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OPERATOR SYSTEM NUCLEARITY VIA C^*-ENVELOPES

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Abstract

We prove that an operator system is (min, ess)-nuclear if its C^*-envelope is nuclear. This allows us to deduce that an operator system associated to a generating set of a countable discrete group by Farenick *et al.* ['Operator systems from discrete groups', *Comm. Math. Phys.* **329**(1) (2014), 207–238] is (min, ess)-nuclear if and only if the group is amenable. We also make a detailed comparison between ess and other operator system tensor products and show that an operator system associated to a minimal generating set of a finitely generated discrete group (respectively, a finite graph) is (min, max)-nuclear if and only if the group is of order less than or equal to three (respectively, every component of the graph is complete).

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1. Introduction

An operator system is a self-adjoint unital subspace of $B(\mathcal{H})$ for some complex Hilbert space $\mathcal{H}$. Choi and Effros [5] obtained an abstract characterization of an operator system and quite recently this abstraction proved very useful in the development of the theory of tensor products in the category of operator systems in a series of papers [7, 9, 13–16]. A short survey of this development is available in [10, Ch. 4]. Essentially, a lattice of five tensor products of operator systems consisting of the so-called *minimal* (min), the *maximal* (max), the *maximal commuting* (c), the *left enveloping* (el) and the *right enveloping* (er) tensor products was introduced in [15] and, unlike the category of C^*-algebras, a variety of nuclearity considerations were made, namely, given two operator system tensor products α and β, an operator system

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$\mathcal{S}$ is said to be (α,β)-nuclear if $\mathcal{S}\otimes_\alpha\mathcal{T}=\mathcal{S}\otimes_\beta\mathcal{T}$ for every operator system $\mathcal{T}$. In the last few years, various characterizations of these notions of nuclearity among the primary tensor products in terms of some intrinsic properties of operator systems, namely, *exactness*, *weak expectation property (WEP)*, *double commutant expectation property (DCEP)*, *operator system local lifting property (OSLLP)* and *completely positive factorization property (CPFP)* have been established [7, 9, 13–16]; also, it is known that the maximum nuclearity expected for a finite-dimensional operator system which is not isomorphic to a C^*-algebra is (min, c)-nuclearity, equivalently, C^*-nuclearity [14, 15].

Very recently, Farenick *et al.* in [8] associated an operator system $\mathcal{S}(\mathfrak{u})$ to every generating set $\mathfrak{u}$ of a discrete group G and, based on the developments of [7, 9, 14], they could relate the longstanding operator algebraic questions of *Connes' embedding problem* and *Kirchberg's conjecture* to the tensor products of such operator systems associated to the free group with two generators. In the same paper they also introduced and initiated the study of the natural operator system tensor product 'ess' arising from the enveloping C^*-algebras, namely,

$$\mathcal{S}\otimes_{\text{ess}}\mathcal{T}\subseteq C^*_e(\mathcal{S})\otimes_{\max} C^*_e(\mathcal{T}).$$

It was shown in [14, Section 10] that the $(2n+1)$-dimensional operator system $\mathcal{S}_n$ associated to the universal generating set of the free group with n generators, for $n\geq 2$, is not exact and hence $\mathcal{S}_n$ is not C^*-nuclear. This also illustrates that unlike C^*-algebras, finite-dimensional operator systems need not be nuclear.

This paper came into existence out of the curiosity to understand nuclearity properties of the operator systems associated to amenable discrete groups. Lance proved that the group C^*-algebra of a discrete group is nuclear if and only if the group is amenable [18]. Also, if $\mathfrak{u}$ is a generating set of a discrete group G and the associated operator system $\mathcal{S}(\mathfrak{u})$ is a C^*-nuclear operator system, then, by [16, Corollary 9.6], the group C^*-algebra of G is a nuclear C^*-algebra and, therefore, the group G is amenable. However, it is not yet clear (at least, to us) whether the converse holds or not. Making some progress in this direction, we deduce in Section 5 that for any generating set $\mathfrak{u}$ of a discrete group G, $\mathcal{S}(\mathfrak{u})$ is (min, ess)-nuclear if and only if the group is amenable.

The tool that helps us to achieve this is the notion of the C^*-envelope of an operator system, which was introduced by Arveson in [3] and whose existence for an arbitrary operator system was first established by Hamana in [11].

It is known that, in general, nuclearity does not pass to C^*-subalgebras [4, Remark 4.4.4]. In particular, the same therefore holds for (min, c)-nuclearity of operator systems. The relationship between an operator system and its C^*-envelope is equally mysterious, which we highlight briefly at the beginning of Section 4. Thereafter, we are able to see that the nuclearity of the C^*-envelope behaves well if we restrict to (min, ess)-nuclearity. Moreover, we prove that if an operator system contains enough unitaries of its C^*-envelope, then its (min, ess)-nuclearity is equivalent to the nuclearity of its C^*-envelope.

After a short section on preliminaries in Section 2, we first make some comparisons between the ess operator system tensor product and other operator system tensor products in Section 3.

In Section 5, apart from the characterization of (min, ess)-nuclearity of the group operator system $\mathcal{S}(\mathfrak{u})$ in terms of amenability of the group, we also provide an exhaustive list of (min, max)-nuclear operator systems associated to minimal generating sets of finitely generated groups.

In Section 6, we characterize (min, max)-nuclear graph operator systems (as introduced in [15]) for finite graphs, purely in terms of graph-theoretic properties. We achieve this characterization using an identification of their C^*-envelopes obtained in [19].

Finally, in Section 7, as yet another application of our results on C^*-envelopes, we discuss nuclearity-related properties of some known examples of operator systems (from [1, 2, 8]) whose C^*-envelopes are either known or whose nuclearity can be easily deduced.

2. Preliminaries

For the basics on operator systems, we refer the reader to [6, 20]. And, in order to avoid repetition, we will freely borrow and follow terminologies and notations for tensor products of operator systems and nuclearity-related properties from [14–16]. However, for the sake of completeness we include certain definitions and results that will be used subsequently.

A *C^*-cover* [11, Section 2] of an operator system $\mathcal{S}$ is a pair (A, i) consisting of a unital C^*-algebra A and a complete order embedding $i : \mathcal{S} \to A$ such that $i(A)$ generates the C^*-algebra A. An *essential extension* [11, Section 2] of an operator system $\mathcal{S}$ is a pair $(\mathcal{W}, j)$ consisting of an operator system $\mathcal{W}$ and a complete order embedding $j : \mathcal{S} \to \mathcal{W}$ such that for any operator system $\mathcal{T}$ and any unital completely positive map $\varphi : \mathcal{W} \to \mathcal{T}$, φ is a complete order embedding whenever $\varphi \circ j$ is. The *injective envelope* [5], denoted by $I(\mathcal{S})$, of an operator system $\mathcal{S}$ is the minimal injective operator system that contains $\mathcal{S}$. Its existence and uniqueness were first proved by Hamana in [11].

We will work with Hamana's version of a C^*-envelope [11], according to which the C^*-envelope of an operator system $\mathcal{S}$ is a C^*-cover defined as the C^*-algebra generated by $\mathcal{S}$ in its injective envelope $I(\mathcal{S})$ and is denoted by $C^*_e(\mathcal{S})$. It is known that the injective envelope $I(\mathcal{S})$ is an essential extension of $\mathcal{S}$ [11, Section 3] and the C^*-envelope $C^*_e(\mathcal{S})$ enjoys the following universal 'minimality' property.

Identifying $\mathcal{S}$ with its image in $C^*_e(\mathcal{S})$, for any C^*-cover (A, i) of $\mathcal{S}$, there is a unique surjective unital $*$-homomorphism $\pi : A \to C^*_e(\mathcal{S})$ such that $\pi(i(s)) = s$ for every s in $\mathcal{S}$ [11, Corollary 4.2].

The following proposition, which is folklore, is an immediate consequence of the above universality and will be used quite often in the coming sections.

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Proposition 2.1. *If an operator system $\mathcal{S}$ possesses a nuclear C^*-cover, then $C_e^*(\mathcal{S})$ is nuclear.*

Proof. Using the universal property of C^*-envelopes and the fact that the quotient of a nuclear C^*-algebra is nuclear [4, Corollary 9.4.4], the statement follows. □

We now list some useful rigidity properties (among which (ii) was also pointed out in [14, Section 1]) and an immediate consequence of the universality of $C_e^*(\mathcal{S})$ and, even though these are folklore, we provide their details for the sake of completeness. We would like to thank K. H. Han for sharing alternate proofs of (i) and (ii), which we have included here for their brevity.

Proposition 2.2. *For an operator system $\mathcal{S}$, the following hold.*

(i) *Suppose that $C_e^*(\mathcal{S}) \subset B(\mathcal{H})$ and $\varphi : C_e^*(\mathcal{S}) \to B(\mathcal{H})$ is a unital completely positive map that fixes $\mathcal{S}$; then φ fixes $C_e^*(\mathcal{S})$.*

(ii) *If $\psi : C_e^*(\mathcal{S}) \to \mathcal{T}$ is a unital completely positive map into an operator system $\mathcal{T}$ such that $\psi_{|\mathcal{S}}$ is a complete order embedding, then ψ is a complete order embedding.*

(iii) *If $\mathcal{S}$ is unitally completely order isomorphic to a unital C^*-algebra, then $\mathcal{S} = C_e^*(\mathcal{S})$.*

Proof. By injectivity of $B(H)$, φ admits a unital completely positive extension $\tilde{\varphi} : I(\mathcal{S}) \to B(H)$ and, for the same reason, the inclusion $\iota : C_e^*(\mathcal{S}) \to I(\mathcal{S})$ also admits a unital completely positive extension $\tilde{\iota} : B(H) \to I(\mathcal{S})$. The composition $\tilde{\iota} \circ \tilde{\varphi} : I(\mathcal{S}) \to I(\mathcal{S})$ clearly fixes $\mathcal{S}$ and is unital completely positive; therefore, by rigidity of $I(\mathcal{S})$ (as in [11, Section 3]), it fixes the whole of $I(\mathcal{S})$; and, since φ agrees with $\tilde{\iota} \circ \tilde{\varphi}$ on $C_e^*(\mathcal{S})$, this proves (i).

Let $\mathcal{T} \subset B(H)$ for some Hilbert space H. Then, by injectivity of $B(H)$, the map $\psi : C_e^*(\mathcal{S}) \to \mathcal{T} \subset B(H)$ admits a unital completely positive extension $\tilde{\psi} : I(\mathcal{S}) \to B(H)$. Since $\tilde{\psi}_{|\mathcal{S}}$ is a complete order embedding and $\mathcal{S} \subset I(\mathcal{S})$ is an essential extension [11, Section 3], it follows that $\tilde{\psi}$ is a complete order embedding and hence (ii) follows.

For (iii), let $\varphi : \mathcal{S} \to A$ be a unital complete order isomorphism for some unital C^*-algebra A. Then (A, φ) being a C^*-cover of $\mathcal{S}$, by universality of $C_e^*(\mathcal{S})$, there exists a surjective unital $*$-homomorphism $\pi : A \to C_e^*(\mathcal{S})$ such that $\pi \circ \varphi(s) = s$ for every $s \in \mathcal{S}$, implying that $\mathcal{S} = C_e^*(\mathcal{S})$. □

We must remark here that Proposition 2.2(iii) turns out to be the main ingredient in the classification of nuclear operator systems associated to finitely generated groups and finite graphs. It also allows us to identify some nonnuclear finite-dimensional operator systems, as we will see in Section 7.

An operator subsystem $\mathcal{S}$ of a unital C^*-algebra A is said to contain enough unitaries of A if the unitaries in $\mathcal{S}$ generate A as a C^*-algebra [16, Section 9]. We will come across quite a few instances where we will have to appeal to the following useful result of Kavruk [14, Proposition 5.6].

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Proposition 2.3. *Suppose that $\mathcal{S} \subset A$ contains enough unitaries. Then, up to a $*$-isomorphism fixing $\mathcal{S}$, we have $A = C_e^*(\mathcal{S})$.*

In order to distinguish between operator system tensor products and C^*-tensor products, we will use $\otimes_{C^*\text{-min}}$ and $\otimes_{C^*\text{-max}}$ to denote the minimal and maximal C^*-tensor products, respectively.

Corollary 2.4. *Let $\mathcal{S} \subset A$ and $\mathcal{T} \subset B$ be operator systems containing enough unitaries of A and B, respectively. Then, up to a $*$-isomorphism fixing $\mathcal{S} \otimes_{\text{ess}} \mathcal{T}$, we have $C_e^*(\mathcal{S} \otimes_{\text{ess}} \mathcal{T}) = A \otimes_{C^*\text{-max}} B$.*

Proof. By the definition of $\otimes_{\text{ess}}$, we note that $\mathcal{S} \otimes_{\text{ess}} \mathcal{T}$ contains enough unitaries of $C_e^*(\mathcal{S}) \otimes_{C^*\text{-max}} C_e^*(\mathcal{T})$ and hence, by Proposition 2.3, there does exist a desired $*$-isomorphism. □

Apart from the C^*-envelope of an operator system, there is one more fundamental C^*-cover associated to an operator system $\mathcal{S}$, namely, the universal C^*-algebra $C_u^*(\mathcal{S})$ introduced by Kirchberg and Wassermann [17, Section 3]. We use this notion and operator system techniques to provide a proof of the following folklore result.

Proposition 2.5. *If A is a nonnuclear unital C^*-algebra, then there exists a unital C^*-algebra B such that there is no $*$-isomorphism between $A \otimes_{C^*\text{-min}} B$ and $A \otimes_{C^*\text{-max}} B$ fixing $A \otimes B$.*

Proof. Since A is a nonnuclear C^*-algebra, by [15, Corollary 6.8], there exists an operator system $\mathcal{S}$ such that $A \otimes_{\min} \mathcal{S} \neq A \otimes_{\max} \mathcal{S}$. Now, using injectivity of $\otimes_{\min}$, $A \otimes_{\min} \mathcal{S} \subseteq A \otimes_{\min} C_u^*(\mathcal{S})$ and, by definition of $\otimes_c$ and the fact that $\otimes_c$ coincides with $\otimes_{\max}$ if one of the factors in the tensor product is a C^*-algebra [15, Theorem 6.7], we have $A \otimes_{\max} \mathcal{S} = A \otimes_{\text{c}} \mathcal{S} \subseteq A \otimes_{\max} C_u^*(\mathcal{S})$, where the last complete order embedding is guaranteed by [16, Lemma 2.5]. Thus, $A \otimes_{\min} C_u^*(\mathcal{S}) \neq A \otimes_{\max} C_u^*(\mathcal{S})$ and [15, Theorem 5.12] then clearly implies that for the unital C^*-algebra $B = C_u^*(\mathcal{S})$, there does not exist a $*$-isomorphism between $A \otimes_{C^*\text{- min}} B$ and $A \otimes_{C^*\text{- max}} B$ fixing $A \otimes B$. □

Kirchberg had realized that exactness is quite a fundamental property in the category of C^*-algebras and over the years it has shown its prominence in the categories of operator spaces and operator systems as well. The notion of exactness saw its relevance in the theory of operator systems after Kavruk *et al.* developed an appropriate formalism of the notion of quotient of operator systems in [16, Section 3].

Given an operator system $\mathcal{S}$, a subspace $J \subseteq \mathcal{S}$ is said to be a *kernel* [16, Definition 3.2] if there exist an operator system $\mathcal{T}$ and a unital completely positive map $\phi : \mathcal{S} \to \mathcal{T}$ such that $J = \ker \phi$. For such a kernel $J \subset \mathcal{S}$, Kavruk *et al.* showed that the quotient space $\mathcal{S}/J$ forms an operator system [16, Proposition 3.4] with respect to the natural involution, whose positive cones are given by

$$\begin{aligned} C_n &= C_n(\mathcal{S}/J) \\ &= \{(s_{ij} + J)_{i,j} \in M_n(\mathcal{S}/J) : (s_{ij}) + \varepsilon(1 + J)_n \in \mathcal{D}_n \text{ for every } \varepsilon > 0\}, \end{aligned}$$

where

$$\mathcal{D}_n = \{(s_{ij} + J)_{i,j} \in M_n(\mathcal{S}/J) : (s_{ij} + y_{ij})_{i,j} \in M_n(\mathcal{S})^+ \text{ for some } y_{ij} \in J\}$$

and the Archimedean unit is the coset $1 + J$.

For an operator system $\mathcal{S}$, a unital C^*-algebra A and a closed ideal I in A, let $\mathcal{S} \bar{\otimes} I$ denote the closure of $\mathcal{S} \otimes I$ in the completion $\mathcal{S} \hat{\otimes}_{\min} A$ of the minimal tensor product $\mathcal{S} \otimes_{\min} A$. Then $\mathcal{S} \bar{\otimes} I$ is a kernel in $\mathcal{S} \hat{\otimes}_{\min} A$ and the induced map $(\mathcal{S} \hat{\otimes}_{\min} A)/(\mathcal{S} \bar{\otimes} I) \to \mathcal{S} \hat{\otimes}_{\min} (A/I)$ is unital and completely positive [14, Section 4].

Definition 2.6 [16]. An operator system $\mathcal{S}$ is said to be exact if for every unital C^*-algebra A and a closed ideal I in A, the induced map

$$(\mathcal{S} \hat{\otimes}_{\min} A)/(\mathcal{S} \bar{\otimes} I) \to \mathcal{S} \hat{\otimes}_{\min} (A/I)$$

is a complete order isomorphism.

Exactness is one of the few intrinsic properties of operator systems that has been used as a tool, by Kavruk *et al.* (see for example [14]), in characterizing nuclearity properties of operator systems. Recall the term *nuclearity* for operator systems, a generalization from the category of C^*-algebras, which was introduced in [15, Section 3].

Given two operator system tensor products α and β, an operator system $\mathcal{S}$ is said to be (α,β)-*nuclear* if the identity map between $\mathcal{S} \otimes_\alpha \mathcal{T}$ and $\mathcal{S} \otimes_\beta \mathcal{T}$ is a complete order isomorphism for every operator system $\mathcal{T}$, that is,

$$\mathcal{S} \otimes_\alpha \mathcal{T} = \mathcal{S} \otimes_\beta \mathcal{T}.$$

Also, an operator system $\mathcal{S}$ is said to be C^*-*nuclear* if

$$\mathcal{S} \otimes_{\min} A = \mathcal{S} \otimes_{\max} A$$

for all unital C^*-algebras A.

The following characterizations established in [16, Section 5] and [14, Section 4] are used quite often.

Theorem 2.7. (i) *An operator system $\mathcal{S}$ is exact if and only if it is* $(\min, \mathrm{el})$-*nuclear.*

(ii) *Exactness passes to operator subsystems, that is, if $\mathcal{S}$ is exact, then every operator subsystem of $\mathcal{S}$ is exact. Conversely, if every finite-dimensional operator subsystem of $\mathcal{S}$ is exact, then $\mathcal{S}$ is exact.*

(iii) *An operator system $\mathcal{S}$ is* $(\min, \mathrm{c})$-*nuclear if and only if $\mathcal{S}$ is C^*-nuclear.*

(iv) *An operator system is* $(\mathrm{c}, \max)$-*nuclear if and only if it is unitally completely order isomorphic to a C^*-algebra.*

We now concentrate on the main class of operator systems that we study in this article, namely, the operator systems associated to generating sets of discrete groups. Let G denote a countable discrete group, $\mathfrak{u}$ denote a generating set of G and $\mathcal{S}(\mathfrak{u})$ denote the operator system associated to $\mathfrak{u}$ by Farenick *et al.* in [8], that is,

$\mathcal{S}(\mathfrak{u}) := \text{span}\{1, u, u^* : u \in \mathfrak{u}\} \subset C^*(G)$, where $C^*(G)$ denotes the full group C^*-algebra of the group G [21, Ch. 8]. It was shown in [8] that if $\mathfrak{u}$ is a generating set of the free group $\mathbb{F}_n$, then $\mathcal{S}(\mathfrak{u})$ is independent of the generating set $\mathfrak{u}$ and is simply denoted by $\mathcal{S}_n$. In general, such independence is not expected.

The following observation of [8] is immediate from Proposition 2.3 and plays a fundamental role in the analysis of nuclearity of operator systems associated to discrete groups.

Proposition 2.8. *Let $\mathfrak{u}$ be a generating set of a discrete group G. Then, up to a $*$-isomorphism that fixes the elements of $\mathcal{S}(\mathfrak{u})$, we have $C^*_e(\mathcal{S}(\mathfrak{u})) = C^*(G)$.*

Since the reduced group C^*-algebra is equally important as the full group C^*-algebra, given any generating set $\mathfrak{u}$ of a discrete group G, we associate another obvious operator system, namely, $\mathcal{S}_r(\mathfrak{u}) := \text{span}\{1, u, u^* : u \in \mathfrak{u}\} \subset C^*_r(G)$. In view of Proposition 2.3, analogous to Proposition 2.8 and the fact that G is amenable if and only if $C^*(G) = C^*_r(G)$ [4, 18], we have the following proposition.

Proposition 2.9. *Let $\mathfrak{u}$ be a generating set of a discrete group G. Then:*

(i) *up to a $*$-isomorphism fixing the elements of $\mathcal{S}_r(\mathfrak{u})$, we have $C^*_e(\mathcal{S}_r(\mathfrak{u})) = C^*_r(G)$;*
(ii) *G is amenable if and only if the identity map on $\mathfrak{u}$ extends to a complete order isomorphism between $\mathcal{S}(\mathfrak{u})$ and $\mathcal{S}_r(\mathfrak{u})$.*

Since finite-dimensional C^*-algebras are nuclear and since $C^*_r(\mathbb{F}_n)$ is exact [4], by Theorem 2.7 and Propositions 2.1, 2.9 and 2.3, we observe the following corollary.

Corollary 2.10. (i) *If $\mathfrak{u}$ is a generating set of a nonamenable discrete group G, then $\mathcal{S}(\mathfrak{u})$ and $\mathcal{S}_r(\mathfrak{u})$ do not possess nuclear C^*-covers.*
(ii) *In particular, the finite-dimensional operator systems $\mathcal{S}(\mathfrak{u})$ (for example, $\mathcal{S}_n$ for $n \geq 2$) and $\mathcal{S}_r(\mathfrak{u})$, for $|\mathfrak{u}| < \infty$, do not admit complete order embeddings into finite-dimensional C^*-algebras.*
(iii) *If $\mathfrak{u}$ is a generating set of a free group with $2 \leq |\mathfrak{u}| \leq \infty$, then $\mathcal{S}_r(\mathfrak{u})$ is exact and does not have any nuclear C^*-cover.*

As in the case of group algebras, Proposition 2.8 also suggests that two nonisomorphic group operator systems can have isomorphic C^*-envelopes: for example, consider the nonabelian group D_8 (that is, the Dihedral group of order eight) with presentation $D_8 = \langle a, b \mid a^2 = b^4 = 1, bab = a\rangle$ and the Quaternion group Q_8 with presentation $Q_8 = \langle x, y \mid x^4 = 1, x^2 = y^2, xyx = y\rangle$. It is well known that

$$C^*(D_8) = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{M}_2 = C^*(Q_8)$$

and yet $D_8 \not\cong Q_8$. Let $\mathfrak{u} = \{a, b\}$ and $\mathfrak{v} = \{x, y\}$ be generating sets of D_8 and Q_8, respectively; then $\mathfrak{u}$ and $\mathfrak{v}$ are both minimal generating sets, but $\mathcal{S}(\mathfrak{u}) \not\cong \mathcal{S}(\mathfrak{v})$ as $\dim(\mathcal{S}(\mathfrak{u})) = 4 \neq 5 = \dim(\mathcal{S}(\mathfrak{v}))$.

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3. A comparison between ess and other tensor products

Recall that, for operator systems $\mathcal{S}$ and $\mathcal{T}$, analogous to the commuting tensor product, their ess tensor product was defined, in [8], via the embedding $\mathcal{S} \otimes_{\rm ess} \mathcal{T} \subset C_e^*(\mathcal{S}) \otimes_{\max} C_e^*(\mathcal{T})$. It was also proved there, in Lemma 3.2, that the operator systems associated to free groups satisfy $\mathcal{S}_n \otimes_{\rm ess} \mathcal{S}_m = \mathcal{S}_n \otimes_c \mathcal{S}_m$ for all $n, m \geq 2$. Analogous to this, we will prove in this section that for operator systems associated to amenable groups the ess tensor product is identical with the maximal injective operator system tensor product 'e'. Before that, we first make some other useful observations about the tensor product ess and compare it with other tensor products.

Analogous to the behavior of $\otimes_{\rm c}$ in [15, Theorem 6.6], we have the following proposition.

Proposition 3.1. *For any two unital C^*-algebras A and B,*

$$A \otimes_{\rm ess} B = A \otimes_{\rm c} B = A \otimes_{\max} B.$$

Proof. By [6, Theorem 6.2.4], the injective envelope $I(A)$ has a canonical C^*-algebraic structure and the mapping $i_A : A \to I(A)$ is a C^*-algebraic isomorphism onto its image. In particular, we can assume that $A \subseteq I(A)$ and, since the C^*-envelope of A is the C^*-algebra generated by A in its injective envelope $I(A)$, we have $C_e^*(A) = A$. Similarly, $C_e^*(B) = B$. Hence, by the definition of $\otimes_{\rm ess}$ and the fact that $A \otimes_c B = A \otimes_{\max} B$ [15, Theorem 6.6], the assertion holds. □

Recall, from [15, Section 3], that for two operator system tensor products σ and τ, one says that $\sigma \leq \tau$ if for any two operator systems $\mathcal{S}$ and $\mathcal{T}$ the identity map from $\mathcal{S} \otimes_\tau \mathcal{T}$ onto $\mathcal{S} \otimes_\sigma \mathcal{T}$ is completely positive. With this notion, the following lattice structure on operator system tensor products is known [14–16]:

$$\min \leq e \leq {\rm el}, {\rm er} \leq {\rm c} \leq \max.$$

Also, it can be easily seen that for three operator system tensor products $\sigma \leq \tau \leq \rho$, an operator system $\mathcal{S}$ is (σ, ρ)-nuclear if and only if it is (σ, τ)- and (τ, ρ)-nuclear.

Further, an operator system tensor product α is said to be *functorial* if for operator systems $\mathcal{S}_1, \mathcal{S}_2, \mathcal{T}_1$ and $\mathcal{T}_2$ and unital completely positive maps $\phi : \mathcal{S}_1 \to \mathcal{S}_2$ and $\psi : \mathcal{T}_1 \to \mathcal{T}_2$, the associated map $\phi \otimes \psi : \mathcal{S}_1 \otimes_\alpha \mathcal{T}_1 \to \mathcal{S}_2 \otimes_\alpha \mathcal{T}_2$ is unital completely positive [15]. A tensor product α is said to be *symmetric* if the flip map $\theta : s \otimes t \mapsto t \otimes s$ extends to a unital complete order isomorphism from $\mathcal{S} \otimes_\alpha \mathcal{T}$ onto $\mathcal{T} \otimes_\alpha \mathcal{S}$, and *associative* if the natural isomorphism from $(\mathcal{S} \otimes \mathcal{T}) \otimes \mathcal{R}$ onto $\mathcal{S} \otimes (\mathcal{T} \otimes \mathcal{R})$ yields a complete order isomorphism from $(\mathcal{S} \otimes_\alpha \mathcal{T}) \otimes_\alpha \mathcal{R}$ onto $\mathcal{S} \otimes_\alpha (\mathcal{T} \otimes_\alpha \mathcal{R})$ for all operator systems $\mathcal{S}, \mathcal{T}$ and $\mathcal{R}$.

Proposition 3.2. *The operator system tensor product* ess *is symmetric and not functorial.*

Proof. Since max is symmetric [15, Theorem 5.5], the flip map

$$C_e^*(\mathcal{S}) \otimes_{\max} C_e^*(\mathcal{T}) \ni a \otimes b \xrightarrow{\Phi} b \otimes a \in C_e^*(\mathcal{T}) \otimes_{\max} C_e^*(\mathcal{S})$$

extends to a complete order isomorphism for any two operator systems $\mathcal{S}$ and $\mathcal{T}$ and, by definition of $\otimes_{\rm ess}$, the restriction of Φ to $\mathcal{S} \otimes \mathcal{T}$ implies the symmetry for $\otimes_{\rm ess}$.

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Then, in view of Proposition 3.1, the facts that c is the minimal symmetric and functorial extension of max which agrees with max on C^*-algebras [15, Theorem 6.7] and that ess $\leq$ c [8, Section 2] imply that ess is not functorial. □

Like the tensor product c, it is not known whether ess is associative or not. As applications of our main results, we will be able to make some more significant comparisons between ess and other tensor products in the following section.

We now review some basic facts about the dual of an operator system from [5, 16].

For an operator system $\mathcal{S}$, the Banach space dual $\mathcal{S}^d$ is a matrix ordered space with ordering

$$M_n(\mathcal{S}^d) \ni (f_{ij}) \geq 0 \text{ if and only if}$$
$$F : \mathcal{S} \to M_n \text{ given by } F(s) = (f_{ij}(s)) \text{ is completely positive.}$$

But $\mathcal{S}^d$ need not be an operator system, as it might not have an Archimedean ordered unit. However, if $\mathcal{S}$ is finite dimensional, the dual $\mathcal{S}^d$ possesses an Archimedean order unit and hence admits an operator system structure [5, Corollary 4.5]. It was shown in [16, Proposition 6.2] that $\mathcal{S}^{dd}$ is always an operator system with a canonical Archimedean ordered unit and the canonical inclusion $\mathcal{S} \subset \mathcal{S}^{dd}$ is a complete order embedding.

As in [16, Definition 6.4], an operator system $\mathcal{S}$ is said to have *the weak expectation property (WEP)* if there exists a complete order embedding $\mathcal{S} \subset B(H)$ such that the canonical map $\iota : \mathcal{S} \to \mathcal{S}^{dd}$ extends to a completely positive map $\tilde{\iota} : B(H) \to \mathcal{S}^{dd}$. It is known that $\mathcal{S}$ has the WEP if and only if it is (el, max)-nuclear (see [16, Section 6] and [13, Section 4]).

Proposition 3.3. *Let $\mathcal{S}$ and $\mathcal{T}$ be operator systems whose C^*-envelopes possess the WEP. Then $\mathcal{S} \otimes_{\mathrm{ess}} \mathcal{T} = \mathcal{S} \otimes_{\mathrm{e}} \mathcal{T}$.*

Proof. By [16, Theorem 6.9] (also see [12, Corollary 4.2] and [4, Corollary 3.6.8]), a unital C^*-algebra A possesses the WEP if and only if $A \otimes_{\max} B \subseteq A_1 \otimes_{\max} B$ for any inclusion $A \subseteq A_1$ and any unital C^*-algebra B (Lance's characterization of WEP). Thus,

$$\mathcal{S} \otimes_{\mathrm{ess}} \mathcal{T} \subseteq C_e^*(\mathcal{S}) \otimes_{\max} C_e^*(\mathcal{T}) \subseteq I(\mathcal{S}) \otimes_{\max} I(\mathcal{T})$$

and consequently $\mathcal{S} \otimes_{\mathrm{ess}} \mathcal{T} = \mathcal{S} \otimes_{\mathrm{e}} \mathcal{T}$. □

This allows us to deduce what we had promised at the beginning of this section, that is, analogous to [8, Lemma 3.2], for amenable groups, we have the following corollary.

Corollary 3.4. *Let $\mathfrak{u}$ and $\mathfrak{v}$ be generating sets of amenable discrete groups G and H, respectively. Then $\mathcal{S}(\mathfrak{u}) \otimes_{\mathrm{ess}} \mathcal{S}(\mathfrak{v}) = \mathcal{S}(\mathfrak{u}) \otimes_{\mathrm{e}} \mathcal{S}(\mathfrak{v})$.*

Proof. Since the full group C^*-algebras of amenable groups are nuclear, and the fact that every nuclear C^*-algebra has the WEP [21, Section 17], by Propositions 2.8 and 3.3, the proof follows. □

In view of Proposition 2.9, Corollary 3.4 holds for $\mathcal{S}_r(\mathfrak{u})$ and $\mathcal{S}_r(\mathfrak{v})$ as well. In [16], a generalization of the notion of WEP was introduced and was called *the double commutant expectation property (DCEP)*. An operator system $\mathcal{S}$ is said to have the DCEP if for every complete order embedding $\mathcal{S} \subset B(H)$, there exists a completely positive map $\varphi : B(H) \to \mathcal{S}''$ fixing $\mathcal{S}$.

Proposition 3.5. *Let A be a unital C^*-algebra; then, for every operator system $\mathcal{S}$ possessing the DCEP,*

$$\mathcal{S} \otimes_{\text{ess}} A = \mathcal{S} \otimes_{\text{el}} A = \mathcal{S} \otimes_{\max} A.$$

Proof. By [16, Theorems 7.1 and 7.3], an operator system $\mathcal{S}$ has the DCEP if and only if for every operator system $\mathcal{S}_1$ with $\mathcal{S} \subseteq \mathcal{S}_1$ and any operator system $\mathcal{R}$, we have $\mathcal{S} \otimes_{\text{c}} \mathcal{R} \subseteq_{\text{coe}} \mathcal{T} \otimes_{\text{c}} \mathcal{R}$. In particular, by [15, Theorem 6.7] and the fact that $A = C_e^*(A)$ (as observed in Proposition 3.1),

$$\mathcal{S} \otimes_{\max} A = \mathcal{S} \otimes_{\text{c}} A \subset C_e^*(\mathcal{S}) \otimes_{\text{c}} A = C_e^*(\mathcal{S}) \otimes_{\max} A \supset \mathcal{S} \otimes_{\text{ess}} A.$$

Also, by [16, Theorem 7.3] again, $\mathcal{S}$ has the DCEP if and only if it is (el, c)-nuclear and we are done. □

4. Nuclearity of an operator system via its C^*-envelope

In this section we compare various notions of nuclearity of operator systems with their C^*-envelopes.

Kirchberg and Wassermann [17] gave an example of a $(\min, \max)$-nuclear operator system whose C^*-envelope, as observed by Kavruk in [14, Section 6], is nonexact and hence nonnuclear. The other direction, in general, is equally mysterious.

Proposition 4.1. *The notions of* $(\min, \text{c})$*-nuclearity,* $(\min, \text{er})$*-nuclearity and* (el, c)*-nuclearity do not pass to an operator system from its C^*-envelope.*

Proof. For the operator system $\mathcal{S}_2$ associated to the free group with two generators, there exists a complete order embedding of its dual $\mathcal{S}_2^d$ into M_4 (see [9, Theorem 4.4] and [14, Theorem 10.11]). And, since $\mathcal{S}_2$ is not exact, by [14, Corollary 10.14], $\mathcal{S}_2^d$ is not $(\min, \text{er})$-nuclear. Hence, it fails to be $(\min, \text{c})$-nuclear as $\text{er} \leq \text{c}$. In particular, as exactness passes to operator subsystems (Theorem 2.7), $\mathcal{S}_2^d \subseteq M_4$ is exact and hence $(\min, \text{el})$-nuclear [16, Section 5]. Therefore, it is not (el, c)-nuclear as well.

By Proposition 2.1, $C_e^*(\mathcal{S}_2^d)$ is nuclear. Thus, none of $(\min, \text{c})$-nuclearity, $(\min, \text{er})$-nuclearity or (el, c)-nuclearity passes to an operator system from its C^*-envelope. □

However, by the very way $\otimes_{\text{ess}}$ is defined, we have the following result.

Proposition 4.2. *An operator system is* $(\min, \text{ess})$*-nuclear if its C^*-envelope is nuclear. Moreover, a unital C^*-algebra is* $(\min, \text{ess})$*-nuclear as an operator system if and only if it is nuclear as a C^*-algebra.*

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Proof. Let $\mathcal{S}$ be an operator system with nuclear C^*-envelope. By injectivity of $\otimes_{\min}$, we have $\mathcal{S} \otimes_{\min} \mathcal{T} \subseteq C_e^*(\mathcal{S}) \otimes_{\min} C_e^*(\mathcal{T})$ and, by the definition of $\otimes_{\rm ess}$, $\mathcal{S} \otimes_{\rm ess} \mathcal{T} \subseteq C_e^*(\mathcal{S}) \otimes_{\max} C_e^*(\mathcal{T})$ for any operator system $\mathcal{T}$. By [15, Corollary 6.8], a nuclear C^*-algebra is $(\min, \max)$-nuclear as an operator system, $C_e^*(\mathcal{S}) \otimes_{\min} C_e^*(\mathcal{T}) = C_e^*(\mathcal{S}) \otimes_{\max} C_e^*(\mathcal{T})$ and, hence, $\mathcal{S}$ is $(\min, \mathrm{ess})$-nuclear.

Conversely, if A is a unital C^*-algebra which is $(\min, \mathrm{ess})$-nuclear, then, by Proposition 3.1, $A \otimes_{\min} B = A \otimes_{\rm ess} B = A \otimes_{\max} B$ for every unital C^*-algebra B. Therefore, by [14, Proposition 4.11], A is $(\min, \mathrm{c})$-nuclear as an operator system and, hence, by [15, Theorem 6.7 and Corollary 6.8], A is nuclear as a C^*-algebra. □

We do not know whether the C^*-envelope of a $(\min, \mathrm{ess})$-nuclear operator system is nuclear in general or not. However, the situation is better in the enough unitaries case.

Theorem 4.3. *Let $\mathcal{S} \subset A$ contain enough unitaries of the unital C^*-algebra A. Then $\mathcal{S}$ is* $(\min, \mathrm{ess})$*-nuclear if and only if A is a nuclear C^*-algebra.*

Proof. If A is nuclear, then it follows from Propositions 2.3 and 4.2 that $\mathcal{S}$ is $(\min, \mathrm{ess})$-nuclear. Conversely, suppose that A is not nuclear. Then, by Proposition 2.5, there exists a unital C^*-algebra B such that the identity map on $A \otimes B$ does not extend to a $*$-isomorphism between $A \otimes_{C^*\text{-}\min} B$ and $A \otimes_{C^*\text{-}\max} B$. Note that, by injectivity of $\otimes_{\min}$, $\mathcal{S} \otimes_{\min} B$ has enough unitaries in $A \otimes_{C^*\text{-}\min} B$ and, by definition of $\otimes_{\rm ess}$, so does $\mathcal{S} \otimes_{\rm ess} B$ in $A \otimes_{C^*\text{-}\max} B$. Thus, by Proposition 2.3 again, $A \otimes_{C^*\text{-}\min} B$ is the C^*-envelope of $\mathcal{S} \otimes_{\min} B$ and likewise $A \otimes_{C^*\text{-}\max} B$ is that of $\mathcal{S} \otimes_{\rm ess} B$. In particular, this implies that $\mathcal{S} \otimes_{\min} A \neq \mathcal{S} \otimes_{\rm ess} A$ and hence $\mathcal{S}$ is not $(\min, \mathrm{ess})$-nuclear. □

Recall from [8, Section 2] that $\mathrm{ess} \leq \mathrm{c}$. However, as an application of our main results (Proposition 4.2 and Theorem 4.3), the next proposition shows that ess does not compare that well with er and el.

Proposition 4.4. *We have* $\mathrm{er} \nleq \mathrm{ess}$, $\mathrm{ess} \nleq \mathrm{er}$ *and* $\mathrm{ess} \nleq \mathrm{el}$.

Proof. We saw in Proposition 4.1 that there exists a complete order embedding of $\mathcal{S}_2^d$ into M_4. So, by Proposition 4.2, $\mathcal{S}_2^d$ is $(\min, \mathrm{ess})$-nuclear. In Proposition 4.1, we also saw that $\mathcal{S}_2^d \subset M_4$ is not $(\min, \mathrm{er})$-nuclear. This implies that $\mathrm{er} \nleq \mathrm{ess}$.

Then, by [16, Proposition 9.9], $\mathcal{S}_2$ is $(\min, \mathrm{er})$-nuclear but, by Theorem 4.3, it is not $(\min, \mathrm{ess})$-nuclear and hence $\mathrm{ess} \nleq \mathrm{er}$. Finally, if $\mathfrak{u}$ is a generating set of a free group $\mathbb{F}$ with $2 \leq |\mathfrak{u}| \leq \infty$, then $C_r^*(\mathbb{F})$ being exact (see [21, Ch. 8] and [4, Proposition 5.1.8]), $\mathcal{S}_r(\mathfrak{u})$ is $(\min, \mathrm{el})$-nuclear by Theorem 2.7; and, on the other hand, by Theorem 4.3 again, $\mathcal{S}_r(\mathfrak{u})$ is not $(\min, \mathrm{ess})$-nuclear, implying that $\mathrm{ess} \nleq \mathrm{el}$. □

It is not clear whether $\mathrm{el} \leq \mathrm{ess}$ or not. However, we will prove later (in Corollary 4.10) that $(\min, \mathrm{ess})$-nuclearity implies $(\min, \mathrm{el})$-nuclearity.

As an immediate consequence of Proposition 4.2, we observe that $(\min, \mathrm{ess})$-nuclearity passes to an operator system from its C^*-envelope. However, since a unital C^*-algebra is $(\min, \mathrm{ess})$-nuclear if and only if it is nuclear, and as nuclearity is not preserved by C^*-subalgebras [4, Section 4], we observe, in general, the following.

Remark 4.5. The notion of $(\min, \mathrm{ess})$-nuclearity does not pass to operator subsystems.

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Proposition 4.6. *The notions of* (el, c)*-nuclearity and* (el, max)*-nuclearity do not pass to operator subsystems.*

Proof. Since a C^*-algebra is nuclear if and only if it is exact and has the WEP [21, Section 17], and as exactness passes to C^*-subalgebras, the failure of passage of nuclearity to C^*-subalgebras [4] also shows that the WEP does not pass to C^*-subalgebras. And, as the WEP and the DCEP are the same for a C^*-algebra [14, 16], this also illustrates that the DCEP (equivalently, (el, c)-nuclearity) and the WEP (equivalently, (el, max)-nuclearity) do not pass to operator subsystems in general. □

On the other hand, the following corollary is an immediate consequence of Propositions 2.1 and 4.2.

Corollary 4.7. *An operator system $\mathcal{S}$ is* (min, ess)*-nuclear if it admits a nuclear C^*-cover.*

In particular, we also see that if $\mathcal{S} \subset A$ is an operator subsystem of a finite-dimensional C^*-algebra A, then $\mathcal{S}$ is (min, ess)-nuclear. We saw in Proposition 4.1 that $\mathcal{S}_2^d \subset M_4$ is not (min, c)-nuclear; so, it also serves as an example of a finite-dimensional (exact) operator system which is not (ess, c)-nuclear. We thus have the following remark.

Remark 4.8. An operator system need not be (ess, c)-nuclear if it possesses a nuclear C^*-envelope. In particular, (ess, c)-nuclearity does not pass to operator subsystems.

By Proposition 4.2, every C^*-algebra which is (min, ess)-nuclear is also nuclear and hence exact. In fact, the same is true for operator systems as well, which will follow from the proposition given below, where the notations $\hat{\otimes}$ and $\bar{\otimes}$ have similar meanings as in Definition 2.6:

Proposition 4.9. *Let $\mathcal{S}$ be an operator system and $\mathcal{I}$ be a closed ideal in a unital C^*-algebra A. Then $\mathcal{S} \bar{\otimes} \mathcal{I}$ is a completely biproximinal kernel in $\mathcal{S} \hat{\otimes}_{\mathrm{ess}} A$ and the induced map*

$$\frac{\mathcal{S} \hat{\otimes}_{\mathrm{ess}} A}{\mathcal{S} \bar{\otimes} \mathcal{I}} \to \mathcal{S} \hat{\otimes}_{\mathrm{ess}} A/\mathcal{I}$$

is a unital complete order isomorphism.

Proof. Since ess is induced by a C^*-algebraic tensor product, by [16, Proposition 5.14], $\mathcal{S} \bar{\otimes} \mathcal{I}$ is a completely biproximinal kernel in $\mathcal{S} \hat{\otimes}_{\mathrm{ess}} A$, that is, $C_n(\mathcal{S} \hat{\otimes}_{\mathrm{ess}} A/\mathcal{S} \bar{\otimes} \mathcal{I}) = \mathcal{D}_n(\mathcal{S} \hat{\otimes}_{\mathrm{ess}} A/\mathcal{S} \bar{\otimes} \mathcal{I})$ for all $n \geq 1$ [16, Definition 4.9], and the induced map $\varphi : \mathcal{S} \hat{\otimes}_{\mathrm{ess}} A/\mathcal{S} \bar{\otimes} \mathcal{I} \to C_e^*(\mathcal{S}) \hat{\otimes}_{\max} A/C_e^*(\mathcal{S}) \hat{\otimes}_{\max} \mathcal{I}$ is a complete order isomorphic inclusion. On the other hand, the canonical map $\theta : (C_e^*(\mathcal{S}) \hat{\otimes}_{\max} A/C_e^*(\mathcal{S}) \hat{\otimes}_{\max} \mathcal{I}) \to C_e^*(\mathcal{S}) \hat{\otimes}_{\max} A/\mathcal{I}$ is a complete order isomorphism (by [16, Corollary 15.6]) and we have $\mathcal{S} \hat{\otimes}_{\mathrm{ess}} A/\mathcal{I} \subseteq C_e^*(\mathcal{S}) \hat{\otimes}_{\max} A/\mathcal{I}$. Clearly, $\theta \circ \varphi$ is a surjection onto $\mathcal{S} \hat{\otimes}_{ess} A/\mathcal{I}$ and agrees with the induced map $\mathcal{S} \hat{\otimes}_{\mathrm{ess}} A/\mathcal{S} \bar{\otimes} \mathcal{I} \to \mathcal{S} \hat{\otimes}_{\mathrm{ess}} A/\mathcal{I}$ and hence the assertion holds. □

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COROLLARY 4.10. *Every operator system which is* (min, ess)-*nuclear is also* (min, el)-*nuclear.*

As remarked earlier, we do not know whether el $\leq$ ess or not, but Corollary 4.10 does hint that it could very well be true.

The converse to Corollary 4.10 is false. Consider the free group $\mathbb{F}_n$ with n generators and $n \geq 2$. Its reduced group C^*-algebra $C_r^*(\mathbb{F}_n)$ is exact, that is, (min, el)-nuclear [4, 21], but not C^*-nuclear and hence, by Proposition 4.2, it is not (min, ess)-nuclear.

Before moving on to the section on operator systems associated to discrete groups, we point out that, unlike the category of C^*-algebras, exactness, (min, ess)-nuclearity, (min, c)-nuclearity and (min, max)-nuclearity do not pass to operator system quotients.

COROLLARY 4.11. *The notion of* (min, ess)-*nuclearity does not pass to operator system quotients.*

PROOF. For each $n \geq 2$, let $J_n \subset M_n(\mathbb{C})$ be the kernel [9, Section 2] in $M_n(\mathbb{C})$ consisting of all diagonal matrices $D \in M_n(\mathbb{C})$ with $\mathrm{tr}(D) = 0$; and $\mathcal{W}_n$ be the operator subsystem of $C^*(\mathbb{F}_{n-1})$ spanned by $\{u_i u_j^* : 1 \leq i, j \leq n\}$, where $u_2, \ldots, u_n$ are the universal unitaries that generate $C^*(\mathbb{F}_{n-1})$ and $u_1 := 1$. For $n \geq 3$, by [9, Theorem 2.4], M_n/J_n is completely order isomorphic to $\mathcal{W}_n$ and $C_e^*(\mathcal{W}_n) = C_e^*(M_n/J_n) = C^*(\mathbb{F}_{n-1})$. By Theorem 4.3, we deduce that the quotient M_n/J_n is not (min, ess)-nuclear. □

In addition, the example of M_n/J_n also shows that $C_e^*(M_n/J_n)$ is isomorphic to $C^*(\mathbb{F}_{n-1})$, which is not exact [21, Section 17]. Now, since $\mathcal{W}_n$ contains enough unitaries of $C^*(\mathbb{F}_{n-1})$, $\mathcal{W}_n$ and hence M_n/J_n is not exact (by [16, Corollary 9.6]). In particular, exactness, (min, c)-nuclearity and (min, max)-nuclearity do not pass to quotients in the category of operator systems as well.

Note that Kavruk [14, Theorem 10.2] showed that the quotient operator system M_3/J_3 has a deep relationship with Kirchberg's conjecture. It was established that Kirchberg's conjecture has a positive answer if and only if M_3/J_3 possesses the DCEP. There is one more quotient operator system which is equally deeply related to Kirchberg's conjecture, namely, T_n/J_n, where $T_n := \{[a_{i,j}] \in M_n : a_{i,j} = 0 \text{ if } |i - j| > 1\}$. It was also established, in [14, Corollary 10.6], that Kirchberg's conjecture is true if and only if T_3/J_3 possesses the DCEP. This was a consequence of the fact [14, Theorem 10.5] that T_n/J_n is completely order isomorphic to $\mathcal{S}_{n-1}$ for all $n \geq 3$. We easily deduce, again from Theorem 4.3, that the quotient operator system T_3/J_3 is not (min, ess)-nuclear.

Furthermore, as observed in Proposition 4.4, the dual $\mathcal{S}_2^d$ of the free group operator system $\mathcal{S}_2$ is (min, ess)-nuclear, whereas $\mathcal{S}_2^{dd} = \mathcal{S}_2$ (up to complete order isomorphism) is not (min, ess)-nuclear and hence we also observe the following corollary.

COROLLARY 4.12. *The notion of* (min, ess)-*nuclearity does not pass to operator system duals.*

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5. Nuclearity of group operator systems

Recall that if $\mathcal{S}(\mathfrak{u})$ is a (min, c)-nuclear operator system, then $C^*(G)$ is a nuclear C^*-algebra [16, Corollary 9.6] and, since the group C^*-algebra of a discrete group is nuclear if and only if the group is amenable [4, 18], it follows that the group is amenable. By Theorem 4.3, we realize that amenability can, in fact, be recovered from (min, ess)-nuclearity itself. Surprisingly, it is not yet clear (at least, to us) whether the amenability of a group guarantees (min, c)-nuclearity of an operator system associated to a generating set of the group. The following theorem can be treated as a small step in this direction, whose proof is now an immediate consequence of Proposition 2.8 and Theorem 4.3:

Theorem 5.1. *Let $\mathfrak{u}$ be a generating set of a discrete group G. Then the operator systems $\mathcal{S}(\mathfrak{u})$ and $\mathcal{S}_r(\mathfrak{u})$ are* (min, ess)*-nuclear if and only if G is amenable.*

Note that Theorem 5.1, kind of, falls in line with the fact that amenability of a discrete group is equivalent to the nuclearity of its reduced group C^*-algebra $C^*_r(G)$ and injectivity of its group von Neumann algebra $L(G)$ (see [4, Section 3] and [18]).

We now move towards identifying the (min, max)-nuclear group operator systems associated to finitely generated groups. For a finite-dimensional operator system $\mathcal{S}$, it is known that $\mathcal{S}$ is (c, max)-nuclear if and only if it is unitally completely order isomorphic to a C^*-algebra [14, Proposition 4.12]. In fact, since ess $\leq$ c, this allows us to conclude the following proposition.

Proposition 5.2. *Let $\mathcal{S}$ be a finite-dimensional operator system. Then the following are equivalent.*

(i) $\mathcal{S}$ *is* (ess, max)*-nuclear.*
(ii) $\mathcal{S}$ *is* (c, max)*-nuclear.*
(iii) $\mathcal{S}$ *is unitally completely order isomorphic to a C^*-algebra.*
(iv) $\mathcal{S}$ *is* (min, max)*-nuclear.*

Since an injective operator system admits the structure of a C^*-algebra [5, Theorem 3.1], Proposition 5.2 has the following immediate consequence.

Corollary 5.3. *Let $\mathfrak{u}$ be a finite generating set of a discrete group G. Then $\mathcal{S}(\mathfrak{u})$ is* (min, max)*-nuclear if and only if $\mathcal{S}(\mathfrak{u})$ is injective.*

We now provide the promised exhaustive list of nuclear group operator systems associated to minimal generating sets of finitely generated groups.

Theorem 5.4. *Let $\mathfrak{u}$ be a generating set of a finitely generated discrete group G.*

(i) *If $\mathfrak{u}$ is finite and $\mathcal{S}(\mathfrak{u})$ is* (min, max)*-nuclear, then G is finite.*
(ii) *If $\mathfrak{u}$ is a minimal generating set, then $\mathcal{S}(\mathfrak{u})$ is* (min, max)*-nuclear if and only if G is of order less than or equal to three.*

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Proof. We prove both assertions simultaneously. Note that since G is finitely generated, $\mathfrak{u}$ is finite in (ii) as well. So, by Proposition 5.2, $\mathcal{S}(\mathfrak{u})$ is (min, max)-nuclear if and only if $\mathcal{S}(\mathfrak{u})$ is completely order isomorphic to a C^*-algebra. Then, by Proposition 2.2(iii), $\mathcal{S}(\mathfrak{u})$ is completely order isomorphic to a C^*-algebra if and only if $\mathcal{S}(\mathfrak{u}) = C^*_e(\mathcal{S}(\mathfrak{u})) = C^*(G)$, which is true if and only if $\mathfrak{u} \cup \mathfrak{u}^{-1} \cup \{e\} = G$, where $\mathfrak{u}^{-1} := \{u^{-1} : u \in \mathfrak{u}\}$.

Suppose that $\mathcal{S}(\mathfrak{u})$ is (min, max)-nuclear. Since $\mathfrak{u}$ is finite, $\mathcal{S}(\mathfrak{u}) = C^*(G)$ is finite dimensional and hence G is finite. Note that if $|\mathfrak{u}| > 1$ and $a, b \in \mathfrak{u}$ with $a \neq b$, then, by minimality of $\mathfrak{u}$, the set $\{ab\} \cup \mathfrak{u} \cup \mathfrak{u}^{-1} \cup \{e\}$ is linearly independent in $C^*(G)$ and therefore $ab \notin \mathcal{S}(\mathfrak{u})$. So, $\mathfrak{u}$ must be a singleton, that is, G must be cyclic. And, clearly, for a singleton $\mathfrak{u}$, the equality $\mathfrak{u} \cup \mathfrak{u}^{-1} \cup \{e\} = G$ holds only if $G = \mathbb{Z}_1, \mathbb{Z}_2$ or $\mathbb{Z}_3$. Conversely, if $G = \mathbb{Z}_1, \mathbb{Z}_2$ or $\mathbb{Z}_3$, then clearly $\mathfrak{u} \cup \mathfrak{u}^{-1} \cup \{e\} = G$ and $\mathcal{S}(\mathfrak{u})$ is then (min, max)-nuclear. □

Note that minimality of $\mathfrak{u}$ cannot be dropped from the statement of Theorem 5.4. For example, if $G = \mathbb{Z}_3 \oplus \mathbb{Z}_3$ and $\mathfrak{u} = \{(1,0), (1,1), (0,1), (1,2)\}$, then $\mathcal{S}(\mathfrak{u})$ is (min, max)-nuclear but G is neither cyclic nor $|G| \leq 3$. On the other hand, it is not yet clear whether $\mathcal{S}(\mathfrak{u})$ is (min, max)-nuclear if $\mathfrak{u}$ is infinite even if it equals G or $G \setminus \{e\}$.

We thus obtain yet another collection of finite-dimensional (min, ess)-nuclear operator systems which are not (min, max)-nuclear.

Corollary 5.5. *For any finitely generated amenable group G with $|G| \geq 4$ and any finite generating set $\mathfrak{u}$ of G such that $\mathfrak{u} \cup \mathfrak{u}^{-1} \cup \{e\} \neq G$, $\mathcal{S}(\mathfrak{u})$ is* (min, ess)*-nuclear but not* (min, max)*-nuclear. In particular,* (ess, max)*-nuclearity does not pass to operator subsystems.*

The preceding corollary also gives examples of operator systems which are not (min, max)-nuclear and yet possess nuclear C^*-envelopes.

6. Nuclearity of graph operator systems

Given a finite graph G with n vertices, Kavruk *et al.* in [15] associated an operator system $\mathcal{S}_G$ as the finite-dimensional operator subsystem of $M_n(\mathbb{C})$ given by

$$\mathcal{S}_G = \text{span}\{\{E_{i,j} : (i, j) \in G\} \cup \{E_{i,i} : 1 \leq i \leq n\}\} \subseteq M_n(\mathbb{C}),$$

where $\{E_{i,j}\}$ is the standard system of matrix units in $M_n(\mathbb{C})$ and (i, j) denotes (an unordered) edge in G. In view of Proposition 4.2, the graph operator system $\mathcal{S}_G \subseteq M_n$ is always (min, ess)-nuclear.

It is known that if G is chordal, that is, no minimal cycle of G has length greater than 3, then $\mathcal{S}_G$ is C^*-nuclear [15, Proposition 6.11]. It is not known whether the converse is true or not. However, motivated by the discussions in [19, Section 3], we obtain the following characterization of the (min, max)-nuclear graph operator systems.

Theorem 6.1. *Let G be a finite graph. Then the associated operator system $\mathcal{S}_G$ is* (min, max)*-nuclear if and only if each component of G is complete.*

Proof. Suppose that each connected component of G is a complete graph. Then we easily see that $\mathcal{S}_G = M_{n_1} \oplus \cdots \oplus M_{n_k} \subset M_n$, where the n_i are the number of vertices in the connected components of G, k is the number of connected components of G and n is the number of vertices in G. Thus, $\mathcal{S}_G$ is (min, max)-nuclear.

Conversely, suppose that $\mathcal{S}_G$ is (min, max)-nuclear. Then, by Proposition 2.2(iii), we have $\mathcal{S}_G = C_e^*(\mathcal{S}_G)$ and Ortiz and Paulsen proved in [19, Theorem 3.2] that $C_e^*(\mathcal{S}_G) = C^*(\mathcal{S}_G) \subseteq M_n(\mathbb{C})$ for any graph G. Let $\mathcal{F}$ be a connected component of G and $v \neq w$ be any two vertices in $\mathcal{F}$. Then there exists a sequence of connected edges $(v, i_1), (i_1, i_2), (i_2, i_3), \ldots, (i_r, w)$ in $\mathcal{F}$ connecting v with w. Further, since $\mathcal{S}_G = C^*(\mathcal{S}_G)$, we get $E_{v,w} = E_{v,i_1} E_{i_1,i_2} \cdots E_{i_{r-1},i_r} E_{i_r,w} \in \mathcal{S}_G$. This implies that $(v, w) \in G$, that is, v and w are connected by an edge and hence $\mathcal{F}$ is complete. □

7. Nuclearity properties of some known examples

7.1. Operator systems of commuting and noncommuting *n*-cubes. Inspired by Tsirelson's noncommutative analogues of n-dimensional cubes, Farenick *et al.* in [8] introduced an $(n + 1)$-dimensional operator system $NC(n)$ as follows.

Let $\mathcal{G} = \{h_1, \ldots, h_n\}$, let $\mathcal{R} = \{h_j^* = h_j, \|h\| \leq 1, 1 \leq j \leq n\}$ be a set of relations in the set $\mathcal{G}$ and let $C^*(\mathcal{G}|\mathcal{R})$ denote the universal unital C^*-algebra generated by $\mathcal{G}$ subject to the relations $\mathcal{R}$. The operator system

$$NC(n) := \operatorname{span}\{1, h_1, \ldots, h_n\} \subset C^*(\mathcal{G}|\mathcal{R})$$

is called *the operator system of the noncommuting n-cube.*

They showed that up to a $*$-isomorphism, $C_e^*(NC(n)) = C^*(*_n\mathbb{Z}_2)$ [8, Corollary 5.6] and that $NC(1)$ is (min, max)-nuclear [8, Proposition 6.1] and $NC(2)$ is (min, c)-nuclear [8, Theorem 6.3]. Further, it follows from [8, Theorem 6.13] that $NC(n)$ is not (min, max)-nuclear for all $n \geq 2$. Proposition 2.2(iii) now provides an alternate proof of this fact.

Farenick *et al.* further introduced the *operator system of the commuting n-cube* as the operator subsystem $C(n) \subset C([-1, 1]^n)$ given by

$$C(n) = \operatorname{span}\{1, x_1, x_2, \ldots, x_n\},$$

where x_i is the ith coordinate function on $[-1, 1]^n$. Clearly, the C^*-algebra generated by $C(n)$ in $C([-1, 1]^n)$ is commutative. As a consequence, the C^*-envelope of $C(n)$ is also commutative and hence nuclear; in particular, $C(n)$ is (min, ess)-nuclear.

There was one more important example introduced in [8], namely, the operator system $\mathcal{V} \subset \ell_4^\infty$ given by

$$\mathcal{V} := \{(a, b, c, d) : a + b = c + d\} \subset \ell_4^\infty.$$

They proved in [8, Theorem 6.11] that $\mathcal{V}$ is not (min, max)-nuclear. It will be interesting to investigate the following question.

Question. Let A be a unital commutative C^*-algebra and $\mathcal{S}$ be an operator subsystem of A. Is $\mathcal{S}$ C^*-nuclear?

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7.2. Operator systems generated by a single operator. Determining the C^*-envelope of a given operator system is in general quite challenging. However, very recently, Argerami and Farenick [1, 2] considered the operator systems generated by single operators of certain classes of operators and successfully calculated their C^*-envelopes. The calculation of these C^*-envelopes together with Proposition 4.2 leads to interesting examples of finite-dimensional (min, ess)-nuclear operator systems. Further, using Proposition 2.2(iii) and simple dimension comparisons, we check whether these singly generated operator systems are (min, max)-nuclear or not.

The operator system generated by a bounded linear operator T acting on a complex Hilbert space $\mathcal{H}$ is defined to be the unital self-adjoint subspace $OS(T) = \operatorname{span}\{1, T, T^*\} \subset B(\mathcal{H})$. Argerami and Farenick exploited the $*$-isomorphism between $C^*_e(OS(T))$ and the quotient of $C^*(T)$ by the Silov boundary ideal of $OS(T)$ given by Arveson [3] in order to do the explicit calculations of the C^*-envelopes. The results below follow from [1, Remarks].

Example 7.1. (i) If T is normal, then $C^*_e(OS(T))$ is commutative and hence nuclear, implying that $OS(T)$ is (min, ess)-nuclear.

(ii) If T is a contraction such that $\mathbb{T} \subset \sigma(T)$, then $C^*_e(OS(T)) = C(\mathbb{T})$ and hence $OS(T)$ is (min, ess)-nuclear.

(iii) If T is an isometry, then $OS(T)$ is (min, ess)-nuclear. This is true because of (i) and the fact that $C^*_e(OS(T)) = C(\mathbb{T})$ if T is not unitary.

Since the dimensions of the operator systems $OS(T)$ in (i), (ii) and (iii) do not equal the dimensions of their respective C^*-envelopes, $OS(T) \neq C^*_e(OS(T))$ in all the three cases and, therefore, the above operator systems are not (min, max)-nuclear.

For the sake of convenience of the reader, we now recall the definitions of the classes of operators whose operator systems were considered in [1] and [2].

Example 7.2. If $\mathbb{C}^\times := \mathbb{C}\setminus\{0\}$ and $\xi = (\xi_1, \xi_2, \ldots, \xi_d) \in (\mathbb{C}^\times)^d$, then *the irreducible weighted unilateral shift* with weights $\xi_1, \xi_2, \ldots, \xi_d$ is the operator $W(\xi)$ on $\mathbb{C}^{d+1}$ given by the matrix

$$W(\xi) = \begin{bmatrix} 0 & & & & 0 \\ \xi_1 & 0 & & & \\ & \xi_2 & \ddots & & \\ & & \ddots & 0 & \\ & & & \xi_d & 0 \end{bmatrix}.$$

By [1, Proposition 3.2], $C^*_e(OS(W(\xi))) = M_{d+1}(\mathbb{C})$, which is a nuclear C^*-algebra and hence $OS(W(\xi))$ is (min, ess)-nuclear. Since $M_{d+1}(\mathbb{C})$ is at least four dimensional, $OS(W(\xi)) \neq C^*_e(OS(W(\xi)))$, which further implies that $OS(W(\xi))$ is not (min, max)-nuclear.

Example 7.3. A *weighted unilateral shift operator* is an operator W on $l^2(\mathbb{N})$ whose action on the standard orthonormal basis $\{e_n : n \in \mathbb{N}\}$ of $l^2(\mathbb{N})$ is given by

$$We_n = w_n e_{n+1}, \quad n \in \mathbb{N},$$

where the weight sequence $\{w_n\}_{n\in\mathbb{N}}$ for W consists of nonnegative real numbers with $\sup_n w_n < \infty$. If there is a $p \in \mathbb{N}$ such that $w_{n+p} = w_n$ for every $n \in \mathbb{N}$, then W is called a periodic unilateral weighted shift of period p. If at least one of $w_1, \dots, w_p$ is not repeated in the list, W is said to be distinct.

By [1, Theorem 3.5], $C_e^*(\mathcal{OS}(W)) = C(\mathbb{T}) \otimes M_p(\mathbb{C})$, which is again a nuclear C^*-algebra and hence $\mathcal{OS}(W)$ is (min, ess)-nuclear and, clearly not (min, max)-nuclear.

Note that Example 7.2 is a special case of this.

Recall that an operator J on an n-dimensional Hilbert space $\mathcal{H}$ is a basic Jordan block if there is an orthonormal basis of $\mathcal{H}$ for which J has a matrix representation of the form

$$J_n(\lambda) := \begin{bmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \dots & \dots & 0 & \lambda \end{bmatrix}$$

for some $\lambda \in \mathbb{C}$.

Example 7.4. Let $J = \bigoplus_{k=1}^{\infty} J_{m_k}(\lambda) \in B(l^2(\mathbb{N}))$ and $m := sup\{m_k : k \in \mathbb{N}\}$. Then, by [2, Proposition 2.2],

$$C_e^*(\mathcal{OS}(J)) = \begin{cases} C(\mathbb{T}) & \text{if } m = \infty, \\ M_m(\mathbb{C}) & \text{if } m < \infty \end{cases}$$

and hence $\mathcal{OS}(J)$ is (min, ess)-nuclear. As in earlier examples, one can see that for $m > 1$, $\mathcal{OS}(J)$ is not (min, max)-nuclear.

An operator $J \in B(l^2(\mathbb{N}))$ is said to be a *Jordan operator* if $J = \bigoplus_j J_{n_j}(\lambda_j)$ for some finite or infinite sequence of basic Jordan operators $J_{n_j}(\lambda_j)$. In this definition, the n_j or the λ_j are not required to be distinct. But repetitions of the same pair n_j, λ_j are not allowed: if a direct sum of d copies of a basic Jordan block $J_n(\lambda)$ is considered, then it is denoted by $J_n(\lambda) \otimes 1_d$.

Example 7.5. Consider the Jordan operator

$$J = \begin{bmatrix} 1 & & & & \\ & \omega & & & \\ & & \omega^2 & & \\ & & & 0 & 1 \\ & & & 0 & 0 \end{bmatrix},$$

where $\omega = (-1 - i\sqrt{3})/2$. Then, by [2, Remark 2.7], $C_e^*(\mathcal{OS}(J)) = \mathbb{C}^3$, which shows that $\mathcal{OS}(J)$ coincides with its C^*-envelope and hence is (min, max)-nuclear.

EXAMPLE 7.6. If $J = \bigoplus_{k=1}^{n}(J_{m_k}(\lambda_k) \otimes I_{d_k})$, with $\lambda_1 > \lambda_2 > \cdots > \lambda_n$ all real and $\max\{m_2, \ldots, m_{n-1}\} \leq \min\{m_1, m_n\}$, then, by [2, Corollary 2.12], $C_e^*(OS(J))$ equals

$$\begin{cases} M_{m_1}(\mathbb{C}) \oplus M_{m_n}(\mathbb{C}) & \text{if } \min\{m_1, m_2\} \geq 2, \\ \mathbb{C} \oplus \mathbb{C} & \text{if } m_1 = m_n = 1, \\ M_{m_n}(\mathbb{C}) & \text{if } m_1 = 1, m_n \geq 2, |\lambda_1 - \lambda_n| \leq \cos \dfrac{\pi}{m_n + 1}, \\ \mathbb{C} \oplus \mathbb{M}_{m_n}(\mathbb{C}) & \text{if } m_1 = 1, m_n \geq 2, |\lambda_1 - \lambda_n| > \cos \dfrac{\pi}{m_n + 1}, \\ M_{m_1}(\mathbb{C}) & \text{if } m_1 \geq 2, m_n = 1, |\lambda_1 - \lambda_n| \leq \cos \dfrac{\pi}{m_1 + 1}, \\ M_{m_1}(\mathbb{C}) \oplus \mathbb{C} & \text{if } m_1 \geq 2, m_n = 1, |\lambda_1 - \lambda_n| > \cos \dfrac{\pi}{m_n + 1}, \end{cases}$$

implying that $OS(J)$ is (min, ess)-nuclear but not (min, max)-nuclear for all of the above cases except for the case when $m_1 = m_n = 1$. If $m_1 = m_n = 1$, $OS(J)$ turns out to be (min, max)-nuclear.

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