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APPROXIMATELY ANGLE PRESERVING MAPPINGS

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Abstract

We study linear mappings which preserve vectors at a specific angle. We introduce the concept of (ε, c)-angle preserving mappings and define $\widehat{\varepsilon}(T, c)$ as the 'smallest' number ε for which T is an (ε, c)-angle preserving mapping. We derive an exact formula for $\widehat{\varepsilon}(T, c)$ in terms of the norm $\|T\|$ and the minimum modulus $[T]$ of T. Finally, we characterise approximately angle preserving mappings.

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1. Introduction

Throughout this paper, $\mathscr{H}, \mathscr{K}$ denote real Hilbert spaces with dimensions greater than or equal to two and $\mathbb{B}(\mathscr{H}, \mathscr{K})$ denotes the Banach space of all bounded linear mappings between the Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$. We write $\mathbb{B}(\mathscr{H})$ for $\mathbb{B}(\mathscr{H}, \mathscr{H})$.

As usual, vectors $x, y \in \mathscr{H}$ are said to be orthogonal, $x \perp y$, if $\langle x, y\rangle = 0$, where $\langle ., .\rangle$ denotes the inner product of $\mathscr{H}$. A mapping $T : \mathscr{H} \longrightarrow \mathscr{K}$ is called orthogonality preserving if it preserves orthogonality, that is,

$$x \perp y \Longrightarrow Tx \perp Ty \quad (x, y \in \mathscr{H}).$$

Orthogonality preserving mappings may be nonlinear and discontinuous (see [2]). Under the additional assumption of linearity, a mapping T is orthogonality preserving if and only if it is a scalar multiple of an isometry, that is, $T = \gamma U$, where U is an isometry and $\gamma \geq 0$ (see [5]).

It is natural to consider approximate orthogonality (ε-orthogonality), $x \perp^{\varepsilon} y$, defined by $|\langle x, y\rangle| \leq \varepsilon\|x\|\,\|y\|$. For $\varepsilon \geq 1$, it is clear that every pair of vectors are ε-orthogonal, so the interesting case is when $\varepsilon \in [0, 1)$.

A mapping $T : \mathscr{H} \longrightarrow \mathscr{K}$ is an approximately orthogonality preserving mapping, or an ε-orthogonality preserving mapping, if

$$x \perp y \Longrightarrow Tx \perp^{\varepsilon} Ty \quad (x, y \in \mathscr{H}).$$

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Obviously, if $\varepsilon = 0$, then T is orthogonality preserving. A natural question is whether a linear ε-orthogonality preserving mapping T must be close to a linear orthogonality preserving mapping (see [1, 4, 7]).

In a Hilbert space $\mathscr{H}$ we can define a relation connected to the notion of angle. Fix $c \in (-1, 1)$. For $x, y \in \mathscr{H}$, we say that $x \angle_c y$ if $\langle x, y\rangle = c\,\|x\|\,\|y\|$. Thus, $c = \cos(\alpha)$, where α is the angle between x and y if $x, y \in \mathscr{H} \setminus \{0\}$.

A mapping $T : \mathscr{H} \longrightarrow \mathscr{K}$ is c-angle preserving if it preserves the angle c, that is,

$$x \angle_c y \implies Tx \angle_c Ty \quad (x, y \in \mathscr{H}).$$

Angle preserving mappings may be far from linear and continuous. There is an (infinite-dimensional) Euclidean space $\mathscr{H}$ and an injective map $T : \mathscr{H} \longrightarrow \mathscr{H}$ such that the condition $x \angle_{1/2} y$ implies that $Tx \angle_{1/2} Ty$, while the map T is discontinuous at all points (see [6, Remark 3]). A characterisation of angle preserving mappings on finite-dimensional Euclidean spaces was obtained in [6] and Chmieliński [3] studied stability of angle preserving mappings on the plane.

In the next section, we present some characterisations of linear mappings preserving certain angles. We show (Theorem 2.4) that a nonzero linear map T is c-angle preserving if and only if T is a scalar multiple of an isometry, generalising [2, Theorem 1] and [12, Theorem 3.8].

Fix $\varepsilon \in [0, 1)$ and define $x \angle_c^{\varepsilon} y$ by

$$|\langle x, y\rangle - c\,\|x\|\,\|y\|| \le \varepsilon\|x\|\,\|y\|,$$

which is equivalent to $c - \varepsilon \leqslant \cos\alpha \leqslant c + \varepsilon$, where α is the angle between x and y. If $c = 0$, then $\angle_0 = \perp$ and $\angle_0^{\varepsilon} = \perp^{\varepsilon}$. It is easy to see that $\angle_c$ and $\angle_c^{\varepsilon}$ are weakly homogeneous in the sense that $x \angle_c y \Leftrightarrow \alpha x \angle_c \beta y$ and $x \angle_c^{\varepsilon} y \Leftrightarrow \alpha x \angle_c^{\varepsilon} \beta y$ for all $\alpha, \beta \in \mathbb{R}^+$. For $\varepsilon \ge 1 + |c|$, it is obvious that $x \angle_c^{\varepsilon} y$ for all $x, y \in \mathscr{H}$. Hence, we shall only consider the case $\varepsilon \in [0, 1 + |c|)$.

A mapping $T : \mathscr{H} \longrightarrow \mathscr{K}$ satisfying the condition

$$x \angle_c y \implies Tx \angle_c^{\varepsilon} Ty \quad (x, y \in \mathscr{H})$$

is called an ε-approximate c-angle preserving mapping or (ε, c)-angle preserving mapping.

Recently, angle preserving mappings have been studied in [8, 9] via an approach different from ours. When $\mathscr{H}, \mathscr{K}$ are finite dimensional, the third author [9] proved that for an arbitrary $\delta > 0$ there exists $\varepsilon > 0$ such that for any linear (ε, c)-angle preserving mapping T there exists a linear c-angle preserving mapping such that

$$\|T - S\| \le \delta \min\{\|T\|, \|S\|\}.$$

Our intention is to obtain a characterisation of approximate angle preserving mappings. If $0 \le \varepsilon_1 \le \varepsilon_2 < 1 + |c|$ and T is an (ε_1, c)-angle preserving mapping, then T is also an (ε_2, c)-angle preserving mapping. This fact motivates us to give the following definition (see also [13]).

Definition 1.1. Let $c \in (-1, 1)$. For each map $T : \mathscr{H} \longrightarrow \mathscr{K}$, let $\widehat{\varepsilon}(T, c)$ be the 'smallest' number ε such that T is (ε, c)-angle preserving, that is,

$$\widehat{\varepsilon}(T, c) := \inf\{\varepsilon \in [0, 1 + |c|] : T \text{ is an } (\varepsilon, c)\text{-angle preserving mapping}\}.$$

Thus, $\widehat{\varepsilon}(T, c) = 1 + |c|$ whenever T is not an approximately c-angle preserving mapping. Also, it is easy to see that $\widehat{\varepsilon}(T, -c) = \widehat{\varepsilon}(T, c) = \widehat{\varepsilon}(\alpha T, c)$ for all $\alpha \in \mathbb{R} \setminus \{0\}$.

In the last section, we state some basic properties of the function $\widehat{\varepsilon}(., c)$. If $T \in \mathbb{B}(\mathscr{H}, \mathscr{K})$, then we derive an exact formula for $\widehat{\varepsilon}(T, c)$ in terms of the norm $\|T\|$ and the minimum modulus $[T]$ of T. Here $[T]$ is the largest number $m \geq 0$ such that $\|Tx\| \geq m\|x\|$ $(x \in \mathscr{H})$. We use this formula to characterise the approximately c-angle preserving mappings (Corollary 3.4) and show that every nonzero linear mapping T is approximately c-angle preserving if and only if T is bounded below.

2. Linear mappings preserving angles

We start our work with the following lemmas. The first follows immediately from the definition of the angle between vectors.

Lemma 2.1. *Let $c \in [0, 1)$. If $x, y \in \mathscr{H}$ are such that $\|x\| = \|y\| = 1$ and $x \perp y$, then:*

(i) $\left(x + \sqrt{1 + c/1 - c}y\right) \angle_c \left(-x + \sqrt{1 + c/1 - c}y\right)$;

(ii) $\left(x + \sqrt{1 - c/1 + c}y\right) \angle_c \left(x - \sqrt{1 - c/1 + c}y\right)$.

Lemma 2.2 [10, Theorem 2.3]. *Let $T \in \mathbb{B}(\mathscr{H}, \mathscr{K})$ be an injective linear map. Suppose that $\dim \mathscr{H} = n$. Then there exists an orthonormal basis $\{x_1, x_2, \ldots, x_n\}$ for $\mathscr{H}$ such that*

$$[T] = \|Tx_1\|, \quad \|Tx_2\| = \|T\| \quad \text{and} \quad Tx_i \perp Tx_j \quad (1 \leq i \neq j \leq n).$$

Corollary 2.3. *Let $T : \mathscr{H} \longrightarrow \mathscr{K}$ be a nonzero injective linear map. Suppose that unit vectors $x, y \in \mathscr{H}$ are linearly independent. Then there exist unit vectors x_1, x_2 such that*

$$x_1 \perp x_2, \quad Tx_1 \perp Tx_2, \quad \|Tx_1\| \leq \|Tx\| \leq \|Tx_2\| \quad \text{and} \quad \|Tx_1\| \leq \|Ty\| \leq \|Tx_2\|.$$

We are now ready to characterise the c-angle preserving mappings. The following result is a generalisation of [2, Theorem 1].

Theorem 2.4. *Let $T : \mathscr{H} \longrightarrow \mathscr{K}$ be a nonzero linear map and let $c \in (-1, 1)$. Then the following statements are equivalent:*

(i) $x \angle_c y \implies Tx \angle_c Ty \quad (x, y \in \mathscr{H})$;

(ii) *there exists $\gamma > 0$ such that $\|Tx\| = \gamma\|x\| \quad (x \in \mathscr{H})$.*

Proof. The implication (ii)⇒(i) follows from the polarisation formula. The implication (i)⇒(ii) follows from a more general theorem (see Corollary 3.5). □

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The following example shows that Theorem 2.4 fails if the assumption of linearity is dropped. Nonlinear mappings satisfying $x \angle_c y \implies Tx \angle_c Ty \quad (x, y \in \mathscr{H})$ may be very strange and even noncontinuous.

Example 2.5. Let $c \in (-1, 1)$. Let $\varphi\colon \mathscr{H} \to \mathbb{R}$ be a fixed nonvanishing function. Define the mapping $T : \mathscr{H} \longrightarrow \mathscr{H}$ by $T(x) := \varphi(x)\cdot x$. Then $x \angle_c y \implies Tx \angle_c Ty$ for all $x, y \in \mathscr{H}$. If φ is not continuous, then T clearly is not continuous. In particular, T is clearly not a similarity.

Taking $\mathscr{K} = \mathscr{H}$ and $T = \mathrm{Id} : (\mathscr{H}, \langle ., .\rangle_1) \longrightarrow (\mathscr{H}, \langle ., .\rangle_2)$, the identity map, we obtain the following result from Theorem 2.4.

Corollary 2.6. *Let $c \in (-1, 1)$. Suppose that $\mathscr{H}$ is a vector space equipped with two (complete) inner products $\langle ., .\rangle_1$, $\langle ., .\rangle_2$ generating the norms $\|.\|_1$, $\|.\|_2$ and c-angle relations $\angle_{c,1}$, $\angle_{c,2}$, respectively. Then the following conditions are equivalent:*

(i) *there exists $\gamma > 0$ such that $\|x\|_2 = \gamma\|x\|_1 \quad (x \in \mathscr{H})$;*
(ii) $x \angle_{c,1} y \implies x \angle_{c,2} y \quad (x, y \in \mathscr{H})$;
(iii) $\sup\{|\langle x, y\rangle_2/\|x\|_2\,\|y\|_2 - c| : x \angle_{c,1} y,\ x, y \in \mathscr{H} \setminus \{0\}\} = 0$.

Corollary 2.7. *Let $T \in \mathbb{B}(\mathscr{H}, \mathscr{K})$ be a bijective linear map and let $c \in (-1, 1)$. Then the following statements are equivalent:*

(i) $x \angle_c y \implies Tx \angle_c Ty \quad (x, y \in \mathscr{H})$;
(ii) *$\|TST^{-1}\| \le \|S\|$ for all invertible linear mappings $S \in \mathbb{B}(\mathscr{H})$.*

Proof. The implication (i)⇒(ii) follows immediately from Theorem 2.4.

(ii)⇒(i). Suppose that (ii) holds. For every $\varepsilon > 0$,

$$\|\varepsilon I + T(x \otimes y)T^{-1}\| = \|T(\varepsilon I + x \otimes y)T^{-1}\| \le \|\varepsilon I + x \otimes y\| \quad (x, y \in \mathscr{H}).$$

Here, $x \otimes y$ denotes the rank-one operator in $\mathbb{B}(\mathscr{H})$ defined by $(x \otimes y)(z) := \langle z, y\rangle x$ for $z \in \mathscr{H}$. Letting $\varepsilon \to 0^+$ yields

$$\|T(x \otimes y)T^{-1}\| \le \|x \otimes y\| \quad (x, y \in \mathscr{H}).$$

This implies that $\|T\|\,\|T^{-1}\| \le 1$. Hence,

$$\|T\|\,\|x\| \le \frac{\|x\|}{\|T^{-1}\|} \le \|Tx\| \le \|T\|\,\|x\| \quad (x \in \mathscr{H}),$$

which gives

$$\|Tx\| = \|T\|\,\|x\| \quad (x \in \mathscr{H}).$$

Now, the equivalence (i)⇔(ii) of Theorem 2.4 gives (i). □

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3. Approximately angle preserving mappings

Our aim in this section is to characterise approximately angle preserving mappings. The following lemma follows immediately from Definition 1.1.

LEMMA 3.1. *Let $T : \mathscr{H} \longrightarrow \mathscr{K}$ be a linear map and let $c \in (-1, 1)$. Then the following statements hold:*

(i) $\widehat{\varepsilon}(T, c) = \sup\{|\langle Tx, Ty\rangle/\|Tx\|\,\|Ty\| - c| : x \angle_c y,\ x, y \in \mathscr{H} \setminus \{0\}\}$*;*
(ii) $\widehat{\varepsilon}(T, c) = \sup\{|\langle Tx, Ty\rangle/\|Tx\|\,\|Ty\| - c| : x \angle_c y,\ \|x\| = \|y\| = 1,\ x, y \in \mathscr{H}\}$.

Our next theorem is a generalisation of [13, Lemma 2.2].

THEOREM 3.2. *Let $T : \mathscr{H} \longrightarrow \mathscr{K}$ be a nonzero linear map and let $c \in (-1, 1)$. If $[T] = 0$, then $\widehat{\varepsilon}(T, c) = 1 + |c|$.*

PROOF. Since $\widehat{\varepsilon}(T, -c) = \widehat{\varepsilon}(T, c)$, we may assume that $c \in [0, 1)$. We consider two cases.

Case 1. T is not injective.

There exists a subspace $\mathscr{H}_1$ such that $2 \leq \dim \mathscr{H}_1 < \infty$ and $T|_{\mathscr{H}_1}$ is not injective, that is, $\{0\} \neq \ker(T|_{\mathscr{H}_1}) \neq \mathscr{H}_1$. (Indeed, if T is injective on every finite-dimensional subspace, then T has to be injective.) Since the set $\ker(T|_{\mathscr{H}_1})$ is not dense, we can find two vectors $x \in (\ker(T|_{\mathscr{H}_1}))^{\perp}$, $y \in \ker(T|_{\mathscr{H}_1})$ such that $\|x\| = \|y\| = 1$ and $x \perp y$. By Lemma 2.1(i),

$$\left|\frac{\langle T(x + \sqrt{\frac{1+c}{1-c}}y), T(-x + \sqrt{\frac{1+c}{1-c}}y)\rangle}{\|T(x + \sqrt{\frac{1+c}{1-c}}y)\|\,\|T(-x + \sqrt{\frac{1+c}{1-c}}y)\|} - c\right| = \left|\frac{-\|Tx\|^2}{\|Tx\|^2} - c\right| = 1 + c.$$

Thus, by Lemma 3.1(i), $\widehat{\varepsilon}(T, c) = 1 + c$.

Case 2. T is injective.

Assume that $\widehat{\varepsilon}(T, c) < 1 + c$. Then there exists $\varepsilon_0 < 1 + c$ such that T is an (ε_0, c)-angle preserving mapping. Consider arbitrary unit vectors $x, y \in \mathscr{H}$. If x and y are linearly dependent, then $\sqrt{(1 - c)(1 + c - \varepsilon_0)/(1 + c)(1 - c + \varepsilon_0)}\|Ty\| \leq \|Tx\|$. If x and y are linearly independent, then, by Corollary 2.3, there exist unit vectors x_1, x_2 such that

$$x_1 \perp x_2, \quad Tx_1 \perp Tx_2, \quad \|Tx_1\| \leq \|Tx\| \leq \|Tx_2\| \quad \text{and} \quad \|Tx_1\| \leq \|Ty\| \leq \|Tx_2\|. \tag{3.1}$$

So, by Lemma 2.1(ii),

$$T\Big(x_1 + \sqrt{\frac{1-c}{1+c}}x_2\Big) \angle_c^{\varepsilon_0} T\Big(x_1 - \sqrt{\frac{1-c}{1+c}}x_2\Big).$$

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Put $u = x_1 + \sqrt{1-c/1+c}x_2$ and $v = x_1 - \sqrt{1-c/1+c}x_2$. Then

$$\begin{aligned}
-\|Tx_1\|^2 + \frac{1-c}{1+c}\|Tx_2\|^2 + c\Big(\|Tx_1\|^2 + \frac{1-c}{1+c}\|Tx_2\|^2\Big) \\
&\le \Big|\|Tx_1\|^2 - \frac{1-c}{1+c}\|Tx_2\|^2 - c\Big(\|Tx_1\|^2 + \frac{1-c}{1+c}\|Tx_2\|^2\Big)\Big| \\
&= |\langle Tu, Tv\rangle - c\|Tu\|\,\|Tv\|| \\
&\le \varepsilon_0\|Tu\|\,\|Tv\| \\
&= \varepsilon_0\Big\|T\Big(x_1 + \sqrt{\frac{1-c}{1+c}}x_2\Big)\Big\|\,\Big\|T\Big(x_1 - \sqrt{\frac{1-c}{1+c}}x_2\Big)\Big\| \\
&\le \varepsilon_0\Big(\|Tx_1\|^2 + \frac{1-c}{1+c}\|Tx_2\|^2\Big).
\end{aligned}$$

Hence,

$$-\|Tx_1\|^2 + \frac{1-c}{1+c}\|Tx_2\|^2 + c\Big(\|Tx_1\|^2 + \frac{1-c}{1+c}\|Tx_2\|^2\Big) \le \varepsilon_0\Big(\|Tx_1\|^2 + \frac{1-c}{1+c}\|Tx_2\|^2\Big)$$

or, equivalently,

$$\sqrt{\frac{(1-c)(1+c-\varepsilon_0)}{(1+c)(1-c+\varepsilon_0)}}\|Tx_2\| \le \|Tx_1\|. \tag{3.2}$$

By combining (3.1) and (3.2),

$$\sqrt{\frac{(1-c)(1+c-\varepsilon_0)}{(1+c)(1-c+\varepsilon_0)}}\|Ty\| \le \sqrt{\frac{(1-c)(1+c-\varepsilon_0)}{(1+c)(1-c+\varepsilon_0)}}\|Tx_2\| \le \|Tx_1\| \le \|Tx\|.$$

By passing to the supremum over y and to the infimum over x in the above inequality, we obtain $\sqrt{(1-c)(1+c-\varepsilon_0)/(1+c)(1-c+\varepsilon_0)}\|T\| \le [T]$. Since $\|T\| > 0$ and $[T] = 0$, this yields $\varepsilon_0 = 1 + c$. This contradiction shows that $\widehat{\varepsilon}(T, c) = 1 + c$. □

Next, we formulate one of our main results.

Theorem 3.3. *Let $c \in (-1, 1)$. Suppose that $T \in \mathbb{B}(\mathscr{H}, \mathscr{K})$ and $[T] \neq 0$. Then*

$$\widehat{\varepsilon}(T, c) = \frac{(1-|c|^2)(\|T\|^2 - [T]^2)}{(1+|c|)\|T\|^2 + (1-|c|)[T]^2}.$$

Proof. We may assume that $c \in [0, 1)$. Since $[T] > 0$, there exist unit vectors x_1, x_2 such that

$$x_1 \perp x_2, \quad Tx_1 \perp Tx_2, \quad [T] = \|Tx_1\| \quad \text{and} \quad \|Tx_2\| = \|T\|. \tag{3.3}$$

It follows from Lemma 2.1(ii) that $\big(x_2 + \sqrt{(1-c)/(1+c)}x_1\big) \angle_c \big(x_2 - \sqrt{(1-c)/(1+c)}x_1\big)$.

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Now, let us put $u = x_2 + \sqrt{(1-c)/(1+c)}x_1$ and $v = x_2 - \sqrt{(1-c)/(1+c)}x_1$. By (3.3),

$$\begin{aligned}
\left|\frac{\langle Tu, Tv\rangle}{\|Tu\|\,\|Tv\|} - c\right| &= \left|\frac{\|Tx_2\|^2 - \frac{1-c}{1+c}\|Tx_1\|^2}{\|Tx_2\|^2 + \frac{1-c}{1+c}\|Tx_1\|^2} - c\right| \\
&= \left|\frac{\|T\|^2 - \frac{1-c}{1+c}[T]^2}{\|T\|^2 + \frac{1-c}{1+c}[T]^2} - c\right| \\
&= \left|\frac{(1+c)\|T\|^2 - (1-c)[T]^2}{(1+c)\|T\|^2 + (1-c)[T]^2} - c\right| \\
&= \frac{(1-c^2)(\|T\|^2 - [T]^2)}{(1+c)\|T\|^2 + (1-c)[T]^2}.
\end{aligned}$$

By Lemma 3.1(i),

$$\widehat{\varepsilon}(T, c) \geq \frac{(1-c^2)(\|T\|^2 - [T]^2)}{(1+c)\|T\|^2 + (1-c)[T]^2}. \tag{3.4}$$

On the other hand, let $x, y \in \mathscr{H}$ be such that $x \angle_c y$ and $\|x\| = \|y\| = 1$. Then

$$\begin{aligned}
\left\|\frac{Tx}{\|Tx\|} + \frac{Ty}{\|Ty\|}\right\|^2 &= \left\|T\Big(\frac{x}{\|Tx\|} + \frac{y}{\|Ty\|}\Big)\right\|^2 \\
&\leq \|T\|^2\left\|\frac{x}{\|Tx\|} + \frac{y}{\|Ty\|}\right\|^2 \\
&= \|T\|^2\Big(\frac{1}{\|Tx\|^2} + \frac{1}{\|Ty\|^2} + \frac{2c}{\|Tx\|\,\|Ty\|}\Big),
\end{aligned}$$

whence

$$\left\|\frac{Tx}{\|Tx\|} + \frac{Ty}{\|Ty\|}\right\|^2 \leq \|T\|^2\Big(\frac{1}{\|Tx\|^2} + \frac{1}{\|Ty\|^2} + \frac{2c}{\|Tx\|\,\|Ty\|}\Big). \tag{3.5}$$

Similarly,

$$\left\|\frac{Tx}{\|Tx\|} - \frac{Ty}{\|Ty\|}\right\|^2 \geq [T]^2\Big(\frac{1}{\|Tx\|^2} + \frac{1}{\|Ty\|^2} - \frac{2c}{\|Tx\|\,\|Ty\|}\Big)$$

and

$$\left\|\frac{Tx}{\|Tx\|} - \frac{Ty}{\|Ty\|}\right\|^2 \leq \|T\|^2\Big(\frac{1}{\|Tx\|^2} + \frac{1}{\|Ty\|^2} - \frac{2c}{\|Tx\|\,\|Ty\|}\Big).$$

Now, let

$$\left\|\frac{Tx}{\|Tx\|} - \frac{Ty}{\|Ty\|}\right\|^2 = \mu[T]^2\Big(\frac{1}{\|Tx\|^2} + \frac{1}{\|Ty\|^2} - \frac{2c}{\|Tx\|\,\|Ty\|}\Big) \tag{3.6}$$

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with $1 \le \mu \le \|T\|/[T]$. It follows from (3.5) and (3.6) that

$$\begin{aligned}
4 &= \Big\|\frac{Tx}{\|Tx\|} + \frac{Ty}{\|Ty\|}\Big\|^2 + \Big\|\frac{Tx}{\|Tx\|} - \frac{Ty}{\|Ty\|}\Big\|^2 \\
&\le (\|T\|^2 + \mu[T]^2)\Big(\frac{1}{\|Tx\|^2} + \frac{1}{\|Ty\|^2}\Big) + (\|T\|^2 - \mu[T]^2)\frac{2c}{\|Tx\|\,\|Ty\|} \\
&\le (\|T\|^2 + \mu[T]^2)\Big(\frac{1}{\|Tx\|^2} + \frac{1}{\|Ty\|^2}\Big) + c(\|T\|^2 - \mu[T]^2)\Big(\frac{1}{\|Tx\|^2} + \frac{1}{\|Ty\|^2}\Big) \\
&= \Big((1+c)\|T\|^2 + (1-c)\mu[T]^2\Big)\Big(\frac{1}{\|Tx\|^2} + \frac{1}{\|Ty\|^2}\Big).
\end{aligned}$$

Hence,

$$\frac{1}{\|Tx\|^2} + \frac{1}{\|Ty\|^2} \ge \frac{4}{(1+c)\|T\|^2 + (1-c)\mu[T]^2}. \tag{3.7}$$

From (3.6) and (3.7),

$$\begin{aligned}
\Big\langle \frac{Tx}{\|Tx\|}, \frac{Ty}{\|Ty\|}\Big\rangle - c &= 1 - c - \frac{1}{2}\Big\|\frac{Tx}{\|Tx\|} - \frac{Ty}{\|Ty\|}\Big\|^2 \\
&= 1 - c + \frac{\mu c[T]^2}{\|Tx\|\,\|Ty\|} - \frac{1}{2}\mu[T]^2\Big(\frac{1}{\|Tx\|^2} + \frac{1}{\|Ty\|^2}\Big) \\
&\le 1 - c + \frac{1}{2}\mu c[T]^2\Big(\frac{1}{\|Tx\|^2} + \frac{1}{\|Ty\|^2}\Big) - \frac{1}{2}\mu[T]^2\Big(\frac{1}{\|Tx\|^2} + \frac{1}{\|Ty\|^2}\Big) \\
&= 1 - c - \frac{1}{2}(1-c)\mu[T]^2\Big(\frac{1}{\|Tx\|^2} + \frac{1}{\|Ty\|^2}\Big) \\
&\le 1 - c - \frac{2(1-c)\mu[T]^2}{(1+c)\|T\|^2 + (1-c)\mu[T]^2} \\
&= \frac{(1-c^2)\|T\|^2 - (1-c^2)\mu[T]^2}{(1+c)\|T\|^2 + (1-c)\mu[T]^2} \quad \text{(since } 1 \le \mu\text{)} \\
&\le \frac{(1-c^2)(\|T\|^2 - [T]^2)}{(1+c)\|T\|^2 + (1-c)[T]^2}.
\end{aligned}$$

Hence,

$$\sup\Big\{\Big|\frac{\langle Tx, Ty\rangle}{\|Tx\|\,\|Ty\|} - c\Big| : x \angle_c y,\ \|x\| = \|y\| = 1,\ x, y \in \mathscr{H}\Big\} \le \frac{(1-c^2)(\|T\|^2 - [T]^2)}{(1+c)\|T\|^2 + (1-c)[T]^2}.$$

From Lemma 3.1(ii),

$$\widehat{\varepsilon}(T, c) \le \frac{(1-c^2)(\|T\|^2 - [T]^2)}{(1+c)\|T\|^2 + (1-c)[T]^2}. \tag{3.8}$$

Combining (3.4) and (3.8) gives the formula for $\widehat{\varepsilon}(T, c)$. □

As an immediate consequence of Theorem 3.3, we get a characterisation of the (ε, c)-angle preserving mappings.

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Corollary 3.4. *Suppose that* $T \in \mathbb{B}(\mathscr{H}, \mathscr{K}) \setminus \{0\}$ *and* $c \in (-1, 1)$. *Then there exists an* $\varepsilon \in [0, 1 + |c|)$ *such that* T *is an* (ε, c)*-angle preserving mapping if and only if* T *is bounded below.*

Corollary 3.5. *Let* $c \in (-1, 1)$ *and* $\varepsilon \in [0, 1 + |c|)$. *Suppose that* $T \in \mathbb{B}(\mathscr{H}, \mathscr{K}) \setminus \{0\}$ *is an* (ε, c)*-angle preserving mapping. Then* T *is injective and the following statements hold:*

(i) $\sqrt{(1 + |c|)(1 - |c| - \varepsilon)/(1 - |c|)(1 + |c| + \varepsilon)}\|T\| \le [T]$;
(ii) $\sqrt{(1 + |c|)(1 - |c| - \varepsilon)/(1 - |c|)(1 + |c| + \varepsilon)}\|Tx\|\,\|y\| \le \|Ty\|\,\|x\| \quad (x, y \in \mathscr{H})$;
(iii) $\sqrt{(1 + |c|)(1 - |c| - \varepsilon)/(1 - |c|)(1 + |c| + \varepsilon)}\|T\|\,\|x\| \le \|Tx\| \le \|T\|\,\|x\| \quad (x \in \mathscr{H})$.

Proof. Since T is an (ε, c)-angle preserving mapping, we have $\widehat{\varepsilon}(T, c) < 1 + |c|$. Theorem 3.2 ensures that T is injective. From Theorem 3.3,

$$\widehat{\varepsilon}(T, c) = \frac{(1 - |c|^2)(\|T\|^2 - [T]^2)}{(1 + |c|)\|T\|^2 + (1 - |c|)[T]^2} \le \varepsilon$$

or, equivalently,

$$\sqrt{\frac{(1 + |c|)(1 - |c| - \varepsilon)}{(1 - |c|)(1 + |c| + \varepsilon)}}\|T\| \le [T].$$

From the above inequality, for $x, y \in \mathscr{H}$,

$$\sqrt{\frac{(1 + |c|)(1 - |c| - \varepsilon)}{(1 - |c|)(1 + |c| + \varepsilon)}}\|Tx\|\,\|y\| \le \sqrt{\frac{(1 + |c|)(1 - |c| - \varepsilon)}{(1 - |c|)(1 + |c| + \varepsilon)}}\|T\|\,\|x\|\,\|y\| \le [T]\,\|y\|\,\|x\| \le \|Ty\|\,\|x\|$$

and

$$\sqrt{\frac{(1 + |c|)(1 - |c| - \varepsilon)}{(1 - |c|)(1 + |c| + \varepsilon)}}\|T\|\,\|x\| \le [T]\,\|x\| \le \|Tx\| \le \|T\|\,\|x\|.$$ □

Corollary 3.6. *Let* $c \in (-1, 1)$. *For* $T, S \in \mathbb{B}(\mathscr{H}) \setminus \{0\}$, *the following statements hold:*

(i) *if* T, S *are left invertible, then* $\widehat{\varepsilon}(ST, c) < 1 + |c|$;
(ii) *if* S *is a scalar multiple of an isometry, then* $\widehat{\varepsilon}(ST, c) = \widehat{\varepsilon}(T, c)$;
(iii) *if* $T^{-1} \in \mathbb{B}(\mathscr{H}) \setminus \{0\}$, *then* $\widehat{\varepsilon}(T^{-1}, c) = \widehat{\varepsilon}(T, c)$.

Proof. (i) Since T and S are left invertible, $[TS] \ge [T]\,[S] > 0$ and, by Theorem 3.3, $\widehat{\varepsilon}(TS, c) < 1 + |c|$.

(ii) This follows because $\|S\| = [S]$, $\|ST\| = \|S\|\,\|T\|$ and $[ST] = [S]\,[T]$.

(iii) To see this, note that $\|T^{-1}\| = 1/[T]$ and $[T^{-1}] = 1/\|T\|$. □

The next corollary gives another property of the function $\widehat{\varepsilon}(., c)$.

Corollary 3.7. *Let* $c \in (-1, 1)$. *The function* $T \mapsto \widehat{\varepsilon}(T, c)$ *is norm continuous at each* $T \in \mathbb{B}(\mathscr{H}, \mathscr{K})$ *with* $[T] > 0$.

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Proof. Suppose that $T_n \in \mathbb{B}(\mathscr{H}, \mathscr{K})$ are such that $\lim_{n\to\infty} \|T_n - T\| = 0$. Since $T \neq 0$, we may assume that $T_n \neq 0$ for all $n \in \mathbb{N}$. Then

$$\lim_{n\to\infty} \|T_n\| = \|T\|, \quad \lim_{n\to\infty} [T_n] = [T] \quad \text{and} \quad (1+c)\|T_n\|^2 + (1-c)[T_n]^2 \neq 0.$$

Thus, by Theorem 3.3,

$$\lim_{n\to\infty} \widehat{\varepsilon}(T_n, c) = \lim_{n\to\infty} \frac{(1-|c|^2)(\|T_n\|^2 - [T_n]^2)}{(1+|c|)\|T_n\|^2 + (1-|c|)[T_n]^2} = \frac{(1-|c|^2)(\|T\|^2 - [T]^2)}{(1+|c|)\|T\|^2 + (1-|c|)[T]^2} = \widehat{\varepsilon}(T, c).$$ □

Remark 3.8. The function $\widehat{\varepsilon}(., c)$ is not continuous at 0 even in the case $c = 0$. Take any mapping T which is not orthogonality preserving. Thus, $\widehat{\varepsilon}(T, c) \neq 0$. Let $T_n = T/n$. Then $\lim_{n\to\infty} \|T_n\| = 0$, but, for every n, $\widehat{\varepsilon}(T_n, c) = \widehat{\varepsilon}(T, c) \neq 0$ (see [13, Remark 2.7]).

Next, we prove that every injective operator preserves approximate orthogonality.

Theorem 3.9. *Suppose that $T \in \mathbb{B}(\mathscr{H}, \mathscr{K})$ and $0 < [T] \leq \|T\|$. Then T satisfies*

$$x \perp y \Longrightarrow Tx \perp^{\varepsilon_T} Ty \quad (x, y \in \mathscr{H})$$

with $\varepsilon_T = 1 - [T]^2/\|T\|^2$.

Proof. Fix two arbitrary nonzero vectors $x, y \in \mathscr{H}$ such that $x \perp y$. Since $0 < [T]$, it follows that T is injective. From Corollary 2.3, there exist unit vectors $a, b \in \operatorname{span}\{x, y\}$ such that

$$a \perp b, \quad Ta \perp Tb, \quad \|Ta\| \leq \|Tx\| \leq \|Tb\| \quad \text{and} \quad \|Ta\| \leq \|Ty\| \leq \|Tb\|. \tag{3.9}$$

Moreover, there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that $x = \alpha a + \beta b$, $y = \gamma a + \delta b$. Since $x \perp y$,

$$\alpha\gamma = -\beta\delta. \tag{3.10}$$

Furthermore, $Tx = \alpha Ta + \beta Tb$ and $Ty = \gamma Ta + \delta Tb$. If $\alpha\beta\gamma\delta = 0$, then it is easy to see that $\langle Tx, Ty\rangle = 0$ and, in particular, $Tx \perp^{\varepsilon_T} Ty$. So, now suppose that $\alpha\beta\gamma\delta \neq 0$. Denote $\theta := \alpha/\beta = -\delta/\gamma$. It follows from (3.9) and (3.10) that

$$\begin{aligned}
\frac{|\langle Tx, Ty\rangle|}{\|Tx\|\,\|Ty\|} &= \frac{|\alpha\gamma\|Ta\|^2 + \beta\delta\|Tb\|^2|}{\sqrt{|\alpha|^2\|Ta\|^2 + |\beta|^2\|Tb\|^2}\,\sqrt{|\gamma|^2\|Ta\|^2 + |\delta|^2\|Tb\|^2}} \\
&= \frac{(\|Tb\|^2 - \|Ta\|^2)|\alpha\gamma|}{\sqrt{|\alpha|^2\|Ta\|^2 + |\beta|^2\|Tb\|^2}\,\sqrt{|\gamma|^2\|Ta\|^2 + |\delta|^2\|Tb\|^2}} \\
&= \frac{1 - \frac{\|Ta\|^2}{\|Tb\|^2}}{\sqrt{\frac{\|Ta\|^2}{\|Tb\|^2} + \frac{1}{|\theta|^2}}\,\sqrt{\frac{\|Ta\|^2}{\|Tb\|^2} + |\theta|^2}} \\
&\leq \frac{1 - \frac{\|Ta\|^2}{\|Tb\|^2}}{\sqrt{\frac{\|Ta\|^4}{\|Tb\|^4} + 1}} \leq 1 - \frac{\|Ta\|^2}{\|Tb\|^2} \leq 1 - \frac{[T]^2}{\|T\|^2} = \varepsilon_T,
\end{aligned}$$

whence $|\langle Tx, Ty\rangle| \leq \varepsilon_T \|Tx\|\,\|Ty\|$. Thus, $Tx \perp^{\varepsilon_T} Ty$. □

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The following result can be considered an extension of Theorem 3.9. More precisely, we show that every injective operator approximately preserves the inner product.

THEOREM 3.10. *Assume that* $\dim \mathscr{H} < \infty$. *Suppose that* $T \in \mathbb{B}(\mathscr{H}, \mathscr{K})$ *and* $0 < [T]$. *Then there exists* γ *such that* T *satisfies*

$$|\langle Tx, Ty\rangle - \gamma\langle x, y\rangle| \leq \Big(1 - \frac{[T]^2}{\|T\|^2}\Big)\|T\|^2\,\|x\|\,\|y\| \quad (x, y \in \mathscr{H}). \tag{3.11}$$

Moreover, $[T]^2 \leq |\gamma| \leq 2\|T\|^2 - [T]^2$.

PROOF. Combining Theorem 3.9 and [11, Theorem 5.5], we immediately get (3.11). Fix $u \in \mathscr{H}$ such that $\|u\| = 1$. Putting u in place of x and y in (3.11) gives $|\|Tu\|^2 - \gamma| \leq (1 - [T]^2/\|T\|^2)\,\|T\|^2$. Choosing u as an arbitrary unit vector and passing to the supremum and infimum over $\|u\| = 1$ gives $[T]^2 \leq |\gamma| \leq 2\|T\|^2 - [T]^2$. □

To end this paper, we show that in the finite-dimensional case Corollary 3.5 can be strengthened. Indeed, as an immediate consequence of Corollary 3.5 and Theorem 3.10, we obtain the following result.

COROLLARY 3.11. *Let* $c \in (-1, 1)$ *and* $\varepsilon \in [0, 1 + |c|)$. *Suppose that* $T \in \mathbb{B}(\mathscr{H}, \mathscr{K}) \setminus \{0\}$ *is an* (ε, c)*-angle preserving mapping. Assume that* $\dim \mathscr{H} < \infty$. *Then there exists* γ *such that* T *satisfies*

$$|\langle Tx, Ty\rangle - \gamma\langle x, y\rangle| \leq \Big(1 - \frac{(1 + |c|)(1 - |c| - \varepsilon)}{(1 - |c|)(1 + |c| + \varepsilon)}\Big)\|T\|^2\,\|x\|\,\|y\| \quad (x, y \in \mathscr{H}).$$

Moreover, $[T]^2 \leq |\gamma| \leq 2\|T\|^2 - [T]^2$.

PROOF. By Corollary 3.5, T is injective and, since $\dim \mathscr{H} < \infty$, $[T] > 0$. The desired conclusion follows from Theorem 3.10. □

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