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ARTICLE

A note on extremal constructions for the Erdős–Rademacher problem

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Abstract

For given positive integers $r \geq 3$, n and $e \leq \binom{n}{2}$, the famous Erdős–Rademacher problem asks for the minimum number of r-cliques in a graph with n vertices and e edges. A conjecture of Lovász and Simonovits from the 1970s states that, for every $r \geq 3$, if n is sufficiently large then, for every $e \leq \binom{n}{2}$, at least one extremal graph can be obtained from a complete partite graph by adding a triangle-free graph into one part.

In this note, we explicitly write the minimum number of r-cliques predicted by the above conjecture. Also, we describe what we believe to be the set of extremal graphs for any $r \geq 4$ and all large n, amending the previous conjecture of Pikhurko and Razborov.

Keywords: Erdős–Rademacher problem; Lovász–Simonovits conjecture; Clique density theorem

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1. Introduction

Given integers $n \geq r \geq 2$, let $T_r(n)$ denote the balanced complete r-partite graph on n vertices, and let $t_r(n)$ denote the number of edges in $T_r(n)$. The celebrated Turán Theorem [24] (with the case $r = 3$ proved earlier by Mantel [13]) states that, for $n \geq r \geq 3$, every n-vertex graph with at least $t_{r-1}(n) + 1$ edges contains a copy of an *r-clique* K_r, that is, a complete graph on r vertices. An unpublished result of Rademacher from 1941 (see [3]) states that, in fact, every n-vertex graph with $t_2(n) + 1$ edges contains at least $\lfloor n/2 \rfloor$ copies of K_3. The graph obtained from $T_2(n)$ by adding one edge to the larger part shows that the bound $\lfloor n/2 \lfloor$ is tight. Rademacher's theorem motivated Erdős [3] to consider the following more general question, now referred to as the *Erdős–Rademacher problem*: determine

$$g_r(n, e) := \min \left\{ N(K_r, G) : \ G \text{ is an } (n, e)\text{-graph} \right\}, \tag{1}$$

where an (n, e)-*graph* means a graph with n vertices and e edges and $N(K_r, G)$ denotes the number of r-cliques in G.

This problem has attracted a lot of attention and has been actively studied since it first appeared. Various results covering special ranges of (n, e) were obtained (see e.g. [2, 4–7, 11, 12, 14, 18–20]) until Razborov [22] determined the asymptotic value of $g_3(n, e)$ using flag algebras. Later, using different methods, Nikiforov [17] determined the asymptotic value of $g_r(n, e)$ for $r = 4$ and Reiher [23] did this for all $r \geq 5$. For some further related results, we refer the reader to [1, 8, 10, 15, 16, 21, 25].

Determining the exact value of $g_r(n, e)$ seems very challenging due to multiple (conjectured) extremal constructions. Given n and e in $\mathbb{N} := \{1, 2, \dots\}$ with $e \le \binom{n}{2}$, let

$$k = k(n, e) := \min\{s \in \mathbb{N} : \ t_s(n) \ge e\}, \tag{2}$$

that is, k is the smallest chromatic number that an (n, e)-graph can have. Let $\mathcal{H}_1(n, e)$ (resp. $\mathcal{K}(n, e)$) denote the family of (n, e)-graphs that can be obtained from a complete $(k-1)$-partite (resp. complete multipartite) graph by adding a triangle-free graph into one part. Note that the only difference between these two definitions is that we restrict the number of parts to $k-1$ when defining $\mathcal{H}_1(n, e)$; thus $\mathcal{H}_1(n, e) \subseteq \mathcal{K}(n, e)$. Lovász and Simonovits [11] conjectured that for every integer $r \ge 3$ there exists n_0 such that, for all positive integers $n \ge n_0$ and $e \le \binom{n}{2}$, it holds that

$$g_r(n, e) = \min\left\{N(K_r, H) : \ H \in \mathcal{K}(n, e)\right\}, \tag{3}$$

that is, at least one $g_r(n, e)$-extremal graph is in $\mathcal{K}(n, e)$. Note that (3) trivially holds for $e \le t_{r-1}(n)$ when $g_r(n, e) = 0$.

Erdős in [3] (resp. [4]) showed that (3) is true for $r = 3$ when $e \le t_2(n) + 3$ (resp. $e \le t_2(n) + cn$ for some constant $c > 0$). Lovász and Simonovits [11] (see also Nikiforov and Khadzhiivanov [19]) extended the result of Erdős to all e satisfying $e \le t_2(n) + \lfloor n/2 \rfloor$. Later, Lovász and Simonovits [12] proved (3) for $r \ge 3$ when $e/\binom{n}{2}$ lies in a small upper neighbourhood of $1 - 1/m$ for some integer $m \ge r-1$. More recently, Liu, Pikhurko and Staden [9] determined $g_3(n, e)$ for all positive integers n when $e \le (1 - o(1))\binom{n}{2}$. Determining the exact value of $g_r(n, e)$ for $r \ge 4$ is still wide open in general.

Given $n, e \in \mathbb{N}$ with $e \le \binom{n}{2}$, let $a^* = a^*(n, e) \in \mathbb{N}^k$ be the unique vector such that

$$a_k^* := \min\left\{a \in \mathbb{N} : \ a(n-a) + t_{k-1}(n-a) \ge e\right\},$$
$$a_1^* + \cdots + a_{k-1}^* = n - a_k^*, \quad \text{and} \quad a_1^* \ge \cdots \ge a_{k-1}^* \ge a_1^* - 1,$$

where $k = k(n, e)$ is as defined in (2). Thus a_k^* is the smallest possible part size that a k-partite (n, e)-graph can have. Also, let

$$m^* = m^*(n, e) := \sum_{\{i,j\} \in \binom{[k]}{2}} a_i^* a_j^* - e, \quad \text{and}$$
$$h_r^*(n, e) := \sum_{I \in \binom{[k]}{r}} \prod_{i \in I} a_i^* - m^* \cdot \sum_{I' \in \binom{[k-2]}{r-2}} \prod_{j \in I'} a_j^*,$$

where $[k] := \{1, \dots, k\}$ and $\binom{X}{k} := \{Y \subseteq X : |Y| = k\}$. Let $T := K[A_1^*, \dots, A_k^*]$ be the complete k-partite graph with parts $A_1^*, \dots, A_k^*$ where $|A_i^*| = a_i^*$ for $i \in [k]$. Let $H^* = H^*(n, e)$ be the graph obtained from T by removing an m^*-edge star whose centre lies in A_k^* and whose leaves lie in A_{k-1}^*. It is not hard to see (see e.g. the calculation in (10)) that $0 \le m^* \le a_{k-1}^* - a_k^*$, so the graph H^* is well-defined. Also, let $\mathcal{H}_1^*(n, e)$ be the family defined as follows: If $m^* = 0$, take all graphs obtained from T by replacing, for some $i \in [k-1]$, the bipartite graph $T[A_i^* \cup A_k^*]$ with an arbitrary triangle-free graph with $a_i^* a_k^*$ edges. If $m^* > 0$, take all graphs obtained from T by replacing $T[A_{k-1}^* \cup A_k^*]$ with an arbitrary triangle-free graph with $a_{k-1}^* a_k^* - m^*$ edges. Observe that $\mathcal{H}_1^*(n, e) \subseteq \mathcal{H}_1(n, e)$ and every graph in $\mathcal{H}_1^*(n, e)$ has the same number of r-cliques (see Fact 2.2); also, the graph $H^* = H^*(n, e)$ is contained in $\mathcal{H}_1^*(n, e)$.

Sharpening the Lovász–Simonovits Conjecture, Pikhurko and Razborov [21, Conjecture 1.4] conjectured that, for $r \ge 4$ and sufficiently large n, every n-vertex graph with $e \le \binom{n}{2}$ edges and that contains the minimum number of K_r is in $\mathcal{K}(n, e)$. However, we show here that this conjecture is false (see Theorem 1.1 and Proposition 1.2) and present an amended version (see Conjecture 1.3) as follows.

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First, we write explicitly the value of $g_r(n, e)$ predicted by the Lovász–Simonovits Conjecture. (We also refer the reader to [9, Proposition 1.5] where similar results are proved for $r = 3$.)

Theorem 1.1. *Suppose that* $r, n, e \in \mathbb{N}$ *satisfy* $n \geq r \geq 3$ *and* $e \leq \binom{n}{2}$*. Then*

$$\min \left\{ N(K_r, G) : G \in \mathcal{K}(n, e) \right\} = h_r^*(n, e). \tag{4}$$

Moreover, if $r \geq 4$ *and* $e > t_{r-1}(n)$*, then*

$$\left\{ G \in \mathcal{K}(n, e) : N(K_r, G) = h_r^*(n, e) \right\} = \mathcal{H}_1^*(n, e). \tag{5}$$

Note that, since $\mathcal{H}_1^*(n, e) \subseteq \mathcal{H}_1(n, e)$, Theorem 1.1 remains true if we replace $\mathcal{K}(n, e)$ by $\mathcal{H}_1(n, e)$. In fact, the later version of the Lovász–Simonovits Conjecture from [12] states that, for all sufficiently large $n \geq n_0(r)$, at least one $g_r(n, e)$-extremal graph is in $\mathcal{H}_1(n, e)$. By (4), these two conjectures are equivalent. One should be able to show with some extra work that (5) also holds for $r = 3$ (it is also implied by the results in [9] that (5) holds for most e, given n). Since our main focus is the case $r \geq 4$, we do not pursue this strengthening here.

Given integers $n, e \in \mathbb{N}$ with $e \leq \binom{n}{2}$, we define the family $\mathcal{H}_2^*(n, e)$ as follows (with k, a^*, m^* being as before). Take those graphs in $\mathcal{H}_1^*(n, e)$ that are k-partite, along with the following family. Take disjoint sets $A_1, \ldots, A_k$ of sizes $a_1^*, \ldots, a_k^*$, respectively, and let $m := m^*$. If $m^* = 0$ and $a_1^* \geq a_k^* + 2$, then we also allow $(|A_1|, \ldots, |A_k|) = \left(a_2^*, \ldots, a_{k-1}^*, a_1^* - 1, a_k^* + 1\right)$ and let $m := a_1^* - a_k^* - 1$. Take all graphs obtained from $K[A_1, \ldots, A_k]$ by removing any m edges, each connecting B_i to A_i for some $i \in I$, where $I := \left\{ i \in [k-1] : |A_i| = |A_{k-1}| \right\}$ and $\{B_i : i \in I\}$ are some pairwise disjoint subsets of A_k. Clearly, every graph in $\mathcal{H}_2^*(n, e)$ is an (n, e)-graph.

Proposition 1.2. *Suppose that* $n \geq r \geq 4$ *and* $t_{r-1}(n) < e \leq \binom{n}{2}$ *are integers. Then*

$$N(K_r, G) = h_r^*(n, e), \qquad \text{for every } G \in \mathcal{H}_2^*(n, e).$$

Also, there are infinitely many pairs $(n, e) \in \mathbb{N}^2$ *with* $t_{r-1}(n) < e \leq \binom{n}{2}$ *such that* $\mathcal{H}_2^*(n, e) \setminus \mathcal{H}_1^*(n, e) \neq \emptyset$*.*

We propose the following amended conjecture.

Conjecture 1.3. *Let* $r \geq 4$ be fixed. *For every sufficiently large integer* n *and every integer* e *with* $t_{r-1}(n) < e \leq \binom{n}{2}$*, it holds that*

$$\left\{ G : G \text{ is an } (n, e)\text{-graph with } N(K_r, G) = g_r(n, e) \right\} = \mathcal{H}_1^*(n, e) \cup \mathcal{H}_2^*(n, e).$$

For comparison with the case $r = 3$, the exact result of Liu, Pikhurko and Staden [9] valid for $e \leq (1 - o(1))\binom{n}{2}$ states that the set of $g_3(n, e)$-extremal graphs is exactly $\mathcal{H}_0^*(n, e) \cup \mathcal{H}_2^*(n, e)$ for a certain explicit family $\mathcal{H}_0^*(n, e) \supseteq \mathcal{H}_1^*(n, e)$, where the inclusion is strict for infinitely many pairs (n, e). However, for $r \geq 4$ and $e > t_{r-1}(n)$, every graph in $\mathcal{H}_0^*(n, e) \setminus \mathcal{H}_1^*(n, e)$ can be shown to have more K_r's than $H^*(n, e)$. (Basically, each such graph is obtained from a complete $(k-1)$-partite graph by adding edges into more than one part and cannot minimise the number of K_r's for $r \geq 4$ by Lemma 2.5.)

For the purposes of this paper (namely for Proposition 1.2), only the difference $\mathcal{H}_2^*(n, e) \setminus \mathcal{H}_1^*(n, e)$ matters; we use the current definitions merely so that the families $\mathcal{H}_i^*(n, e)$ and $\mathcal{H}_i(n, e)$ are the same as in [9].

The rest of the paper of organised as follows. In the next section, we present some definitions and preliminary results. As a step towards proving Theorem 1.1, we first find extremal graphs in a certain family $\mathcal{H}_0(n, e)$ in Section 3 (see Proposition 3.1 for the exact statement). We derive Theorem 1.1 in Section 4. The proof of Proposition 1.2 is presented in Section 5.

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2. Preliminaries

Given ℓ pairwise disjoint sets $A_1, \ldots, A_\ell$, we use $K[A_1, \ldots, A_\ell]$ to denote the complete ℓ-partite graph with parts $A_1, \ldots, A_\ell$; if we care only about the isomorphism type of this graph (i.e. only the sizes of the parts matter), we may instead write $K_{a_1,\ldots,a_\ell}$, where $a_i := |A_i|$ for $i \in [\ell]$.

Let $G = (V, E)$ be a graph. By $|G|$ we denote the number of edges in G. Let $\overline{G} := \left(V, \binom{V}{2} \setminus E\right)$ denote the *complement* of G. The subgraph of G *induced* by a set $A \subseteq V$ is $G[A] := \left(A, \binom{A}{2} \cap E\right)$. For disjoint $A, B \subseteq V$, we use $G[A, B]$ to denote the induced bipartite graph with parts A and B (which consists of edges connecting A to B).

In the remainder of this note, we assume unless it is stated otherwise that $r, n, e \in \mathbb{N}$ satisfy $r \geq 3$ and $e \leq \binom{n}{2}$ (and we minimise the number of r-cliques over (n, e)-graphs). Also, $k = k(n, e)$ is defined in (2).

Given a family $\mathcal{F}$ of (n, e)-graphs, we use $\mathcal{F}^{\min}$ to denote the collection of graphs $F \in \mathcal{F}$ with the minimum number of K_r's (over all graphs in $\mathcal{F}$). For convenience, we set $N(K_0, G) := 1$ and $N(K_{-1}, G) := 0$ for all graphs G.

Let the family $\mathcal{H}_0(n, e)$ be the collection of all (n, e)-graphs that can be obtained from an n-vertex complete $(k-1)$-partite graph by adding a (possibly empty) triangle-free graph into each part. It is clear from the definition that $\mathcal{H}_1(n, e) \subseteq \mathcal{H}_0(n, e)$.

The following fact follows from some simple calculations (with the argument for Part (i) being the same as in (10)).

Fact 2.1. *Let k, a^*, m^*, H^*, and $h_r^*(n, e)$ be as defined in Section 1. Then it holds for all $r \geq 3$ that*

(i) $0 \leq m^* \leq a^*_{k-1} - a^*_k$,

(ii) $|K_{a^*_1,\ldots,a^*_k}| - |K_{a^*_1,\ldots,a^*_{k-2},a^*_{k-1}+1,a^*_k-1}| = a^*_{k-1} - a^*_k + 1$,

(iii) $N(K_r, H^*) = h^*_r(n, e) \geq g_r(n, e)$.

We also need the following simple facts for counting r-cliques in some special classes of graphs.

Fact 2.2. *Let G be a graph, $S \subseteq V(G)$ be a vertex set, and $\overline{S} := V(G) \setminus S$. Suppose that the induced subgraph $G[S]$ is triangle-free, and the induced bipartite graph $G[S, \overline{S}]$ is complete. Then*

$$N(K_r, G) = |G[S]| \cdot N(K_{r-2}, G[\overline{S}]) + |S| \cdot N(K_{r-1}, G[\overline{S}]) + N(K_r, G[\overline{S}]).$$

Fact 2.3. *Suppose that G is a graph obtained from $K[V_1, \ldots, V_\ell]$ by adding a triangle-free graph. Let $S := V_1 \cup V_2$ and $\overline{S} := V(G) \setminus S$. Then*

$$\begin{aligned} N(K_r, G) &= |G[V_1]| \cdot |G[V_2]| \cdot N(K_{r-4}, G[\overline{S}]) \\ &\quad + (|G[V_1]| \cdot |V_2| + |G[V_2]| \cdot |V_1|) \cdot N(K_{r-3}, G[\overline{S}]) \\ &\quad + |G[S]| \cdot N(K_{r-2}, G[\overline{S}]) + |S| \cdot N(K_{r-1}, G[\overline{S}]) + N(K_r, G[\overline{S}]). \end{aligned}$$

Fact 2.4. *Let G be a graph, $S \subseteq V(G)$, and $\overline{S} := V(G) \setminus S$. Suppose that the induced subgraph $G[S]$ is 3-partite, and the induced bipartite subgraph $G[S, \overline{S}]$ is complete. Then*

$$\begin{aligned} N(K_r, G) = N(K_3, G[S]) \cdot N(K_{r-3}, G[\overline{S}]) + |G[S]| \cdot N(K_{r-2}, G[\overline{S}]) \\ + |S| \cdot N(K_{r-1}, G[\overline{S}]) + N(K_r, G[\overline{S}]). \end{aligned}$$

We will also use the following results.

Lemma 2.5. *Let $r \geq 4$ and let $n, e \in \mathbb{N}$ satisfy $t_{r-1}(n) < e \leq \binom{n}{2}$. Suppose that $G \in \mathcal{H}_0^{\min}(n, e)$ is a graph with a vertex partition $V(G) = B_1 \cup \ldots \cup B_{k-1}$ such that G is the union of $K[B_1, \ldots, B_{k-1}]$ with a triangle-free graph. Then G contains at most one part B_i which is partially full, meaning that $0 < |G[B_i]| < t_2(|B_i|)$.*

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Proof. Suppose to the contrary that G contains two partially full parts B_i and B_j for some $1 \le i < j \le k-1$. Let $x := |G[B_i]|$, $\sigma := |G[B_i]| + |G[B_j]|$ and $H := G[V(G) \setminus (B_i \cup B_j)]$. Observe from Fact 2.3 that there exist constants C_2, C_3, C_4 depending on $|B_i|$, $|B_j|$ and H (but not on x) such that

$$N(K_r, G) = N(K_{r-4}, H) \cdot x(\sigma - x) + C_2 x + C_3(\sigma - x) + C_4 =: P(x).$$

Let G_i be the graph obtained from G by moving one edge from $G[B_j]$ to $G[B_i]$ and rearranging the latter graph to be still K_3-free, which is possible by Mantel's theorem. Similarly, let G_j be the graph obtained from G by moving one edge from $G[B_i]$ to $G[B_j]$. Note that $N(K_r, G_i) = P(x+1)$ and $N(K_r, G_j) = P(x-1)$. Since $e > t_{r-1}(n)$, we have

$$P(x+1) + P(x-1) - 2P(x) = -2N(K_{r-4}, H) < 0. \tag{6}$$

Thus $\min \{N(K_r, G_i), N(K_r, G_j)\} < N(K_r, G)$, contradicting the minimality of G. □

The following simple inequality from [9] will be useful. For completeness, we include its short proof here.

Lemma 2.6 ([9, Lemma 4.5]). *For all integers $a \ge 1$, $k \ge 2$, and $n \ge ak$, we have*

$$a(n-a) + t_{k-1}(n-a) > (a-1)(n-a+1) + t_{k-1}(n-a+1). \tag{7}$$

Proof. Let $a_1 \ge \cdots \ge a_{k-1}$ denote the part sizes of $T_{k-1}(n-a)$. If we increase its number of vertices by one, then the part sizes of the new Turán graph, up to reordering, can be obtained by increasing a_{k-1} by one. Thus the difference between the expressions in (7) is

$$|K_{a_1,\ldots,a_{k-1},a}| - |K_{a_1,\ldots,a_{k-2},a_{k-1}+1,a-1}| = a_{k-1}a - (a_{k-1}+1)(a-1) = a_{k-1} - a + 1, \tag{8}$$

which is positive since $a_{k-1} \ge \lfloor (n-a)/(k-1) \rfloor \ge \lfloor (ak-a)/(k-1) \rfloor = a$. □

3. Extremal graphs in $\mathcal{H}_0(n, e)$

As an intermediate step towards Theorem 1.1, we will first prove the following result, which determines the extremal graphs in $\mathcal{H}_0(n, e)$.

Proposition 3.1. *For all integers $n \ge r \ge 4$ and $t_{r-1}(n) < e \le \binom{n}{2}$, we have that $\mathcal{H}_0^{\min}(n, e) = \mathcal{H}_1^*(n, e)$.*

We will use this result later to prove Theorem 1.1 by induction on the number of parts in a graph in $\mathcal{K}(n, e)$. Note that, in general, neither $\mathcal{K}(n, e)$ nor $\mathcal{H}_0(n, e)$ is a subfamily of the other. However, when we work on the structure of extremal graphs in $\mathcal{K}(n, e)$ in the proof of Theorem 1.1, some intermediate graphs may be in $\mathcal{H}_0(n, e)$.

We need some further preliminaries before we can prove Proposition 3.1.

Given a graph $G \in \mathcal{H}_0^{\min}(n, e)$ with partition $B_1, \ldots, B_{k-1}$, we apply the following modification to G to obtain a new graph $H' = H'(G) \in \mathcal{H}_0^{\min}(n, e)$. Note that, in fact, these steps do not depend on r.

Step 1: If there is a part B_i that is partially full in G, then let $B := B_i$ (by Lemma 2.5, such B_i is unique if it exists). Otherwise, take an arbitrary $i \in [k-1]$ with $|G[B_i]| = t_2(|B_i|)$ and let $B := B_i$. Since $|G| > t_{k-1}(n)$, $|G[B_i]|$ cannot be 0 for all $i \in [k-1]$. Thus, the set B is well-defined.

Step 2: Note that $G - B$ is a complete multipartite graph. Let $A_1, \ldots, A_{t-2}$ denote its parts. Let $a_i := |A_i|$ for $i \in [t-2]$ and assume that $a_1 \ge \cdots \ge a_{t-2}$. Note that each original part B_ℓ is either B, some A_i, or the union of two parts A_i and A_j.

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Step 3: Choose integers $a_{t-1} \geq a_t \geq 1$ such that

$$a_{t-1} + a_t = |B| \quad \text{and} \quad (a_{t-1}+1)(a_t - 1) < |G[B]| \leq a_{t-1}a_t.$$

Note that this is possible by Mantel's theorem since $G[B]$ is triangle-free. Let $A_{t-1} \sqcup A_t = B$ be a partition with $|A_{t-1}| = a_{t-1}$ and $|A_t| = a_t$. If $|G[B]| = t_2(|B|)$, then $a_{t-1} = \lceil |B|/2 \rceil$ and $a_t = \lfloor |B|/2 \rfloor$ and we assume that $A_{t-1} \sqcup A_t = B$ is the original partition of $G[B]$ with the two parts labelled so that $|A_{t-1}| \geq |A_t|$.

Step 4: Let H' be obtained from $K[A_1, \ldots, A_t]$ by removing a star whose centre lies in A_t and m' leaves lie in A_{t-1}, where

$$m' := \sum_{ij \in \binom{[t]}{2}} a_i a_j - e = a_{t-1}a_t - |G[B]|. \tag{9}$$

This is possible because, by Step 3,

$$0 \leq m' = a_{t-1}a_t - |G[B]| \leq a_{t-1}a_t - ((a_{t-1}+1)(a_t-1)+1) = a_{t-1} - a_t. \tag{10}$$

Notice that to obtain H' we only change the structure of G on B while keeping $|G[B]| = |H'[B]|$. Thus, $H' \in \mathcal{H}_0(n, e)$ and, since $G[B, V(G) \setminus B]$ is complete bipartite and $G[B]$ is triangle-free, it follows from Fact 2.2 that $N(K_r, H') = N(K_r, G)$, and hence, $H' \in \mathcal{H}_0^{\min}(n, e)$.

Lemma 3.2. *For all $r \geq 3$, integers n and e with $t_{r-1}(n) < e \leq \binom{n}{2}$ and $G \in \mathcal{H}_0^{\min}(n, e)$, the graph H' produced by Steps 1–4 above is isomorphic to $H^*(n, e)$.*

Proof. To prove that $H' \cong H^*(n, e)$, it suffices to show that $t = k$ and $(|A_1|, \ldots, |A_t|) = a^*$, where k and a^* are as defined in Section 1.

Claim 3.3. *If $m' = 0$, then $|H'[A_h \cup A_i \cup A_j]| > t_2(a_h + a_i + a_j)$ for all $\{h, i, j\} \in \binom{[t]}{3}$. If $m' > 0$, then $|H'[A_h \cup A_{t-1} \cup A_t]| > t_2(a_h + a_{t-1} + a_t)$ for all $h \in [t-2]$.*

Proof. Let $S := A_h \cup A_i \cup A_j$, with $\{i, j\} = \{t-1, t\}$ if $m' > 0$. Suppose to the contrary that $|H'[S]| \leq t_2(|S|)$. Then let G_1 be a new graph obtained from H' by replacing $H'[S]$ with a bipartite graph of the same size. Note that the induced bipartite graph $H'[S, \overline{S}]$ is complete. (Indeed, this is trivially true if $m' = 0$ as then $H' = K[A_1, \ldots, A_t]$; if $m' > 0$, then the only non-complete pair is $[A_{t-1}, A_t]$, but both sets lie in S.) Since H' is t-partite, the graph G_1 is $(t-1)$-partite (and with at most one non-complete pair of parts). By Steps 2–3, we have $t \leq 2(k-1)$. So we can represent G_1 as the union of a complete $(k-1)$-partite graph and a triangle-free graph, which implies that $G_1 \in \mathcal{H}_0(n, e)$. It is easy to see from Fact 2.4 that $N(K_r, G_1) \leq N(K_r, H')$, since $0 = N(K_3, G_1[S]) \leq N(K_3, H'[S])$. So it follows from the minimality of H' that $N(K_3, H'[S]) = 0$. If $\{t-1, t\}$ is not a subset of $\{h, i, j\}$, then $H'[S]$ is a complete 3-partite graph and contains at least one traingle, contradicting $N(K_3, H'[S]) = 0$. Therefore, $\{t-1, t\} \subseteq \{h, i, j\}$. By symmetry, we may assume that $\{t-1, t\} = \{i, j\}$ (thus being consistent with our earlier assumption if $m' > 0$). Note that $|H[A_{t-1}, A_t]| \geq 1$, since otherwise, we would have $m' \geq a_{t-1}a_t > a_{t-1} - a_t$, contradicting (10). Note that each edge in $H[A_{t-1}, A_t]$ is in $|A_h|$ triangles in $H[S]$, contradicting $N(K_3, H'[S]) = 0$. □

Claim 3.4. *If $m' > 0$, then $a_{t-2} \geq a_{t-1}$.*

Proof. Suppose to the contrary that $a_{t-2} \leq a_{t-1} - 1$. Then let G_2 be a new graph obtained from H' by moving edges from $[A_{t-2}, A_t]$ to $[A_{t-1}, A_t]$ until this is no longer possible. Let $S := A_{t-2} \cup A_{t-1} \cup A_t$. If $A_{t-2} \cup A_t$ is an independent set in G_2 (i.e. if $m' \geq a_{t-2}a_t$), then $|H'[S]| = |G_2[S]| \leq t_2(|S|)$, contradicting Claim 3.3. Thus $G_2[S]$ can be viewed as a graph obtained from $K[A_{t-2}, A_{t-1}, A_t]$ by removing m' edges from $K[A_{t-2}, A_t]$. So $G_2 \in \mathcal{H}_0(n, e)$. Note that

$$N(K_3, G_2[S]) - N(K_3, H'[S]) = m' \, (a_{t-2} - a_{t-1}) < 0,$$

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which combined with Fact 2.4 implies that $N(K_r, G_2) - N(K_r, H') < 0$, contradicting the minimality of H'. □

If $m' > 0$, let $C_i := A_i$ for $i \in [t]$. If $m' = 0$, let $C_1, \ldots, C_t$ be a relabelling of $A_1, \ldots, A_t$ so that the sizes of the sets are non-increasing. Regardless of the value of m', the following statements clearly hold:

(i) $c_1 \geq \cdots \geq c_t$, where $c_i := |C_i|$ for $i \in [t]$,
(ii) $0 \leq m' \leq c_{t-1} - c_t$,
(iii) Claim 3.3 applies to all triples $\{C_i, C_{t-1}, C_t\}$ for $i \in [t-2]$.

The rest of the proof is written so that it works for both $m' = 0$ and $m' > 0$.

Claim 3.5. *We have $c_1 \leq c_{t-1} + 1$.*

Proof. Let $S := C_1 \cup C_{t-1} \cup C_t$. Note that

$$|K_{c_1-1, c_{t-1}+1, c_t}| - |H'[S]| = m' - c_{t-1} + c_1 - 1 =: m''.$$

Suppose to the contrary that $c_1 \geq c_{t-1} + 2$. Then $m'' \geq m' + 1$. Take a partition $C'_1 \cup C'_{t-1} \cup C'_t = S$ of sizes $c_1 - 1, c_{t-1} + 1, c_t$, respectively. Let H_S be the graph obtained from $K[C'_1, C'_{t-1}, C'_t]$ by removing m'' edges between C'_{t-1} and C'_t. This is possible since $m'' \leq (c_{t-1} + 1)c_t$. (Indeed, otherwise $|H'[S]| \leq (c_1 - 1)(c_{t-1} + c_t + 1) \leq t_2(|S|)$, contradicting Claim 3.3.) We have $|H_S| = |H'[S]|$. Let H'' be the graph obtained from H' by replacing $H'[S]$ with H_S. Note that $H'' \in \mathcal{H}_0(n, e)$. It follows from $m' \leq c_{t-1} - c_t$ that

$$\begin{aligned} N(K_3, H'[S]) - N(K_3, H''[S]) &= \big(c_1 c_{t-1} c_t - m' c_1\big) \\ &\quad - \big((c_1 - 1)(c_{t-1} + 1)c_t - (m' - c_{t-1} + c_1 - 1)(c_1 - 1)\big) \\ &\geq (c_1 - c_t)(c_1 - c_{t-1} - 2) + 1 \ \geq 1, \end{aligned}$$

which combined with Fact 2.4 implies that $N(K_r, H') - N(K_r, H'') > 0$, contradicting the minimality of H'. □

Claim 3.6. *We have $t = k$.*

Proof. It suffices to show that $t_{t-1}(n) < e \leq t_t(n)$. The upper bound $e \leq t_t(n)$ is trivial, since H' is t-partite. So it remains to show that $e > t_{t-1}(n)$. Let $T := H'[C_1 \cup \cdots \cup C_{t-1}]$. It follows from Claim 3.5 that $T \cong T_{t-1}(n - c_t)$. Therefore,

$$|H'| - t_{t-1}(n - c_t) = |H' \setminus T| = c_t(n - c_t) - m'. \tag{11}$$

On the other hand, by viewing $T_{t-1}(n)$ as a graph obtained from $T_{t-1}(n - c_t)$ by adding c_t new vertices into some parts, we obtain

$$t_{t-1}(n) - t_{t-1}(n - c_t) \leq c_t(n - c_{t-1} - 1).$$

By combining these two inequalities, we obtain

$$|H'| - t_{t-1}(n) \geq c_t(c_{t-1} + 1 - c_t) - m' \geq (c_t - 1)(c_{t-1} - c_t) + c_t > 0,$$

proving that $e > t_{t-1}(n)$. □

Claim 3.7. *The sequence $(|C_1|, \ldots, |C_k|)$ of part sizes is equal to $a^* = a^*(n, e)$.*

Proof. Recall that $t = k$ and, by (11), we have that

$$|H'| - t_{k-1}(n - c_k) = c_k(n - c_k) - m' \leq c_k(n - c_k). \tag{12}$$

Let us show that c_k is the smallest nonnegative integer a satisfying

$$f(a) := a(n - a) + t_{k-1}(n - a) \geq e.$$

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This inequality holds for $a = c_k$ by (12). Note that $c_k \leq n/k$ as it is the smallest among $c_1 + \cdots + c_k = n$. Thus, by Lemma 2.6, it is enough to check that $a = c_k - 1$ violates this condition. Notice that

$$f(c_k - 1) - f(c_k) \leq 2c_k - n - 1 + (n - c_k - c_{k-1}) = c_k - c_{k-1} - 1.$$

Therefore, it follows from $m' \leq c_{k-1} - c_k$ that

$$f(c_k - 1) \leq f(c_k) - (m' + 1) \leq |H'| + m' - (m' + 1) < |H'|,$$

as desired.

Thus $c_k = a^*_k$ and (since $t = k$ by Claim 3.6) we have $(c_1, \ldots, c_k) = a^*$ by Claim 3.5, as desired. □

Also, it follows from the definitions that $m' = m^*$ and thus H' is isomorphic to $H^*(n, e)$. This completes the proof of Lemma 3.2. □

Now we are ready to prove Proposition 3.1.

Proof of Proposition 3.1. Let $G \in \mathcal{H}_0^{\min}(n, e)$ be arbitrary. Let $B_1, \ldots, B_{k-1}$ be a vertex partition such that G is the union of $K[B_1, \ldots, B_{k-1}]$ with a triangle-free graph J. Let $b_i := |B_i|$ for $i \in [k-1]$. Apply Steps 1–4 to G to obtain a k-partite graph H' with parts $A_1, \ldots, A_k$. By Lemma 3.2, we have $H' \cong H^* := H^*(n, e)$. Assume that $|A_i| = a^*_i$ for $i \in [k]$ and that all missing edges of H' (if any exist) go between A_{k-1} and A_k.

The following claim follows from the definitions of Steps 1–4.

Claim 3.8. *If $i \in [k-1]$ satisfies $|G[B_i]| \in \{0, t_2(b_i)\}$, then $H'[B_i] = G[B_i]$.*

Since H' is k-partite, it follows from the definitions of Steps 1–4 that exactly one part B_p of G is divided into $A_q \cup A_s$ in Steps 2–3, where, say, $1 \leq q < s \leq k$, while the remaining parts of G correspond to the remaining parts of H'. In particular, $b_p = a^*_q + a^*_s$.

Claim 3.9. *We have $|G[B_p]| > 0$.*

Proof. It follows from $m^* \leq a^*_{k-1} - a^*_k$ that

$$|H'[B_p]| = a^*_q a^*_s - m^* \geq a^*_q a^*_s - (a^*_{k-1} - a^*_k) > 0.$$

Combined with Claim 3.8, we see that $|G[B_p]| > 0$. □

Suppose first that $m^* = 0$. Then $H' = K[A_1, \ldots, A_k]$, and G can be obtained from H' by replacing $H'[A_q \cup A_s]$ with $G[B_p]$. Moreover, $G[B_p]$ is a triangle-free graph with $a^*_q + a^*_s$ vertices and $a^*_q a^*_s$ edges. If $a^*_s = a^*_k$, then it follows from the definition of $\mathcal{H}^*_1(n, e)$ that $G \in \mathcal{H}^*_1(n, e)$. Otherwise, $|a^*_q - a^*_s| \leq 1$ (by the definition of a^*), and hence, $G[B_p] \cong T_2(a^*_q + a^*_s)$. This implies that G does not contain any partially full part, and hence, $G = H' \in \mathcal{H}^*_1(n, e)$.

Suppose that $m^* > 0$. Since $G[A_i, A_j]$ is complete for all $\{i, j\} \neq \{q, s\}$ and $H'[A_i, A_j]$ is complete iff $\{i, j\} \neq \{k-1, k\}$, we have $\{q, s\} = \{k-1, k\}$. Thus G can be obtained from $K[A_1, \ldots, A_k]$ by replacing $K[A_{k-1} \cup A_k]$ with a triangle-free graph with $a^*_{k-1}a^*_k - m^*$ edges. This gives $G \in \mathcal{H}^*_1(n, e)$. We conclude that $\mathcal{H}_0^{\min}(n, e) \subseteq \mathcal{H}^*_1(n, e)$. Since $\mathcal{H}^*_1(n, e) \subseteq \mathcal{H}_0(n, e)$ and every graph in $\mathcal{H}^*_1(n, e)$ contains the same number of K_r's, we have $\mathcal{H}_0^{\min}(n, e) = \mathcal{H}^*_1(n, e)$. □

4. Proof of Theorem 1.1

With Proposition 3.1 in hand, we are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Fix integers $n \geq r \geq 3$ and $e \leq \binom{n}{2}$. Notice that (4) can be reduced to $\min\{N(K_r, G) : G \in \mathcal{K}(n, e)\} \geq h^*_r(n, e)$, since the other direction is trivially true. Suppose that

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$G \in \mathcal{K}^{\min}(n, e)$ is a graph obtained from a complete ℓ-partite graph by adding a triangle-free graph to one part. We aim to show that $N(K_r, G) \geq h_r^*(n, e)$ when $r \geq 3$ and, in addition, $G \in \mathcal{H}_1^*(n, e)$ when $r \geq 4$ and $e > t_{r-1}(n)$. We prove this statement by induction on $\ell + r$. Notice that if $\ell = k - 1$ (where $k = k(n, e)$) and $r \geq 4$, then $G \in \mathcal{H}_0(n, e)$, and it follows from Proposition 3.1 that $G \in \mathcal{H}_1^*(n, e)$, as desired. If $\ell = k - 1$ and $r = 3$, then $G \in \mathcal{H}_0(n, e)$, and it follows from [9, Proposition 1.5] that $N(K_3, G) \geq h_3^*(n, e)$. So the statement is true for all pairs (ℓ, r) with $\ell = k - 1$ and $r \geq 3$, and this serves as our base case.

Assume that $\ell \geq k$ and $r \geq 3$. Let $U_1 \cup \cdots \cup U_\ell = V(G)$ be a partition such that G is obtained from the complete ℓ-partite graph $K[U_1, \ldots, U_\ell]$ by adding a triangle-free graph into U_ℓ. We can assume that U_ℓ is not an independent set (otherwise consider instead the $(\ell - 1)$-partition of $V(G)$ where $U_{\ell-1}$ and U_ℓ are merged together).

First, we prove (4). Assume that $\ell \geq r - 1$, as otherwise $h_r^*(n, e) = 0$ and there is nothing to do. Note that U_ℓ is as large as any other part: if some part U_i has strictly larger size then by moving all edges from U_ℓ to U_i (by $|U_i| > |U_\ell|$ there is enough space for this) we strictly decrease the number of r-cliques (since $\ell \geq r - 1$), a contradiction. By relabelling parts $U_1, \ldots, U_{\ell-1}$, we may assume that U_1 is of smallest size among $U_1, \ldots, U_{\ell-1}$. Let $\hat{G}$ denote the induced subgraph of G on $U_2 \cup \cdots \cup U_\ell$. Let $\hat{n} := n - |U_1|$ and $\hat{e} := |\hat{G}|$. Let $\hat{k} := k(\hat{n}, \hat{e})$ be as defined in (2) (while we reserve k for $k(n, e)$).

Claim 4.1. *We have $\hat{k} \leq k$.*

Proof. Let $H^* = H^*(n, e)$ be the k-partite graph as defined in Section 1. Assume that $A_1^*, \ldots, A_k^*$ are the corresponding parts of H^* of sizes $a_1^* \geq \cdots \geq a_k^*$, respectively. It is clear that $|A_1^*| \geq \frac{n}{k}$. It follows from the minimality of U_1 that $|U_1| \leq \frac{n-|U_\ell|}{\ell-1} \leq \frac{n}{k} \leq |A_1^*|$. Let $W_1 \subseteq A_1^*$ be a set of size $|U_1|$ and let H' be the induced subgraph of H^* on $V(H) \setminus W_1$. Observe that H' is still a k-partite graph and $|H'| \geq |\hat{G}|$. So it follows from the definition that $\hat{k} \leq k$. □

Note that $\hat{G}$ can be viewed as a graph obtained from a complete $(\ell - 1)$-partite graph by adding a triangle-free graph into one part; in particular, $\hat{G} \in \mathcal{K}(\hat{n}, \hat{e})$. Let $\hat{H}$ be $H^*(\hat{n}, \hat{e})$ and let G' be the graph obtained from G by replacing $\hat{G}$ with $\hat{H}$. It follows from the inductive hypothesis that

$$N(K_r, \hat{H}) = h_r^*(\hat{n}, \hat{e}) \leq N(K_r, \hat{G}) \quad \text{and} \quad N(K_{r-1}, \hat{H}) \leq N(K_{r-1}, \hat{G}).$$

Hence,

$$\begin{aligned} h_r^*(n, e) \leq N(K_r, G') &= N(K_r, \hat{H}) + |U_1| \cdot N(K_{r-1}, \hat{H}) \\ &\leq N(K_r, \hat{G}) + |U_1| \cdot N(K_{r-1}, \hat{G}) = N(K_r, G), \end{aligned}$$

finishing the inductive step for proving (4).

Now suppose that $r \geq 4$ and $e > t_{r-1}(n)$, and suppose for contradiction that $G \notin \mathcal{H}_1^*(n, e)$. Reusing the notation introduced above, let us first derive a contradiction from assuming that $\hat{G} \notin \mathcal{H}_1^*(\hat{n}, \hat{e})$.

If $\hat{e} > t_{r-1}(\hat{n})$, then it follows from the inductive hypothesis that

$$N(K_r, \hat{H}) < N(K_r, \hat{G}) \quad \text{and} \quad N(K_{r-1}, \hat{H}) \leq N(K_{r-1}, \hat{G}).$$

Therefore,

$$\begin{aligned} N(K_r, G') &= N(K_r, \hat{H}) + |U_1| \cdot N(K_{r-1}, \hat{H}) \\ &< N(K_r, \hat{G}) + |U_1| \cdot N(K_{r-1}, \hat{G}) = N(K_r, G), \end{aligned} \tag{13}$$

contradicting the minimality of G.

So suppose that $\hat{e} \leq t_{r-1}(\hat{n})$. We have that $\ell \geq k \geq r$. Recall that $\hat{G}$ is a graph obtained from an $(\ell - 1)$-partite graph by adding a non-empty triangle-free graph. Thus, we have $N(K_r, \hat{H}) = 0 <$

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$N(K_r, \hat{G})$. In addition, by (4), we have $N(K_{r-1}, \hat{H}) = h^*_{r-1}(\hat{n}, \hat{e}) \le N(K_{r-1}, \hat{G})$. But then the same calculation as in (13) gives a contradiction to the minimality of G.

Thus we have that $\hat{G} \in \mathcal{H}^*_1(\hat{n}, \hat{e})$. Let $\hat{A}^*_1 \cup \ldots \cup \hat{A}^*_{\hat{k}} = V(\hat{G})$ be the partition of $\hat{G}$ as in the definition of $\mathcal{H}^*_1(\hat{n}, \hat{e})$. Let $B_1 := U_1 \cup \hat{A}^*_1$, $B_i := \hat{A}^*_i$ for $2 \le i \le \hat{k} - 2$, and $B_{\hat{k}-1} := \hat{A}_{\hat{k}-1} \cup \hat{A}_{\hat{k}}$. We can view G as a graph obtained from $K[B_1, \ldots, B_{\hat{k}-1}]$ by adding triangle-free graphs into two parts, namely $G[B_1]$ and $G[B_{\hat{k}-1}]$. Since $\hat{k} \le k$ by Claim 4.1, it holds that $G \in \mathcal{H}_0(n, e)$. Therefore, it follows from Proposition 3.1 that $G \in \mathcal{H}^*_1(n, e)$, finishing the proof of Theorem 1.1. □

Let us remark that if we replace the family $\mathcal{K}(n, e)$ in Theorem 1.1 by the larger family $\mathcal{K}'(n, e)$ that consists of all graphs obtained from a complete partite graph by adding a triangle-free graph (that is, we allow to add edges into more than one part) then the theorem will remain true. Indeed, for $r \ge 4$, the proof of Lemma 2.5 (which in fact works for any number of parts) shows that every extremal graph $\mathcal{K}'(n, e)$ has at most one partially full part and thus belongs to $\mathcal{K}(n, e)$. For $r = 3$, the equality in (4), will also remain true (again by the proof of Lemma 2.5 except the inequality in (6) becomes equality).

5. Proof of Proposition 1.2

Proof of Proposition 1.2. First, we prove that $N(K_r, H) = h^*_r(n, e)$ for all $H \in \mathcal{H}^*_2(n, e)$. Fix $H \in \mathcal{H}^*_2(n, e)$.

First consider the case when $(|A_1|, \ldots, |A_k|) = a^*$, where the sets $A_1, \ldots, A_k$ are as in the definition of $\mathcal{H}^*_2(n, e)$. Let $K := K[A_1, \ldots, A_k]$, and $m^*_i := |\overline{H}[B_i, A_i]|$ for $i \in I := \{j \in [k-1] : |A_j| = |A_{k-1}|\}$. Note from the definition of I that for all $i \in I$, we have that

$$N(K_{r-2}, K[A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_{k-1}]) = N(K_{r-2}, K[A_1, \ldots, A_{k-2}]),$$

because we count r-cliques in two isomorphic graphs. Therefore,

$$\begin{aligned} N(K_r, K) - N(K_r, H) &= \sum_{i \in I} m^*_i \cdot N(K_{r-2}, K[A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_{k-1}]) \\ &= \sum_{i \in I} m^*_i \cdot N(K_{r-2}, K[A_1, \ldots, A_{k-2}]) \\ &= m^* \cdot N(K_{r-2}, K[A_1, \ldots, A_{k-2}]) = N(K_r, K) - N(K_r, H^*). \end{aligned} \tag{14}$$

It follows that $N(K_r, H) = N(K_r, H^*) = h^*(n, e)$, as desired.

Now suppose that $(|A_1|, \ldots, |A_k|) \ne a^*$. Recall that then $m^* = 0$, $(|A_1|, \ldots, |A_k|) = (a^*_2, \ldots, a^*_{k-1}, a^*_1 - 1, a^*_k + 1)$, $m = a^*_1 - a^*_k + 1$, and H is a graph obtained from $K[A_1, \ldots, A_k]$ by removing some m edges. We may assume that these m edges were removed from parts $[A_{k-1}, A_k]$, since this does not affect the value of $N(K_r, H)$ by the calculation in (14). Now, by viewing H as a graph obtained from $K[A_1, \ldots, A_k]$ by replacing $K[A_{k-1}, A_k]$ with a triangle-free graph, we see that $H \in \mathcal{H}^*_1(n, e)$, and hence, $N(K_r, H) = h^*(n, e)$.

Next, we show that there are infinitely many pairs $(n, e) \in \mathbb{N}^2$ with $t_{r-1}(n) < e \le \binom{n}{2}$ such that $\mathcal{H}^*_2(n, e) \setminus \mathcal{H}^*_1(n, e) \ne \emptyset$. It is enough to chose (n, e) so that $a^*_{k-2} = a^*_{k-1}$ and $m^*, a^*_k \ge 2$; the choice that we use (in (15) below) is rather arbitrary.

Take any integers $p \ge r - 1$, $q \ge 100$, and $2 \le m \le q$. Let $n := 2pq + q$ and $e := \binom{p}{2}(2q)^2 + 2pq^2 - m$. Note that $e + m$ is the number of edges in the complete $(p+1)$-partite graph $K_{2q,\ldots,2q,q}$ with p parts of size $2q$ and one part of size q. The choice of (p, q, m) ensures that

$$e = \binom{p}{2}(2q)^2 + 2pq^2 - m > \binom{p}{2}\left(\frac{2pq + q}{p}\right)^2 \ge t_p(n).$$

By $e < e + m \le t_{p+1}(n)$, we have that $k(n, e) = p$.

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Let us show that $a_p^* = q$. By Lemma 2.6, it is enough to show that $(q-1)(n-q-1) + t_{k-1}(n-q-1) < e$. The left-hand side here is the size of the graph obtained from the complete partite graph $K_{2q,\ldots,2q,q}$ by moving a vertex from the part of size q into one of size $2q$. This results in losing $q+1 > m$ edges, giving the required. Thus,

$$a_1^* = \cdots = a_{p-1}^* = 2q, \quad a_p^* = q, \quad \text{and} \quad m^* = m. \tag{15}$$

Let $V_1 \cup \cdots \cup V_{p+1} = [n]$ be a partition such that $|V_1| = \cdots = |V_p| = 2q$ and $|V_{p+1}| = q$. Fix m distinct vertices $v_1, \ldots, v_m \in V_{p+1}$, and choose a vertex $u_i \in V_i$ for every $i \in [m]$. Let G be the graph obtained from $K[V_1, \ldots, V_{p-1}]$ by removing pairs in $\{\{v_i, u_i\} : i \in [m]\}$. It is easy to see that $G \in \mathcal{H}_2^*(n,e) \setminus \mathcal{H}_1^*(n,e)$, proving Proposition 1.2. □

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