UNIQUE ADDITION RINGS

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1. Introduction. A semigroup $(R, \cdot)$ is said to be a unique addition ring (UA-ring) if there exists a unique binary operation $+$ making $(R, \cdot, +)$ into a ring. All our results can be presented in this semigroup theoretic setting. However, we prefer the following equivalent ring theoretic formulation: a ring R is a UA-ring if and only if any semigroup isomorphism α: $(R, \cdot) \cong (S, \cdot)$ with another ring S is always a ring isomorphism.

UA-rings have been studied in (**8**; **4**) and are also touched on in (**1**; **2**; **6**; **7**). In this note we generalize Rickart's methods to much wider classes of rings. In particular, we show that, for a ring R with a 1 and $n \geqq 2$, the $(n \times n)$ matrix ring over R and its subring of lower triangular matrices are UA-rings. Further examples of UA-rings are "infinite" matrix rings and certain subclasses of the classes of prime rings, von Neumann regular rings, left self-injective rings and semiperfect rings.

2. Notation. For a ring R we let $J(R)$ denote its Jacobson radical and for $n \geqq 1$ we let R_n denote the ring of all $(n \times n)$ matrices over R and $T_n(R)$ the subring of lower triangular matrices. If A is a subset of R, then $l(A) = (x \in R: xA = 0)$ and $r(A) = (x \in R: Ax = 0)$ will denote the left and right annihilators of A, respectively.

3. Basic lemmas.

LEMMA 1. *Let R and S be rings and α: $(R, \cdot) \cong (S, \cdot)$ a semigroup isomorphism. Suppose that A and B are left ideals of R such that $A \oplus B = Re$ for some idempotent $e \in R$. Then*

(1) $(a + b)^\alpha = a^\alpha + b^\alpha$ *for any* $a \in A, b \in B$;

(2) *if s: $A \rightarrow B$ is an R-homomorphism, then* $(a^s + b)^\alpha = (a^s)^\alpha + b^\alpha$ *for any* $a \in A, b \in B$;

(3) α *is additive on* $\sum(A^s: s \in \mathrm{Hom}_R(A, B))$.

Proof. (1) Let $g + h = e$, where $g \in A$, $h \in B$. Then $g^2 = g$, $h^2 = h$, and $gh = hg = 0$. Further, $A = Rg$ and $B = Rh$.

If $f^\alpha = g^\alpha + h^\alpha, f \in R$, then $(fe)^\alpha = (ge)^\alpha + (he)^\alpha = g^\alpha + h^\alpha = f^\alpha$, and thus $f = fe$. On the other hand, $(fg)^\alpha = g^\alpha$ and $(fh)^\alpha = h^\alpha$, hence

$$f = fe = fg + fh = g + h = e.$$

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If $x = a + b$, where $a \in A$ and $b \in B$, then $a = xg$ and $b = xh$. Hence

$$a^\alpha + b^\alpha = x^\alpha g^\alpha + x^\alpha h^\alpha = x^\alpha(g^\alpha + h^\alpha) = x^\alpha e^\alpha = (a + b)^\alpha.$$

(2) Let $s: A \to B$ be an R-homomorphism. Then it is easily verified that $A \oplus B = A^{(1+s)} \oplus B = Re$. Let $a \in A$, $b \in B$. Applying (1) three times we have:

$$a^\alpha + (a^s + b)^\alpha = (a + a^s + b)^\alpha = (a^{(1+s)})^\alpha + b^\alpha = a^\alpha + (a^s)^\alpha + b^\alpha.$$

Hence $(a^s + b)^\alpha = (a^s)^\alpha + b^\alpha$.

(3) follows immediately from (2).

Corollary 1. *If $s: A \cong B$, then α is additive on $A \oplus B = Re$.*

An immediate consequence of Corollary 1 is the following result.

Theorem 1. *If R is a ring with a* 1 *and $n \geqq 2$, then R_n is a* UA-*ring.*

Proof. If e_{ij} $(1 \leqq i, j \leqq n)$ are the usual matrix units for R_n, then $R_n = \oplus_1^n R_n e_{ii}$, where $R_n e_{ii} \cong R_n e_{jj}$ (as R_n-modules) for $1 \leqq i$, $j \leqq n$. The result then follows from Corollary 1. Next we prove a slight generalization of a theorem in (**8**).

Lemma 2. *Let R and S be rings and α: $(R, \cdot) \cong (S, \cdot)$ be a semigroup isomorphism. Suppose that C and C^α are abelian subgroups of R and S, respectively. Further, suppose that $(A_i)_{i\in I}$ and $(B_j)_{j\in J}$ are abelian subgroups of R such that $CA_i \subseteq A_i$ and $B_jC \subseteq B_j$ and such that α is additive on each A_i and on each B_j. If $A = \bigcup_I A_i$ and $B = \bigcup_J B_j$ and $C \cap l(A) \cap r(B) = 0$, then α is additive on C.*

Proof. Let $c_1, c_2 \in C$. Then $c_1^\alpha + c_2^\alpha \in C^\alpha$ and there exists $c \in C$ such that $c^\alpha = c_1^\alpha + c_2^\alpha$.

For any $a \in A$, we have:

$$[(c - c_1 - c_2)a]^\alpha = (ca)^\alpha - (c_1a)^\alpha - (c_2a)^\alpha = (c^\alpha - c_1^\alpha - c_2^\alpha)a^\alpha = 0 = 0^\alpha.$$

Since α is one-to-one, $c - c_1 - c_2 \in l(A)$ and similarly $c - c_1 - c_2 \in r(B)$. Thus $c_1 + c_2 = c$ and $(c_1 + c_2)^\alpha = c^\alpha = c_1^\alpha + c_2^\alpha$.

Corollary 1. *Using the same notation, if each A_i is a left ideal and each B_j is a right ideal and $l(A) \cap r(B) = 0$, then α is a ring isomorphism.*

Proof. Take $C = R$ in Lemma 2.

Corollary 2. *The direct product of* UA-*rings with a* 1 *is a* UA-*ring.*

Proof. Let $R = \prod_{i\in I} R_i$, where for each $i \in I$, R_i is a UA-ring with a 1. Suppose that α: $(R, \cdot) \cong (S, \cdot)$ is a semigroup isomorphism with another ring S and let $e_i \in R$ be the central idempotent with 1 in the ith position and zeros elsewhere. If $f_i = e_i^\alpha$, then f_iSf_i is a ring and α induces $(e_iRe_i, \cdot) \cong (f_iSf_i, \cdot)$.

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However, $e_iRe_i \cong R_i$ and thus α is additive on Re_i for all $i \in I$. Since $l(\bigcup_I Re_i) = 0$, it follows from Corollary 1 that α is a ring isomorphism, hence R is a UA-ring.

Example. Let $M = P_1 \oplus P_2 \oplus Q$ be a left R-module where P_1, P_2, and Q are submodules such that $P_1 \cong P_2$ ($\cong P$) and $\sum(P^f: f \in \mathrm{Hom}_R(P, Q)) = Q$ (for example if $P \cong {}_RR$). Then $S = \mathrm{End}_R(M)$ is a UA-ring.

Proof. Let e_1, e_2, and f be the projections of M onto P_1, P_2, and Q, respectively. Then e_1, e_2, and f are a set of orthogonal idempotents of S such that

$$S = e_1S \oplus e_2S \oplus fS \quad \text{and} \quad e_1S \cong e_2S.$$

Further, $Me_1S = M$. If $a \in r(e_1S)$, then $Ma = Me_1Sa = 0$ and hence $r(e_1S) = 0$. By Lemma 1 and Corollary 1 of Lemma 2, S is a UA-ring.

This result should be compared with those of Mihalev (**6**, see, for example, Proposition 1 and Theorem 3).

4. Unique addition rings. Throughout this section, all rings will be assumed to contain a 1.

LEMMA 3. *Suppose that R and S are rings and α: $(R, \cdot) \cong (S, \cdot)$ is a semigroup isomorphism. If $e \in R$ is an idempotent, then α is additive on $ReR(1 - e)$ and hence on*

$$A = (ReR(1 - e)R + R(1 - e)ReR).$$

Proof. $\mathrm{Hom}(Re, R(1 - e))$ is given by right multiplication by elements of $eR(1 - e)$. Hence

$$ReR(1 - e) = \sum[(Re)^s: s \in \mathrm{Hom}_R(Re, R(1 - e))]$$

and thus by Lemma 1 (3), α is additive on $ReR(1 - e)$. Similarly, α is additive on $R(1 - e)Re$ and therefore on

$$A = (ReR(1 - e)R + R(1 - e)ReR) = ReR(1 - e) \oplus R(1 - e)Re.$$

Combining Lemma 3 and the above corollary, we obtain the following result.

THEOREM 2. *Let R be a ring (with a 1) and G the set of all idempotents of R. If*

$$A = \sum_{g \in G} RgR(1 - g)R \quad \textit{and} \quad l(A) \cap r(A) = 0,$$

then R is a UA-ring.

We now consider rings R such that any non-zero two-sided ideal of the form $l(A) \cap r(A)$, A a non-zero two-sided ideal, contains a non-zero idempotent. For lack of a better name, we call such rings X-rings.

Examples. Recall that a ring R is prime if $l(A) = 0$ for every non-zero two-sided ideal of R. R is said to be semiprime if it contains no non-zero nilpotent ideals. Finally, R is called left P.P. (principal ideals projective) if for any $a \in R$, $l(a) = Re$ for some idempotent $e \in R$. The following are X-rings:

(1) Prime rings and, in particular, simple and primitive rings;

(2) Semisimple I-rings, that is, rings in which every non-zero left ideal contains a non-zero idempotent, in particular, von Neumann regular rings;

(3) Semiprime left P.P. rings. In particular, semiprime semihereditary and hereditary rings. (To show this, note that in a semiprime ring, $l(A) = r(A)$ for any two-sided ideal A and further, for $x \in r(A)$, $x \in rl(x) \subseteq rlr(A) = r(A)$.)

A non-zero central idempotent of a ring R is called abelian if every idempotent of eRe is central in eRe (and hence in R).

THEOREM 3. *Let R be an X-ring containing no abelian central idempotents. Then R is a* UA-*ring.*

Proof. Let $A = \sum_{g \in G} RgR(1 - g)R$, where G is the set of all idempotents of R, and note that $A \neq 0$ since 1 is not an abelian idempotent. Suppose that $l(A) \cap r(A) \neq 0$; then by hypothesis there is a non-zero idempotent $e \in l(A) \cap r(A)$.

Since R has no abelian central idempotents, we can always assume that e is not central. However, $eA = Ae = 0$ implies that $eR(1 - e) = (1 - e)Re = 0$, contradicting the fact that e is not central. Therefore $l(A) \cap r(A) = 0$ and, by Theorem 2, R is a UA-ring.

COROLLARY 1. *An X-ring with no non-trivial central idempotents and at least one non-trivial idempotent is a* UA-*ring.*

Since a prime ring is an X-ring which has no non-trivial central idempotents, we have the following result.

COROLLARY 2. *A prime ring* (*with a* 1) *containing at least one non-trivial idempotent is a* UA-*ring.*

Remark. If S is not a UA-ring, then it is easy to see that, for any ring R, $R \times S$ is not a UA-ring. Since there are fields which are not UA-rings (see **8**), the restriction on abelian central idempotents cannot be omitted.

A ring R is called left self-injective if R considered as a left R-module is injective. Results of Utumi (**9**) yield the following theorem.

THEOREM 4. *Any left self-injective ring R such that $R/J(R)$ contains no abelian central idempotents is a* UA-*ring.*

Proof. Since R is left self-injective, $\bar{R} = R/J(R)$ is a left self-injective, von Neumann regular ring (**9**, Theorems 4.7, 4.8 and Lemma 4.1). By hypothesis, $\bar{R}$ contains no abelian central idempotents and hence, by (**9**, Lemma 7.7) there exist orthogonal idempotents $e_1, e_2, e_3 \in \bar{R}$ such that

$$\bar{R}e_1 \oplus \bar{R}e_2 \oplus \bar{R}e_3 = \bar{R}, \qquad \bar{R}e_1 \cong \bar{R}e_2 \qquad \text{(as left } \bar{R}\text{-modules)}$$

and $\bar{R}e_1 \oplus \bar{R}e_2$ contains a submodule A isomorphic to $\bar{R}e_3$. Since A is injective, being isomorphic to $\bar{R}e_3$, it is a direct summand of $\bar{R}$. Hence for some idempotent e_4 orthogonal to e_3, $A = \bar{R}e_3$ and $\bar{R}e_3 \cong \bar{R}e_4$. By (**9**, Corollaries 3.2 and 4.12), we can lift the idempotents e_i $(i = 1, 2, 3, 4)$ to idempotents $f_i \in R$ such that the sets of idempotents $\{f_1, f_2, f_3\}$ and $\{f_3, f_4\}$ are orthogonal and $R = Rf_1 \oplus Rf_2 \oplus Rf_3$. Further, by (**3**, Proposition 1, 3.8), we can also lift the isomorphisms to obtain $Rf_1 \cong Rf_2$ and $Rf_3 \cong Rf_4$. The result then follows by Corollary 1 of Lemma 1.

5. Matrix rings. For a ring R and set I, let $R_{\mathrm{rf}I}$ denote the ring of "row finite" $(I \times I)$ matrices over R. That is, $R_{\mathrm{rf}I} = ((a_{ij})$: for each $i \in I$, $a_{ij} = 0$ for almost all $j \in I)$. For $r \in R$ we let re_{ij} denote the matrix with r in the (i, j)th place and zeros elsewhere and write $1e_{ij} = e_{ij}$, when R has a 1.

THEOREM 5. *Let R be a ring with a* 1 *and I a set containing at least two elements. Suppose that T is a subring of $R_{\mathrm{rf}I}$ (not necessarily with a* 1*), which contains e_{ki}, e_{kj} for all $k \in I$, where i and j are distinct elements of I. Then T is a* UA-*ring.*

Proof. We have $Te_{ii} \cap Te_{jj} = 0$ and $Te_{ii} \oplus Te_{jj} = T(e_{ii} + e_{jj})$, where $(e_{ii} + e_{jj})$ is an idempotent. Further, $e_{ii} = e_{ij}e_{ji}$, $e_{jj} = e_{ji}e_{ij}$ and $e_{ij}, e_{ji} \in T$. Hence, as left T-modules, $Te_{ii} \cong Te_{jj}$. However, $l(Te_{ii}) = 0$ and thus, by Corollary 1 of Lemma 1 and Corollary 1 of Lemma 2, T is a UA-ring.

A slightly stronger form of Theorem 5, in the case that R is a division ring, was proved by Gluskin (**1**, Theorem 5.10.1).

Examples. (1) If ${}_RF$ is a free left R-module of rank at least 2, then $\mathrm{End}_R(F)$ is a UA-ring. This shows that there is no gain in generality in studying semigroup isomorphisms rather than ring isomorphisms of the endomorphism rings of free modules of rank at least 2; see, for example (**6**; **7**; **2**, 3.13).

Further examples of UA-rings are:

(2) $R_{\mathrm{f}I}$ denotes the ring of "finite" $(I \times I)$ matrices; $R_{\mathrm{f}I} = ((a_{ij})$: $a_{ij} = 0$ for almost all $i, j \in I)$;

(3) $R_{\mathrm{rb}I}$ denotes the ring of "row bounded" $(I \times I)$ matrices; $R_{\mathrm{rb}I} = ((a_{ij})$: $a_{ij} = 0$ for all $j \notin F$, where F is some finite subset of $I)$.

Similar results hold for the rings of "column finite" and "column bounded" $(I \times I)$ matrices.

A ring R is said to have non-trivial matrix rank if $R \cong \prod_{i=1}^{n} (R^{(i)})_{n_i}$ for rings $R^{(i)}$ and integers $n \geqq 1$ and $n_i \geqq 2$ for $i = 1, \ldots, n$.

THEOREM 6. *Let R be a ring with a* 1 *for which idempotents can be lifted modulo $J(R)$. If $R/J(R)$ has non-trivial matrix rank, then R is a* UA-*ring.*

Proof. Suppose that $R/J(R) = \prod_{i=1}^{n} (R^{(i)})_{n_i}$, where $n_i \geqq 2$ and $n \geqq 1$. Then

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lifting the matrix units e_{jj} $(1 \leqq j \leqq n_i)$ of each $(R^{(i)})_{ni}$ to idempotents f_{ij} $(1 \leqq j \leqq n_i)$, we have

$$R = \bigoplus_1^n \left(\bigoplus_1^{ni} Rf_{ij} \right),$$

where for each i, $Rf_{ia} \cong Rf_{ib}$ for all $1 \leqq a, b \geqq n_i$; see (**3**, Proposition 1.3.8). The result follows from Corollary 1 of Lemma 1.

Example. Any left-Artinian ring (or, more generally, any semiperfect ring (see **5**, 3.6)) such that $R/J(R)$ has non-trivial matrix rank, is a UA-ring.

So far, triangular matrix rings have escaped our net. For if Δ is a division ring and $T = T_n(\Delta)$, where $n \geqq 2$, then T is left-Artinian, but $T/J(T) \cong \prod_1^n \Delta$ does not have non-trivial matrix rank. Further, $A = \sum_{g \in G} TgT(1-g)T = J(T)$, where G is the set of all idempotents of T, and $l(A) \cap r(A) = J(T)^{n-1} \neq 0$. T, although hereditary, is not an X-ring. Nevertheless, T is a UA-ring.

To prove this result we begin with a lemma.

LEMMA 4. *Let T be a ring with a* 1 *and let $e \in T$ be an idempotent. If* $l((1-e)Te) \cap (1-e)T(1-e) = 0$ and $r((1-e)Te) \cap eTe = 0$, *then T is a* UA-ring.

Proof. Suppose that α: $(T, \cdot) \cong (S, \cdot)$ is a semigroup isomorphism with another ring S. By Lemma 3, α is additive on $(1-e)Te$ and $eT(1-e)$. Letting $C = (1-e)T(1-e)$ and $A = (1-e)Te$ in Lemma 2, we see that α is additive on $(1-e)T(1-e)$. Similarly, α is additive on eTe and thus on T.

If R and S are rings with a 1 and ${}_SM_R$ is a unital $S-R$ bimodule, faithful both as a right R-module and as a left S-module, then we denote the ring of generalized triangular matrices $\left[\begin{smallmatrix} R & 0 \\ {}_SM_R & S \end{smallmatrix}\right]$ by $T(R, S, M)$. It is easy to see that a ring T is isomorphic to $T(R, S, M)$ for some R, S, M if and only if there is an idempotent $e \in T$ such that $eT(1-e) = 0$ and $eTe \cong R$, $(1-e)T(1-e) \cong S$ and $(1-e)Te \cong M$, and $l((1-e)Te) \cap (1-e)T(1-e) = 0$ and $r((1-e)Te) \cap eTe = 0$.

An immediate consequence of Lemma 4 is the following result.

THEOREM 7. *Any generalized triangular matrix ring $T(R, S, M)$ is a* UA-*ring.*

COROLLARY 1. *If R is a ring with a* 1 *and $n \geqq 2$, then $T_n(R)$ is a* UA-*ring.*

6. Classical left quotient rings. For a ring R let

$$I(R) = (x \in R\colon l(x) = r(x) = 0)$$

denote the set of regular elements of R. As is well known, R possesses a classical left quotient ring Q if and only if the left Ore condition is satisfied (for the concept of classical left quotient ring and the results used in this section see (**3**,

Appendix B; **5**, 4.6)). Namely, for any $a \in R$, $b \in I(R)$ there exist $a' \in R$, $b' \in I(R)$ such that $b'a = a'b$.

Let α: $(R, \cdot) \cong (R', \cdot)$ be a semigroup isomorphism with another ring R'. If R satisfies the left Ore condition, then so does R'. Hence R and R' have classical left quotient rings Q and Q', respectively. The construction of the classical left quotient ring Q by means of an equivalence relation on $R \times I(R)$ (see **3**) and the definition of multiplication in Q involve only the multiplicative structure of R. Hence α: $(R, \cdot) \cong (R', \cdot)$ can be extended to α: $(Q, \cdot) \cong (Q', \cdot)$. The following theorem is then immediate.

THEOREM 8. *If R has a classical left quotient ring Q which is a* UA-*ring, then R is a* UA-*ring.*

Example. Any left order R in a semiperfect ring Q such that $Q/J(Q)$ has non-trivial matrix rank is a UA-ring. A particular example is any prime left Goldie ring of dimension at least 2. This case and the case that R is a semiprime left Goldie ring also follow from (**4**).

It would be interesting to know whether Theorem 8 could be generalized to rings R with zero singular ideal and so include all the results of Johnson (**4**). Q in this case would be the complete ring of quotients of R (see **5**, 4.3) and hence would be a left self-injective von Neumann regular ring, which, barring abelian central idempotents, is a UA-ring.

REFERENCES

1. L. M. Gluskin, *Semigroups and rings of endomorphisms of linear spaces*, Izv. Akad. Nauk. SSSR Ser. Mat. *23* (1959), 841–870 = Amer. Math. Soc. Transl. (2) *45* (1965), 105–137.

2. ——— *Endomorphisms of modules*, Algebra and Mathematical Logic, pp. 3–20, Studies in Algebra (Izdat. Kiev. Univ., Kiev, 1966).

3. N. Jacobson, *Structure of rings*, Rev. ed., Amer. Math. Soc. Colloq. Publ., Vol. 37 (Amer. Math. Soc., Providence, R.I., 1964).

4. R. E. Johnson, *Rings with unique addition*, Proc. Amer. Math. Soc. *9* (1958), 55–61.

5. J. Lambek, *Lectures on rings and modules* (Blaisdell, Waltham, Massachusetts, 1966).

6. A. V. Mihalev, *Isomorphisms of semigroups of endomorphisms of modules*, Algebra i Logika Sem. *5* (1966), no. 5, 59–67.

7. ——— *Isomorphisms of semigroups of endomorphisms of modules*. II, Algebra i Logika Sem. *6* (1967), no. 2, 35–47.

8. C. E. Rickart, *One-to-one mappings of rings and lattices*, Bull. Amer. Math. Soc. *54* (1948), 758–764.

9. Y. Utumi, *On continuous and self-injective rings*, Trans. Amer. Math. Soc. *118* (1965), 158–173.

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