}(N), \text{Int}(N))$ is a separation of $M - p$, so there is a component D of $M - p$ such that $D \subset \text{Int}(N)$. $\bar{D} = D \cup p$ is a nondegenerate nodal subset of M and $\bar{D} \subset N$. By assumption, $\bar{D}$ is not a node of M, so $\bar{D}$ has a cut point x. Since p is not a cut point of $\bar{D}$, $x \neq p$. Since $\bar{D}$ is an A-set in M, x is a cut point of M. Let (U, V) be a separation of $M - x$ such that $p \in U$. Since $(M - N) \cup p$ is connected and contained in $M - x$, $(M - N) \cup p \subset U$. Thus $V \subset \text{Int}(N)$. Let C be a component of $M - x$ such that $C \subset V$. Then $(x, C) \in \mathscr{P}$, so $\mathscr{P} \neq \emptyset$.

Define a relation "$>$" on $\mathscr{P}$ by $(x_1, C_1) > (x_2, C_2)$ if and only if $\bar{C}_1 \subset C_2$. Let $(x_1, C_1) \in \mathscr{P}$. Then $\bar{C}_1 = C_1 \cup x_1$ is a nodal subset of M and $\bar{C}_1 \subset \text{Int}(N)$. By assumption, $\bar{C}_1$ is not a node of M, and it follows as in the above paragraph that there is a cut point x_2 of M and a component C_2 of $M - x_2$ such that $x_2 \in C_1$ and $\bar{C}_2 \subset C_1$. Then $(x_2, C_2) \in \mathscr{P}$, and $(x_2, C_2) > (x_1, C_1)$. Thus the relation $>$ is nonempty, and it is not difficult to show that $>$ is a partial order. Further, we have shown that $(\mathscr{P}, >)$ has no maximal element.

Let $\mathscr{M}$ be a maximal chain in $(\mathscr{P}, >)$. Since $(\mathscr{P}, >)$ has no maximal element, $\mathscr{M}$ has no maximum element. Let $C^* = \bigcap \{\bar{C} : \text{for some } x \in M, (x, C) \in \mathscr{M}\}$. Since $\{\bar{C} : \text{for some } x \in M, (x, C) \in \mathscr{M}\}$ is simply ordered by inclusion and M is compact, C^* is nonempty and connected. Also, $C^* = \bigcap \{C : \text{for some } x \in M, (x, C) \in \mathscr{M}\}$; for if $t \in C^*$ and $(x_1, C_1) \in \mathscr{M}$, then since (x_1, C_1) is not maximum in $\mathscr{M}$, there is a member (x_2, C_2) of $\mathscr{M}$ such that $(x_2, C_2) > (x_1, C_1)$ and $t \in \bar{C}_2 \subset C_1$. Thus $C^* \subset \bigcap \{C : \text{for some } x \in M, (x, C) \in \mathscr{M}\} \subset \bigcap \{\bar{C} : \text{for some } x \in M, (x, C) \in \mathscr{M}\} = C^*$. Suppose C^* contains a cut point t of M. Let (U, V) be a separation of $M - t$ such that $p \in U$. If $(x_1, C_1) \in \mathscr{M}$, then $M - C_1$ is connected, and $p \in M - C_1$, so $M - C_1 \subset U$ and $V \subset C_1$. Thus $V \subset C^*$. Let D^* be a component of $M - t$ contained in V and consider the pair (t, D^*) of $\mathscr{P}$. If $(x, c) \in \mathscr{M}$ then $\bar{D}^* = D^* \cup t \subset C$, $(t, D^*) > (x, C)$. But then $(t, D^*) \in \mathscr{M}$ and is a maximum element of $\mathscr{M}$, and this is a contradiction. Thus no point of C^* is a cut point of M.

Since C^* is an intersection of A-sets of M, C^* is an A-set of M. Then every cut point of C^* is a cut point of M, so C^* has no cut point of itself. If C^* is either nondegenerate or contains a point p which is neither a cut point nor an end point of M, then for some E_0-set E of M, $C^* \subset E$ or $p \in E$. In either case, $E = C^*$. But by assumption, E is not a node of M so contains cut points of M and this is a contradiction. Thus C^* consists of a single end point of M.

4.16. COROLLARY. *If M is compact and has a cut point, then M has at least two nodes.*

4.17. COROLLARY. *If M is compact, H is an H-set of M, and C is a component of $M - H$, then C contains a point a that is a non-cut point of M belonging to a node of M.*

Proof. Let $b = \bar{C} \cap \bar{H}$. By 4.9, $C \cup b$ is a nodal set, so by 4.15, $C \cup b$ contains a node N of M. By 4.5, N contains a non-cut point a of M and since b is a cut point of M, $a \in C$.

4.18. THEOREM. *If M is compact, then every point of M belongs to a cyclic chain $C(a, b)$ of M where a and b are non-cut points of M which belong to nodes of M, and if M has a cut point, then a and b can be chosen to belong to distinct nodes of M.*

Proof. If M has no cut point, the result is immediate; so assume that M has a cut point and let $x \in M$.

If x belongs to a node N_1 of M, let a be any non-cut point of M belonging to N_1. Then there is a node N_2 of M distinct from N_1. Let b be any non-cut point of M belonging to N_2. Then $a \neq b$ and $N_1 \subset C(a, b)$, so $x \in C(a, b)$.

Suppose now that x belongs to no node of M. We consider two cases.

Case 1. x is a cut point of M. Let (U, V) be a separation of $M - x$. Then $U \cup x$ and $V \cup x$ are nodal subsets of M and by 4.15 contain nodes N_1 and

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N_2 respectively. By 4.5, N_1 and N_2 contain points a and b, respectively, such that a and b are non-cut points of M. Then $a \neq b$ and $a \in C(a, b) \cap U$, $b \in C(a, b) \cap V$. Since $C(a, b)$ is connected, $x \in C(a, b)$.

Case 2. x is not a cut point of M. Then since x belongs to no node of M, x is not an end point of M. Let E be the unique E_0-set containing x. Then E is not a node, so E contains two distinct cut points of M, x_1 and x_2. For each $i = 1, 2$, in $M - x_i$ let C_i be a component which does not contain $E - x_i$. Then C_1, C_2 are distinct components of $M - E$ and $\partial(C_1) \neq \partial(C_2)$. Now for each $i = 1, 2$, $C_i \cup x_i$ is a nodal subset of M so contains a node N_i. Let a and b be non-cut points of M belonging to N_1 and N_2, respectively. Then by 2.11, $E \subset C(a, b)$ so $x \in C(a, b)$.

4.19. THEOREM. *If M is compact and H is an H-set of M, C a component of $M - H$ and $\bar{C} \cap \bar{H} = \{b\}$; then if $x \in C$, there is a non-cut point a of M such that $a \in C$ and belongs to a node of M and $x \in C(a, b) \subset C \cup b$.*

Proof. If x belongs to a node N of M, then there is a non-cut point a of M such that $a \in N$. By 4.8, $N \subset C(a, b) \subset C \cup b$. Suppose, then, that x belongs to no node of M. Again we consider two cases.

Case 1. x is a cut point of M. Let (U, V) be a separation of $M - x$ such that $b \in U$. Then $H \subset U$. Since $V \cup x$ is connected and contained in $M - H$, $V \cup x \subset C$. Let a be a non-cut point of M belonging to $V \cup x$. Then $C(a, b)$ meets both U and V, so $x \in C(a, b)$. Since $a, b \in C \cup b$, $C(a, b) \subset C \cup b$.

Case 2. x is not a cut point of M. Since x belongs to no node of M, x is not an end point of M. Now x belongs to an E_0-set E of M. Then $E \subset C \cup b$ and E contains two distinct cut points of M. If $b \in E$, let t be a cut point of M distinct from b. If $b \notin E$, let C_b be the component of $M - E$ containing b and let t be a cut point of M in E such that $t \neq \partial(C_b)$. Let D be a component of $M - t$ such that $b \notin D$. Then $D \subset C$ and D contains a non-cut point a of M belonging to a node of M. Now $C(a, b) \subset C \cup b$. Further, $E \subset C(a, b)$ so $x \in C(a, b)$.

4.20. COROLLARY. *If M is compact, A an A-set of M, C a component of $M - A$ and $b \neq \partial(C)$, then if $x \in C$, there is a non-cut point a of M belonging to a node of M such that $x \in C(a, b) \subset C \cup b$.*

5.0. Null families. The following definition is to be found in Wilder [**8**, p. 106].

Definition. If $\mathscr{G}$ is a covering of a space S, then a point set E of S is said to be of *diameter* $< \mathscr{G}$ if some element of $\mathscr{G}$ contains E.

5.1. *Definition.* Let $\mathscr{F}$ be a family of subsets of a topological space S. Then $\mathscr{F}$ is called a *null family* if and only if for every open cover $\mathscr{G}$ of S, all but a finite number of members of $\mathscr{F}$ have diameter $< \mathscr{G}$.

The next two theorems are easy consequences of Definition 5.1.

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5.2. THEOREM. *Every subfamily of a null family of a topological space S is a null family.*

5.3. THEOREM. *If $\mathscr{F}$ is a null family in a regular space S and $\overline{\mathscr{F}} = \{\bar{F} : F \in \mathscr{F}\}$, then $\overline{\mathscr{F}}$ is a null family.*

Remark. In the proof of the next lemma we make use of the following result which is found in [**3**]: A space S is locally connected if and only if whenever $X \subset S$ and C is a component of $S - X$, then $\operatorname{Int}(C) = C - \bar{X}$.

5.4. LEMMA. *Let H be an H-set of M, $\mathscr{C}$ a collection of components of $M - H$, p a limit point of $\bigcup \mathscr{C}$ such that $p \notin \bigcup \mathscr{C}$ and $\{p_\alpha : \alpha \in (\mathscr{A}, \geqq)\}$ a net in $\bigcup \mathscr{C}$ converging to p. Then $p \in \bar{H}$, and if for each $\alpha \in \mathscr{A}$, $C_\alpha \in \mathscr{C}$ such that $p_\alpha \in C_\alpha$ and $b_\alpha = \bar{C}_\alpha \cap \bar{H}$, then $b_\alpha \rightarrow p$. Further, if M is locally compact, $p \notin \bigcup \{\bar{C} : C \in \mathscr{C}\}$, and $\{q_\alpha : \alpha \in (\mathscr{A}, \geqq)\}$ is a net such that $q_\alpha \in \bar{C}_\alpha$ for each $\alpha \in \mathscr{A}$, then $q_\alpha \rightarrow p$.*

Proof. If $p \notin \bar{H}$, there is a component C of $M - H$ such that $p \in C - \bar{H} = \operatorname{Int}(C)$ and C meets no member of $\mathscr{C}$. Since this is a contradiction, $p \in \bar{H}$.

Now let O be any open set such that $p \in O$, and let V be a connected open set such that $p \in V$ and $V \subset O$. There is an $\alpha^* \in \mathscr{A}$ such that if $\alpha \geqq \alpha^*$, $p_\alpha \in V$. If $\alpha \geqq \alpha^*$, then since $p \notin C_\alpha$, V meets C_α and $M - C_\alpha$ so V meets $\partial(C_\alpha) = b_\alpha$. Thus $b_\alpha \rightarrow p$.

Assume now that M is locally compact, that if $C \in \mathscr{C}$, then $p \notin \bar{C}$, and that $\{q_\alpha : \alpha \in (\mathscr{A}, \geqq)\}$ is a net such that for each $\alpha \in \mathscr{A}$, $q_\alpha \in \bar{C}_\alpha$. Suppose $q_\alpha \nrightarrow p$. Then there is an open set V such that $p \in V$ and the net $\{q_\alpha : \alpha \in (\mathscr{A}, \geqq)\}$ is frequently in $M - V$. Further, we may assume that $\bar{V}$ is compact.

Let $\alpha^* \in \mathscr{A}$ such that if $\alpha \geqq \alpha^*$ then $p_\alpha, b_\alpha \in V$. Let $\mathscr{B} = \{\alpha \in \mathscr{A} : \alpha \geqq \alpha^*$ and $q_\alpha \notin V\}$. By definition of V, $\mathscr{B}$ is a cofinal subset of $\mathscr{A}$ and for each $\alpha \in \mathscr{B}$, $C_\alpha \not\subset V$. Now $\{C_\alpha : \alpha \in \mathscr{B}\}$ must be infinite; for otherwise, $p \in \overline{\bigcup_{\alpha \in \mathscr{B}} C_\alpha} = \bigcup_{\alpha \in \mathscr{B}} \bar{C}_\alpha$, so for some $\alpha \in \mathscr{B}$, $p \in \bar{C}_\alpha$ and this is a contradiction.

Now $\{p_\alpha : \alpha \in (\mathscr{B}, \geqq)\}$ and $\{b_\alpha : \alpha \in (\mathscr{B}, \geqq)\}$ are subnets, respectively, of $\{p_\alpha : \alpha \in (\mathscr{A}, \geqq)\}$ and $\{b_\alpha : \alpha \in (\mathscr{A}, \geqq)\}$, so each converges to p. For each $\alpha \in \mathscr{B}$, since $C_\alpha \not\subset V$ and $p_\alpha \in C_\alpha \cap V$, there is a point $y_\alpha \in C_\alpha \cap \partial(V)$, and since for each $\alpha \in \mathscr{B}$, $\bar{C}_\alpha \cap \bar{H} = b_\alpha \in V$, $y_\alpha \notin \bar{H}$. Since $\partial(V)$ is compact, the net $\{y_\alpha : \alpha \in (\mathscr{B}, \geqq)\}$ has a convergent subnet $y_{\alpha_\beta} \rightarrow y$, $y \in \partial(V)$. Then y is a limit point of $\bigcup_{\alpha \in \mathscr{B}} C_\alpha$. Now if $y \in C_{\bar{\alpha}}$ for some $\bar{\alpha} \in \mathscr{B}$, then $y \in C_{\bar{\alpha}} - \bar{H} = \operatorname{Int} C_{\bar{\alpha}}$. Since $V - \bar{C}_{\bar{\alpha}}$ is open, $p \in V - \bar{C}_{\bar{\alpha}}$, and the net $\{b_\alpha : \alpha \in (\mathscr{B}, \geqq)\}$ converges to p; for some $\gamma^* \in \mathscr{B}$ if $\gamma \geqq \gamma^*$, $\gamma \in \mathscr{B}$, then $C_\alpha \neq C_{\bar{\alpha}}$. But there is a $\partial^* \in \mathscr{B}$ such that $\partial^* \geqq \gamma^*$ and $y_{\partial^*} \in C_{\bar{\alpha}}$, so $C_{\alpha^*} = C_{\bar{\alpha}}$. Since this is a contradiction, $y \notin \bigcup_{\alpha \in \mathscr{B}} C_\alpha$. It follows from the first part of this proof that $y \in \bar{H}$ and $b_{\alpha_\beta} \rightarrow y$. But $b_{\alpha_\beta} \rightarrow p$. The lemma follows.

5.5. THEOREM. *Let M be locally compact, H an H-set of M, $p \in \bar{H}$, and V an open set containing p. Let $\mathscr{C}_p = \{C : C$ is a component of $M - H$ and $\bar{C} \cap \bar{H} = p\}$. Then all but a finite number of members of $\mathscr{C}_p$ are contained in V.*

Proof. Let $\mathscr{C}_p^* = \{C \in \mathscr{C}_p : C \not\subset V\}$ and suppose $\mathscr{C}_p^*$ is infinite. Let G be an open set containing p such that $\bar{G} \subset V$ and $\bar{G}$ is compact. Then for each C in $\mathscr{C}_p^*$, there is a point $y_C \in C \cap \partial(G)$ and $\{y_C : C \in \mathscr{C}_p^*\}$ is infinite. There is a point y in $\partial(G)$ such that y is a limit point of $\{y_C : C \in \mathscr{C}_p^*\}$. Then y is a limit point of $\bigcup C_p^*$ and $y \notin \bigcup \mathscr{C}_p^*$. It follows from 5.4 that y is a limit point of $\bigcup \{\bar{C} \cap \bar{H} : C \in \mathscr{C}_p^*\} = \{p\}$, so $y = p$. This is a contradiction.

5.6. COROLLARY. *If M is locally compact, H an H-set of M and $\mathscr{F}$ is any collection of components of $M - H$ with a common boundary point, then $\mathscr{F}$ is a null family.*

5.7. COROLLARY. *If M is locally compact and $\mathscr{E}$ is any collection of E_0-sets of M such that $\bigcap \mathscr{E} \neq \emptyset$, then $\mathscr{E}$ is a null family.*

Proof. Since $\bigcap \mathscr{E} \neq \emptyset$, there is a point $p \in M$ such that $\bigcap \mathscr{E} = \{p\}$. Let $E^* \in \mathscr{E}$. For each E in $\mathscr{E}$ such that $E \neq E^*$, $E - E^* = E - p$ is connected so is contained in a component C_E of $M - E^*$. Further, p is a boundary point of C_E for each E in $\mathscr{E}$, $E \neq E^*$. By 5.6, $\{C_E : E \in \mathscr{E}, E \neq E^*\}$ is a null family. It follows that $\mathscr{E}$ is a null family.

5.8. THEOREM. *If M is compact, H an H-set of M, and $\mathscr{F} = \{C : C$ is a component of $M - H\}$, then $\mathscr{F}$ is a null family.*

Proof. Suppose not. Then there is an open cover $\mathscr{G}$ of M and an infinite collection $\mathscr{F}' \subset \mathscr{F}$ such that no member of $\mathscr{F}'$ is contained in a member of $\mathscr{G}$. Then for each $C \in \mathscr{F}'$, C is nondegenerate and $\bar{C} \cap \bar{H}$ is degenerate, so there is a point $p_C \in C - \bar{H}$. $\{p_C : C \in \mathscr{F}'\}$ is infinite and M is compact, so for some $p \in M$, p is a limit point of $\{p_C : C \in \mathscr{F}'\}$.

If $p \notin \bar{H}$, then p belongs to $C - \bar{H} = \mathrm{Int}(C)$ for some component C of $M - H$. But then $\mathrm{Int}(C)$ is an open set containing p and meeting $\{p_C : C \in \mathscr{F}'\}$ in at most one point and this is a contradiction. Thus $p \in \bar{H}$.

Since no member of $\mathscr{F}'$ is contained in a member of $\mathscr{G}$ and $\mathscr{G}$ is an open cover of M, it follows from 5.5 that only a finite number of members of $\mathscr{F}'$ have p as a boundary point. Thus we may assume that for each $C \in \mathscr{F}'$, $p \notin \bar{C}$. Let $\{p_\alpha : \alpha \in (\mathscr{A}, \geqq)\}$ be a net in $\{p_C : C \in \mathscr{F}'\}$ such that $p_\alpha \rightarrow p$. For each $\alpha \in (\mathscr{A}, \geqq)$, let $C_\alpha \in \mathscr{F}'$ such that $p_\alpha \in C_\alpha$. Let $G \in \mathscr{G}$ such that $p \in G$. Since for each $\alpha \in \mathscr{A}$, $C_\alpha \not\subset G$, there is a point $q_\alpha \in C_\alpha - G$. Then $M, H, \mathscr{F}', p$, $\{p_\alpha : \alpha \in (\mathscr{A}, \geqq)\}$, and $\{\mathrm{q}_\alpha : \alpha \in (\mathscr{A}, \geqq)\}$ satisfy the conditions of Lemma 5.4, so $q_\alpha \rightarrow p$ and therefore $p \in M - G$. Since this is a contradiction, the theorem follows.

5.9. COROLLARY. *If M is compact and A is an A-set of M, then $\mathscr{F} = \{C : C$ is a component of $M - A\}$ is a null family.*

5.10. *Definitions.* A nondegenerate continuum K in a topological space S is a *continuum of convergence* if and only if there is a net $\{K_\alpha : \alpha \in (\mathscr{A}, \geqq)\}$ of continua such that for each $\alpha \in \mathscr{A}$, $K \cap K_\alpha = \emptyset$ and $K = \lim_\alpha K_\alpha$. A net of sets $\{K_\alpha : \alpha \in (\mathscr{A}, \geqq)\}$ is *almost distinct* (*almost pairwise disjoint*) if and only if

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for each $\alpha \in \mathscr{A}$ there is a $\beta \in \mathscr{A}$ such that if $\gamma \in \mathscr{A}$ and $\gamma \geqq \beta$ then $K_\alpha \neq K_\gamma$ ($K_\alpha \cap K_\gamma = \emptyset$).

5.11. LEMMA. *If K is a continuum of convergence in a locally compact Hausdorff space S and $\{K_\alpha : \alpha \in (\mathscr{A}, \geqq)\}$ is a net of continua such that $K \cap K_\alpha = \emptyset$ and $K = \lim_\alpha K_\alpha$, then the net $\{K_\alpha : \alpha \in (\mathscr{A}, \geqq)\}$ is almost pairwise disjoint and thus almost distinct.*

Proof. We need only note that for each $\alpha \in \mathscr{A}$, $M - K_\alpha$ is open, contains K and the net is eventually in $M - K_\alpha$.

The proof of the next lemma is easy.

5.12. LEMMA. *If K is a continuum of convergence in a connected T_1-space S, then every two points of K are conjugate in S.*

5.13. THEOREM. *If K is a continuum of convergence in M, $\{K_\alpha : \alpha \in (\mathscr{A}, \geqq)\}$ a net of continua such that for each $a \in \mathscr{A}$, $K_\alpha \cap K = \emptyset$ and $K = \lim_\alpha K_\alpha$, then there is an E_0-set E of M such that $K \subset E$ and $K = \lim_\alpha (E \cap K_\alpha)$.*

Proof. Since K is a continuum of convergence, there is an E_0-set E of M such that $K \subset E$. If $k \in \lim_\alpha \sup(E \cap K_\alpha)$, then $k \in \lim_\alpha \sup K_\alpha$, so $\lim_\alpha \sup(E \cap K_\alpha) \subset K$. We show that $K \subset \lim_\alpha \inf(E \cap K_\alpha)$.

Suppose not, and let $k \in K - \lim_\alpha \inf(E \cap K_\alpha)$. Let $y \in K$ such that $y \neq k$. Then there is an open set O such that $k \in O$, $y \notin \bar{O}$, and $\{\alpha \in \mathscr{A} : K_\alpha \cap E \cap O = \emptyset\}$ is cofinal in $\mathscr{A}$. Let $\mathscr{D} = \{(\alpha, V, W) : \alpha \in \mathscr{A}$ and $K_\alpha \cap E \cap O = \emptyset$; V is open, $k \in V \subset O$, and $K_\alpha \cap V \neq \emptyset$; W is open, $y \in W$, and $K_\alpha \cap W \neq \emptyset\}$. It is easy to see that $\mathscr{D} \neq \emptyset$.

Define the relation $>$ on $\mathscr{D}$ by $(\alpha_1, V_1, W_1) > (\alpha_2, V_2, W_2)$ if and only if $\alpha_1 \geqq \alpha_2$, $V_1 \subset V_2$, and $W_1 \subset W_2$. Again it is not difficult to show that $>$ is nonempty and directs $\mathscr{D}$. Also, if we define for $\delta = (\alpha, V, W) \in \mathscr{D}$, $N(\delta) = \alpha$, then $\{K_{N(\delta)} : \delta \in (\mathscr{D}, >)\}$ is a subnet of $\{K_\alpha : \alpha \in (\mathscr{A}, \geqq)\}$ so $\lim_\delta K_{N(\delta)} = K$.

For each $\delta = (\alpha, V, W) \in \mathscr{D}$, let $x_\delta \in K_\alpha \cap V$, and $y_\delta \in K_\alpha \cap W$. Then $\{x_\delta : \delta \in (\mathscr{D}, >)\}$ and $\{y_\delta : \delta \in (\mathscr{D}, >)\}$ are nets converging respectively to k and y, and for each $\delta \in \mathscr{D}$, $x_\delta \notin E$.

For each $\delta \in \mathscr{D}$, let C_δ be the component of $M - E$ such that $x_\delta \in C_\delta$, and let $b_\delta = \partial(C_\delta)$. Since $k \in E$ and $k = \lim_\delta x_\delta$, k is a limit point of $\bigcup \{C_\delta : \delta \in \mathscr{D}\}$ and $k \notin \bigcup \{C_\delta : \delta \in \mathscr{D}\}$. It follows from 5.4 that $k = \lim_\delta b_\delta$.

If $\{\delta \in \mathscr{D} : K_{N(\delta)} \subset C_\delta$ is not bounded in $\mathscr{D}$, then y is a limit point of $\bigcup \{C_\delta : \delta \in \mathscr{D}\}$ and the net $\{b_\delta : \delta \in (\mathscr{D}, >)\}$ converges to y. Since $y \neq k$, this is a contradiction. Thus for some $\delta^* \in \mathscr{D}$, if $\delta > \delta^*$, then $K_{N(\delta)} \not\subset C_\delta$. Since for each $\delta \in \mathscr{D}$, $x_\delta \in K_{N(\delta)} \cap C_\delta$, $b_\delta \in K_{N(\delta)}$ for each $\delta > \delta^*$. But $b_\delta \in K_{N(\delta)} \cap E$ and this yields a contradiction since the net $\{b_\delta : \delta \in (\mathscr{D}, >)\}$ is eventually in O and for all $\delta \in \mathscr{D}$, $K_{N(\delta)} \cap E \cap O = \emptyset$. The theorem follows.

5.14. COROLLARY. *Any continuum of convergence of M is a continuum of convergence of some single E_0-set of M.*

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5.15. COROLLARY. *M has no continuum of convergence if and only if every cyclic element of M has no continuum of convergence.*

The proof of the next theorem is similar to that of 4.2 page 71 of [**6**].

5.16. THEOREM. *If M is compact and $\mathscr{E} = \{E \subset M : E$ is an E_0-set of $M\}$, then $\mathscr{E}$ is a null family.*

6.0. Cyclic chain development theorem. In this section we prove a theorem which is analogous to the Cyclic Chain Approximation Theorem [**6**, Theorem 7.1, p. 73].

6.1. THEOREM. *If M is compact, then there exist a well-ordered set $(\mathscr{A}, \geqq)$, a net $\{p_\alpha : \alpha \in (\mathscr{A}, \geqq)\}$ of non-cut points of M belonging to nodes of M, and a net $\{q_\alpha : \alpha \in (\mathscr{A}, \geqq)\}$ in M such that the net of cyclic chains $\{C(p_\alpha, q_\alpha) : \alpha \in (\mathscr{A}, \geqq)\}$ has the following properties:*

a) *For each $\alpha \in \mathscr{A}$, $H_\alpha = \bigcup_{\gamma<\alpha} C(p_\gamma, q_\gamma)$ is an H-set of M.*

b) *For each $\alpha \in \mathscr{A}$, if α is not the first element of $\mathscr{A}$, then $C(p_\alpha, q_\alpha) \cap \bar{H}_\alpha = \{q_\alpha\}$.*

c) *$M = \bigcup_{\alpha\in\mathscr{A}} C(p_\alpha, q_\alpha)$.*

d) *For every open cover $\mathscr{G}$ of M, there is an $\alpha_{\mathscr{G}} \in \mathscr{A}$ such that if $\alpha \geqq \alpha_{\mathscr{G}}$ and C is a component of $M - H_\alpha$, then* diam $C < \mathscr{G}$.

Proof. If M has no cut point, we let $\mathscr{A} = \{1\}$, and let p_1 and q_1 be any two distinct points of M. Assume, then, that M has a cut point. Our proof has three steps. We first define the well-ordered set $(\mathscr{A}, \geqq)$ and the net $\{p_\alpha : \alpha \in (\mathscr{A}, \geqq)\}$. Next, we define the net $\{q_\alpha : \alpha \in (\mathscr{A}, \geqq)\}$ by induction on $\mathscr{A}$. Finally, we show that the net of cyclic chains $C(p_\alpha, q_\alpha)$ has the properties a)–d).

1. $(\mathscr{A}, \geqq)$ and the net $\{p_\alpha : \alpha \in (\mathscr{A}, \geqq)\}$. Since M has a cut point, M has at least two nodes. Let $\mathscr{N}$ be the set of all nodes of M and let $(\mathscr{A}, \geqq)$ be the set of all ordinals whose cardinal is less than that of $\mathscr{N}$. Let N^* be any (fixed) node of M and let $\{N_\alpha : \alpha \in \mathscr{A}\}$ be an indexing of $\mathscr{N} - \{N^*\}$ by $\mathscr{A}$. For each $\alpha \in \mathscr{A}$, let p_α be a non-cut point of M belonging to N_α. Then the net $\{p_\alpha : \alpha \in (\mathscr{A}, \geqq)\}$ has been defined.

2. The net $\{q_\alpha : \alpha \in (\mathscr{A}, \geqq)\}$. Let q_1 be an non-cut point of M belonging to N^*. It follows from 4.10 that if $\delta \in \mathscr{A}$, $\delta > 1$, then $p_\delta \notin C(p_1, q_1)$. In $M - C(p_1, q_1)$, let C_2 be the component which contains p_2 and define $q_2 = \delta(C_2)$. Then $H_3 = C(p_1, q_1) \cup C(p_2, q_2)$ is an H-set of M, and it follows from 4.10 that if $\delta > 2$, then $p_\delta \notin C(p_1, q_1) \cup C(p_2, q_2)$. Since $C(p_2, q_2) \subset C_2 \cup q_2$, $C(p_2, q_2) \cap C(p_1, q_1) = \{q_2\}$.

Suppose that for some $\beta \in \mathscr{A}$, $\beta \geqq 2$, we have defined q_α for each $\alpha \in \mathscr{A}$, $\alpha < \beta$, in such a way that if $1 < \alpha$, then

1. $\bigcup_{\gamma\leqq\alpha} C(p_\gamma, q_\gamma)$ is an H-set of M;
2. $C(p_\alpha, q_\alpha) \cap \bar{H}_\alpha = \{q_\alpha\}$, $(H_\alpha = \bigcup_{\gamma<\alpha} C(p_\gamma, q_\gamma))$;
3. if $\delta \in \mathscr{A}$, $\delta > \alpha$, then $p_\delta \notin \bigcup_{\gamma\leqq\alpha} C(p_\alpha, q_\alpha)$.

It follows easily from hypotheses 1 and 3 that H_β is an H-set of M and does not contain p_β. In $M - H_\beta$, let C_β be the component which contains p_β, and let q_β be the unique point in $\bar{C}_\beta \cap \bar{H}_\beta$. Since both $H_\beta \cup q_\beta$ and $C(p_\beta, q_\beta)$ are H-sets of M and q_β belongs to each, their union $\bigcup_{\alpha \leqq \beta} C(p_\alpha, q_\alpha)$ is by 3.14 an H-set of M. Also, since $C_\beta \cup q_\beta$ is an A-set of M containing p_β and q_β, $C(p_\beta, q_\beta) \cap \bar{H}_\beta \subset \bar{C}_\beta \cap \bar{H}_\beta = \{q_\beta\}$. Further, by 4.10, if $\delta \in \mathscr{A}$, $\delta > \beta$, then $p_\delta \notin \bigcup_{\alpha \leqq \beta} C(p_\alpha, q_\alpha)$. Thus for each $\alpha \in \mathscr{A}$, q_α is defined.

3. We now show that $\{C(p_\alpha, q_\alpha) : \alpha \in (\mathscr{A}, \geqq)\}$ satisfies conditions a)–d). It follows from the definition of the net $\{q_\alpha : \alpha \in (\mathscr{A}, \geqq)\}$ that a) and b) are satisfied. Also, it is not difficult to show that $H = \bigcup_{\alpha \in \mathscr{A}} C(p_\alpha, q_\alpha)$ is an H-set of M. Now if C is a component of $M - H$, then it follows from 4.19 that C contains a point p which belongs to a node of M. But H contains every node of M, and it follows that $M = H$.

It remains to show that d) is satisfied. If $\mathscr{A}$ is finite, the result is immediate since then $\mathscr{A}$ has a maximum. Suppose, then, that $\mathscr{A}$ is infinite. Then $\mathscr{A}$ has no maximum. Suppose further there is an open cover $\mathscr{G}$ of M such that for each $\alpha \in \mathscr{A}$ there is a component R_α of $M - H_\alpha$ such that diam $R_\alpha > \mathscr{G}$. For each $\alpha \in \mathscr{A}$, let $a_\alpha \in R_\alpha$. Then $\{a_\alpha : \alpha \in (\mathscr{A}, \geqq)\}$ is a net in M so there is a point $a \in M$ and a subnet $\{a_{\alpha_\beta} : \beta \in (\mathscr{B}, \gg)\}$ converging to a. Let $G \in \mathscr{G}$ such that $a \in G$. Now for all $\beta \in \mathscr{B}$, there is a point $b_\beta \in R_{\alpha_\beta} - G$ and a subnet $\{b_{\beta_\delta} : \delta \in (\mathscr{D}, \succ)\}$ converging to $b \in M$. Now for some $\alpha^* \in \mathscr{A}$, $a, b \in H_{\alpha^*}$, and if $\alpha > \alpha^*$, $\alpha \in \mathscr{A}$, then $M - H_\alpha \subset M - H_{\alpha^*}$ and R_α is contained in some component of $M - H_{\alpha^*}$. Since only a finite number of components of $M - H_{\alpha^*}$ have diameter $> \mathscr{G}$ and every R_α is contained in such a component for $\alpha > \alpha^*$, it follows that for some component C of $M - H_{\alpha^*}$, $\{\delta \in (\mathscr{D}, \succ) : R_{\alpha_{\beta_\delta}} \subset C\}$ is cofinal in $(\mathscr{D}, \succ)$. But then both a and b are limit points of C, so $a, b \in \bar{C} \cap \bar{H}_{\alpha^*}$. This is a contradiction. It follows that if $\mathscr{G}$ is any open cover of M, then for some $\alpha_{\mathscr{G}} \in \mathscr{A}$, every component of $M - H_{\alpha_{\mathscr{G}}}$ has diameter $< \mathscr{G}$, so if $\alpha > \alpha_{\mathscr{G}}$ then every component of $M - H_\alpha$ has diameter $< \mathscr{G}$.

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