Proceedings of the Edinburgh Mathematical Society (2010) **53**, 211–237 ©
DOI:10.1017/S0013091507000867 Printed in the United Kingdom

ON THE REGULARITY OF THE HARDY–LITTLEWOOD MAXIMAL OPERATOR ON SUBDOMAINS OF $\mathbb{R}^n$

HANNES LUIRO

Department of Mathematics and Statistics, University of Jyväskylä, PO Box 35 (MaD), 40014 University of Jyväskylä, Finland (haluiro@maths.jyu.fi)

(Received 23 July 2007)

Abstract We establish the continuity of the Hardy–Littlewood maximal operator on $W^{1,p}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is an arbitrary subdomain and $1 < p < \infty$. Moreover, boundedness and continuity of the same operator is proved on the Triebel–Lizorkin spaces $F^p_{s,q}(\Omega)$ for $1 < p, q < \infty$ and $0 < s < 1$.

Keywords: maximal function; Sobolev spaces; regularity

2000 *Mathematics subject classification:* Primary 42B25, 46E35, 47H99

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a subdomain. The local Hardy–Littlewood maximal operator M_Ω is defined for measurable $f : \Omega \mapsto \mathbb{R}$ by

$$M_\Omega f(x) = \sup_{0<r<d(x,\Omega^c)} \fint_{B(x,r)} |f(z)|\, dz. \tag{1.1}$$

In the case where $\Omega = \mathbb{R}^n$ it is understood that the supremum above is taken over all $r > 0$, and in this case we denote $M_{\mathbb{R}^n} = M$.

The maximal operator is one of the most important tools in modern harmonic analysis. Often it is used to estimate the absolute size of functions, but it is also natural to inquire how the maximal operator preserves the differentiability properties of functions. As a first step in this direction, Kinnunen observed [**6**] that M is bounded on $W^{1,p}(\mathbb{R}^n)$. Several articles have since been dedicated to this issue. Korry proved in [**10**] that M is also bounded on the Triebel–Lizorkin spaces $F^p_{s,q}(\mathbb{R}^n)$ when $1 < p, q < \infty$ and $0 < s < 1$. The fractional maximal operator was studied in [**8**]. The study of the local operator brings in new difficulties, and its boundedness was proved by Kinnunen and Lindqvist [**7**]. Later, Hajlasz and Onninen [**5**] found a simpler proof of this result.

For nonlinear operators it is important to note that continuity is not implied by boundedness. An important example is provided by the symmetric decreasing rearrangement on $W^{1,p}(\mathbb{R}^n)$, as was shown by Almgren and Lieb [**1**]. Accordingly, it was asked in [**5**] (the authors attribute the question to Tadeusz Iwaniec) whether the maximal operator

M is continuous on $W^{1,p}(\mathbb{R}^n)$ when $1 < p < \infty$. This question was answered by the author in [**12**]. The proof of the positive answer required the development of new techniques which allow more careful analysis of the radii for which the supremum is attained in (1.1).

Our main result, Theorem 2.12, establishes the continuity of the local maximal operator M_Ω on the Sobolev spaces $W^{1,p}(\Omega)$ when $1 < p < \infty$. The proof starts along the same lines as in [**12**], but the case of subdomains introduces new fairly subtle difficulties arising from the boundary effects (see Remark 2.13). This accounts for the length of the argument. The result is actually formulated more generally for Orlitz–Sobolev spaces, but the main difficulties are already present in the Sobolev space case.

We also extend the works of Korry [**10**] and Kinnunen and Lindqvist [**7**] by establishing the boundedness and continuity of M_Ω on the Triebel–Lizorkin spaces $F^p_{s,q}(\Omega)$. As in the case of Sobolev spaces, the known arguments for M do not transfer easily for M_Ω. The basic obstacle is that in the local case the maximal operator does not commute with translations. However, the fact that we deal with smoothness $s \in (0,1)$ makes the proof of continuity somewhat easier than in the case of standard Sobolev spaces.

The structure of paper is as follows. In § 2 we first recall the definitions of the Orlitz–Sobolev spaces and extend several auxiliary results introduced in [**12**] to the case of subdomains. We have included complete proofs for the sake of readability. The remaining lemmas in this section are specific to the local case and allow us to complete the proof of the continuity of M_Ω in Sobolev spaces. In § 3 we recall the definition of Triebel–Lizorkin spaces and prove the boundedness of M_Ω on these spaces. Finally, § 4 treats the continuity of M_Ω on Triebel–Lizorkin spaces.

2. Continuity of the maximal operator in $W^{1,\psi}(\Omega)$

Let us first introduce some notation. If $A \subset \mathbb{R}^n$ and $r \in \mathbb{R}^n$, we define

$$d(r, A) := \inf_{a \in A} |r - a| \quad \text{and} \quad A_{(\lambda)} := \{x \in \mathbb{R}^n : d(x, A) \leqslant \lambda\} \text{ for } \lambda \geqslant 0.$$

When $\Omega \subset \mathbb{R}^n$, we denote $d(x, \Omega^{\mathrm{c}})$ by $\delta(x)$. The notation $K \subset\subset \Omega$ means that K is open and bounded and $\bar{K} \subset \Omega$.

As mentioned in § 1 we consider the main question in a more general setting of Orlicz–Sobolev spaces. The kind of Orlicz spaces in which M_Ω is bounded is well known (see, for example, [**9**, § 1.2]). We will see that if ψ is such that M_Ω is bounded in $L^\psi(\Omega)$, then the boundedness in Orlicz–Sobolev space $W^{1,\psi}(\Omega)$ (see Lemma 2.1) is an easy corollary of the result of [**5**]. As our first main theorem, we prove the continuity of the maximal operator in these spaces. Of course, this also proves the continuity of the maximal operator in $W^{1,p}(\Omega)$, $1 < p < \infty$, which corresponds to the case when $\psi(t) = t^p$. This case also includes all the real difficulties, and the reader, if desired, may assume that we are working with Sobolev spaces.

Let us now define the Orlicz–Sobolev spaces. To this end, let us assume that ψ is an increasing convex function on $[0, \infty)$ such that

$$\lim_{t \to 0_+} \frac{\psi(t)}{t} = \lim_{t \to \infty} \frac{t}{\psi(t)} = 0.$$

https://doi.org/10.1017/S0013091507000867 Published online by Cambridge University Press

The Orlicz space $L^{\psi}(\Omega)$ (see [**9**, § 1.1]) consists of those measurable functions $f : \Omega \mapsto \mathbb{R}$ such that

$$\int_{\Omega} \psi\left(\frac{|f(x)|}{\lambda}\right) \mathrm{d}x < \infty \tag{2.1}$$

for some $\lambda > 0$. This expression does not give us the norm, but, setting

$$\|f\|_{L^{\psi}} = \inf\left\{\lambda > 0 : \int_{\Omega} \psi\left(\frac{|f(x)|}{\lambda}\right) \mathrm{d}x \leqslant 1\right\}, \tag{2.2}$$

we get a norm (the so-called Luxemburg norm) on $L^{\psi}(\Omega)$. The Orlicz–Sobolev space $W^{1,\psi}(\Omega)$ (see, for example, [**11**, § 7.1]) consists of functions $f \in L^{\psi}(\Omega)$ for which the weak partial derivatives $D_i f \in L^{\psi}(\Omega)$ for all $i \leqslant n$. We define the norm by setting

$$\|f\|_{1,\psi} = \|f\|_{L^{\psi}} + \|\nabla f\|_{L^{\psi}},$$

where ∇f is the weak gradient of f. Let us also denote by $\|f\|_{\psi,A}$ the L^{ψ}-norm of $\chi_A f$ for all measurable sets $A \subset \Omega$.

The boundedness of the maximal operator in Orlicz space $L^{\psi}(\Omega)$ holds if and only if the function ψ^{α} is quasiconvex for some $\alpha \in (0, 1)$ [**9**, § 1.2.1]. In this case, for simplicity, we say that ψ satisfies property (Q). Here quasiconvexity means that there exists a convex function w such that

$$w(t) \leqslant \psi^{\alpha}(t) \leqslant cw(ct) \tag{Q}$$

for some constant $c > 0$. In particular, when (Q) holds, we can exploit in proofs the fact that

$$L^{\psi}(\Omega) \subset L^{p}_{\mathrm{loc}}(\Omega) \tag{2.3}$$

for some $p > 1$, where p depends on α. This follows from the fact that if w is convex, then, for some $c > 0$, $w(t) \geqslant ct$ when $t > 1$. Therefore, quasiconvexity of ψ^{α} says that $\psi(t)^{\alpha} \geqslant w(t) \geqslant ct$, implying that $\psi(t) \geqslant (ct)^{1/\alpha}$.

The following lemma guarantees that when (Q) holds we also have the boundedness of M_{Ω} in $W^{1,\psi}(\Omega)$.

Lemma 2.1. *M_{Ω} is bounded in $W^{1,\psi}(\Omega)$ if it is bounded in $L^{\psi}(\Omega)$.*

Proof. Let $f \in W^{1,\psi}(\Omega)$. Boundedness in $L^{\psi}(\Omega)$ implies that $M_{\Omega} f < \infty$ almost everywhere (a.e.). Moreover, $M_{\Omega}(\nabla f) \in L^{p}_{\mathrm{loc}}(\Omega)$, for some $p > 1$. This follows from the observation (2.3). Then by [**5**, Theorem 3] we have that

$$|\nabla M_{\Omega} f(x)| \leqslant 2 M_{\Omega} |\nabla f|(x) \tag{2.4}$$

for almost every $x \in \Omega$. Now the boundedness in $L^{\psi}(\Omega)$ implies the boundedness in $W^{1,\psi}(\Omega)$. □

Let us then define the sets of the 'best radii' for function $f \in L^{1}_{\mathrm{loc}}(\mathbb{R}^n)$ at point $x \in \Omega$. This useful concept was introduced in [**12**, § 2.1] in the case of $\mathbb{R}^n$. For the definition for every $x \in \Omega$ attach the function $u_x : [0, \delta(x)] \mapsto \mathbb{R}$ such that

$$u_x(0) = |f(x)| \quad \text{and} \quad u_x(r) = \fint_{B(x,r)} |f(y)| \,\mathrm{d}y \text{ when } r \in (0, \delta(x)], \tag{2.5}$$

whence it holds that

$$M_\Omega f(x) = \sup_{r\in(0,\delta(x))} u_x(r),$$

and simply define

$$\mathcal{R}f(x) = \{r \in [0,\delta(x)] : M_\Omega f(x) = u_x(r)\}. \tag{2.6}$$

In the other words, $\mathcal{R}f(x)$, the set of the best radii at point x, consists of those $r \in [0,\delta(x)]$ for which the maximal average is attained. In the above definition the central fact is that functions u_x are continuous for almost all x. The continuity on $(0,\delta(x)]$ is trivial (holds for every x) and at 0 it follows a.e., since almost every $x \in \Omega$ is a Lebesgue point for f. Therefore, if $x \in \Omega$ is a Lebesgue point of f, then function u_x reaches its maximum on a closed interval $[0,\delta(x)]$; thus, $\mathcal{R}f(x)$ is non-empty. Moreover, it is easy to see that $\mathcal{R}f(x)$ is also closed.

In the next lemma we show how the sets $\mathcal{R}f(x)$ and $\mathcal{R}g(x)$ are close to each other when $\|f-g\|_\psi$ is small. This lemma is a counterpart for the result in [**12**, Lemma 2.2] in the global case. Also, the proof is essentially the same as in the case of $L^p(\mathbb{R}^n)$, but for the reader's convenience we give here the whole proof (moreover, we need a part of the proof when proving Lemma 2.3).

Lemma 2.2. *Assume that M_Ω is bounded in $L^\psi(\Omega)$ and suppose that $f_j \to f$ in $L^\psi(\Omega)$ when $j \to \infty$. Then for all $R > 0$ and $\lambda > 0$ it holds that*

$$m(\{x \in \Omega \cap B_R : \mathcal{R}f_j(x) \not\subset \mathcal{R}f(x)_{(\lambda)}\}) \to 0 \quad \text{if } j \to \infty.$$

Proof. First we remark that one can verify the measurability of the above set whenever f_j and f are locally integrable functions (see [**12**, Lemma 2.2]). We may assume that the functions f and f_j are non-negative, since $\mathcal{R}f(x) = \mathcal{R}|f|(x)$. Moreover, assume that $f \not\equiv 0$, since the case $f \equiv 0$ is trivial ($\mathcal{R}0(x) = [0,\delta(x)]$ for all x). Fix $\lambda > 0$, $R > 0$ and $\varepsilon > 0$ and define $\Omega_R = B_R \cap \Omega$. For almost every $x \in \Omega_R$ we find a natural number $i(x) \in \mathbb{N}$ so that

$$⨍_{B(x,r)} f(y)\,\mathrm{d}y < M_\Omega f(x) - \frac{1}{i(x)}, \quad \text{when } d(r,\mathcal{R}f(x)) > \lambda. \tag{2.7}$$

Let us verify this: if this is not possible, there is a sequence of radii $(r_k)_{k=1}^\infty$ with

$$⨍_{B(x,r_k)} f(y)\,\mathrm{d}y \to M_\Omega f(x) \quad \text{and} \quad d(r_k,\mathcal{R}f(x)) > \lambda.$$

Since the sequence $(r_k)_{k=1}^\infty$ is bounded, by moving to a subsequence, if desired, we may assume that $r_k \to r$ as $k \to \infty$. Then it is clear that $r \in \mathcal{R}f(x)$. On the other hand, r satisfies $d(r,\mathcal{R}f(x)) \geqslant \lambda$, whence we obtain the desired contradiction.

It follows from (2.7) that there exists $i \in \mathbb{N}$ so that

$$\Omega_R \subset \left\{x : ⨍_{B(x,r)} f(y)\,\mathrm{d}y < M_\Omega f(x) - \frac{1}{i} \text{ if } d(r,\mathcal{R}f(x)) > \lambda\right\} \cup E =: A \cup E, \tag{2.8}$$

https://doi.org/10.1017/S0013091507000867 Published online by Cambridge University Press

where E is a measurable set with $m(E) < \varepsilon$. Furthermore, when i is fixed, let us define

$$A_j = \left\{ x \in \Omega_R : 2M_\Omega(f - f_j)(x) \geqslant \frac{1}{4i} \right\}.$$

The boundedness of M_Ω in $L^\psi(\Omega)$ implies that there exists $j_0 \in \mathbb{N}$ so that $m(A_j) < \varepsilon$, when $j \geqslant j_0$.

Finally, suppose that $x \in \Omega_R \cap (A_j \cup E)^{\mathrm{c}}$ and $r \in [0, \delta(x)]$ such that $d(r, \mathcal{R}f(x)) > \lambda$. Then

$$\begin{aligned} \fint_{B(x,r)} |f_j(y)| \,\mathrm{d}y &= M_\Omega f_j(x) - (M_\Omega f_j(x) - M_\Omega f(x)) \\ &\quad - \left(M_\Omega f(x) - \fint_{B(x,r)} |f(y)| \,\mathrm{d}y \right) \\ &\quad - \left(\fint_{B(x,r)} |f(y)| \,\mathrm{d}y - \fint_{B(x,r)} |f_j(y)| \,\mathrm{d}y \right) \\ &=: M_\Omega f_j(x) - s_1 - s_2 - s_3. \end{aligned}$$

Above we have $|s_1|, |s_3| \leqslant M_\Omega(f_j - f)(x)$ (for s_1 we use the sublinearity of M_Ω), which implies that (since $x \in A_j^{\mathrm{c}}$) $|s_1| + |s_3| \leqslant 1/4i$. On the other hand, since $x \in E^{\mathrm{c}}$ and $d(r, \mathcal{R}f(x)) > \lambda$, we get that $s_2 > 1/i$. Combining these estimates, we conclude that

$$\fint_{B(x,r)} |f_j(y)| \,\mathrm{d}y \leqslant M_\Omega f_j(x) + \frac{1}{4i} - \frac{1}{i} < M_\Omega f_j(x). \tag{2.9}$$

In particular, $r \notin \mathcal{R}f_j(x)$ and we have proved that $\mathcal{R}f_j(x) \subset \mathcal{R}f(x)_{(\lambda)}$. This verifies that

$$\{x \in \Omega_R : \mathcal{R}f_j(x) \not\subset \mathcal{R}f(x)_{(\lambda)}\} \subset A_j \cup E. \tag{2.10}$$

Here $m(A_j \cup E) < 2\varepsilon$, when $j \geqslant j_0$. This completes the proof. □

Before the next lemma, let us introduce some notation. Assume that $f \in L^p(\Omega)$, $1 \leqslant p < \infty$. Let e_i be one of the standard basis vectors of $\mathbb{R}^n$. For $h \in \mathbb{R}$, $h \neq 0$, we define the functions f_h^i and $f_{\tau(h)}^i$ by setting

$$f_h^i(x) = \frac{f(x + he_i) - f(x)}{h} \quad \text{and} \quad f_{\tau(h)}^i(x) = f(x + he_i). \tag{2.11}$$

These functions are well defined in the set $\{x \in \Omega : \delta(x) > |h|\}$. In particular, for all $K \subset\subset \Omega$ these functions are well defined in K when $|h|$ is small enough. We know that for all $K \subset\subset \Omega$ we have $f_{\tau(h)}^i \to f$ in $L^p(K)$ when $h \to 0$ and, if $p > 1$, for functions $f \in W^{1,p}(\Omega)$ [**3**, § 7.11], that $f_h^i \to D_i f$ in $L^p(K)$ when $h \to 0$.

Next we prove a lemma which is very similar to the previous one. We study how close the sets $\mathcal{R}f(x + he_i)$ and $\mathcal{R}f(x)$ are when h is small. In the case when $\Omega = \mathbb{R}^n$ this is obvious by using Lemma 2.2. Hypothetically this is also the case if $\Omega \neq \mathbb{R}^n$ but technically one has to be very careful with translations in the case of subdomains and therefore we found it necessary to treat this case separately.

Lemma 2.3. *Let $f \in L^p(\Omega)$, $1 < p < \infty$. Then for all i, $1 \leqslant i \leqslant n$, $\lambda > 0$ and $K \subset\subset \Omega$ one has*

(i) $m(\{x \in K : \mathcal{R}f(x+he_i) \not\subset \mathcal{R}f(x)_{(\lambda)}\}) \xrightarrow{h\to 0} 0$,

(ii) $m(\{x \in K : \mathcal{R}f(x) \not\subset \mathcal{R}f(x+he_i)_{(\lambda)}\}) \xrightarrow{h\to 0} 0$.

Proof. Let us first prove (i). The proof is very similar to that of Lemma 2.2. We repeat the first part of the proof and assume that the sets A and E, depending on f, ε and λ, are chosen (see (2.8)). Then, define

$$A_h = \left\{x \in K : M(\chi_\Omega f^i_{\tau(h)} - \chi_\Omega f)(x) > \frac{1}{2i}\right\},$$
$$B_h = \left\{x \in K : |M_\Omega f(x+he_i) - M_\Omega f(x)| > \frac{1}{4i}\right\}$$

and

$$C_h = \left\{x \in \Omega : \exists r \in [\delta(x) - 2|h|, \delta(x)] \quad \text{such that } \left| \fint_{B(x,r)} f(y)\,\mathrm{d}y - \fint_{B(x,\delta(x)-|h|)} f(y)\,\mathrm{d}y \right| > \frac{1}{8i}\right\},$$

and define our final exceptional set to be

$$E_h := E \cup A_h \cup B_h \cup (C_h - he_i).$$

We observe that the measure of the sets A_h, B_h and C_h tends to zero when $h \to 0$. Therefore, for the claim, it suffices to prove that

$$\{x \in K : \mathcal{R}f(x+he_i) \not\subset \mathcal{R}f(x)_{(2\lambda)}\} \subset E_h$$

for h small enough. Let us prove this: choose $h_0 > 0$ so that $K_{(2h_0)} \subset \Omega$ and $h_0 < \lambda$. Then let $|h| < h_0$ and $x \in A \setminus E_h$ so that there exists $r \in \mathcal{R}f(x+he_i)$ such that $d(r, \mathcal{R}f(x)) > 2\lambda$. We treat separately the (harder) case where r is 'too' close to $\delta(x)$. More precisely, suppose first that

$$r \in [\delta(x) - |h|, \delta(x+he_i)].$$

Then, we have $d(\delta(x) - |h|, \mathcal{R}f(x)) > \lambda$ and we get

$$\begin{aligned} M_\Omega f(x+he_i) &= \fint_{B(x+he_i,r)} f(y)\,\mathrm{d}y \\ &\leqslant \left| \fint_{B(x+he_i,r)} f(y)\,\mathrm{d}y - \fint_{B(x+he_i,\delta(x+he_i)-|h|)} f(y)\,\mathrm{d}y \right| \\ &\qquad + \fint_{B(x+he_i,\delta(x+he_i)-|h|)} f(y)\,\mathrm{d}y \end{aligned}$$

https://doi.org/10.1017/S0013091507000867 Published online by Cambridge University Press

$$
\begin{aligned}
&\leqslant \frac{1}{8i} + \fint_{B(x,\delta(x+he_i)-|h|)} f(y+he_i)\,\mathrm{d}y \\
&\leqslant \frac{1}{8i} + \fint_{B(x,\delta(x+he_i)-|h|)} |f(y+he_i)-f(y)|\,\mathrm{d}y \\
&\qquad\qquad\qquad\qquad + \fint_{B(x,\delta(x+he_i)-|h|)} f(y)\,\mathrm{d}y \\
&\leqslant \frac{1}{8i} + \frac{1}{2i} + \fint_{B(x,\delta(x+he_i)-|h|)} f(y)\,\mathrm{d}y \\
&\leqslant \frac{1}{8i} + \frac{1}{2i} + M_\Omega f(x) - \frac{1}{i} \\
&< M_\Omega f(x) - \frac{1}{4i}.
\end{aligned}
$$

This implies that $|M_\Omega f(x+he_i) - M_\Omega f(x)| > 1/4i$ and contradicts the assumption that $x \notin B_h$. The remaining case, $r < \delta(x) - |h|$, is the easier one: with the same reasoning as above we observe that

$$
\begin{aligned}
M_\Omega f(x+he_i) &= \fint_{B(x+he_i,r)} f(y)\,\mathrm{d}y \\
&= \fint_{B(x,r)} f(y+he_i)\,\mathrm{d}y \\
&\leqslant \fint_{B(x,r)} |f(y+he_i)-f(y)|\,\mathrm{d}y + \fint_{B(x,r)} f(y)\,\mathrm{d}y \\
&\leqslant \frac{1}{2i} + \biggl(M_\Omega f(x) - \frac{1}{i}\biggr),
\end{aligned}
$$

which leads to the same contradiction as above. This completes the proof.

Finally, we observe that (ii) is an easy corollary of (i). For that, let us choose $\varepsilon > 0$ such that $K_{(2\varepsilon)} \subset \Omega$. When $|h| < \varepsilon$ we observe that

$$
\begin{aligned}
&\{x \in K : \mathcal{R}f(x) \not\subset \mathcal{R}f(x+he_i)_{(\lambda)}\} \\
&\qquad = \{x \in K + he_i : \mathcal{R}f(x-he_i) \not\subset \mathcal{R}f(x)_{(\lambda)}\} - he_i \\
&\qquad\qquad \subset \{x \in K_{(\varepsilon)} : \mathcal{R}f(x-he_i) \not\subset \mathcal{R}f(x)_{(\lambda)}\} - he_i.
\end{aligned}
$$

Now the claim follows from (i) and the translation invariance of the measure. □

Remark 2.4. The importance of the previous lemma lies in the following observation. We denote by

$$\pi(A,B) := \inf\{\delta > 0 : A \subset B_{(\delta)} \text{ and } B \subset A_{(\delta)}\}$$

the Hausdorff distance of the sets A and B. Using this notation, Lemma 2.3 appears in the form

$$m(\{x \in K : \pi(\mathcal{R}f(x), \mathcal{R}f(x+he_i)) > \lambda\}) \to 0 \quad \text{when } h \to 0.$$

This guarantees that we find a sequence $(h_k)_{k=1}^{\infty}$, $h_k > 0$ with $h_k \to 0$, and such that $\pi(\mathcal{R}f(x), \mathcal{R}f(x + h_k e_i)) \to 0$ as $k \to \infty$ for almost every $x \in K$. Using this observation we next establish, generalizing [**12**, Theorem 3.1] (the argument of the proof is essentially the same), a useful formula for the derivative of the maximal function.

Theorem 2.5. *Assume that (Q) holds and that $f \in W^{1,\psi}(\Omega)$. Then for almost all $x \in \Omega$ it holds that*

(a) $D_i M_\Omega f(x) = \fint_{B(x,r)} D_i|f|(y)\,\mathrm{d}y$ *for all* $r \in \mathcal{R}f(x)$, $0 < r < \delta(x)$ *and*

(b) $D_i M_\Omega f(x) = D_i|f|(x)$ *if* $0 \in \mathcal{R}f(x)$.

Proof. It is sufficient to prove the claim for non-negative functions, because $M_\Omega f = M_\Omega|f|$ and $|f| \in W^{1,\psi}(\Omega)$ if $f \in W^{1,\psi}(\Omega)$ (because $|\nabla|f|| = |\nabla f|$ a.e.). Let us also first assume that f is in some $W^{1,p}(\Omega)$ for some $1 < p < \infty$. Let $K \subset\subset \Omega$. We start by choosing a sequence $(h_k)_{k=1}^{\infty}$, $h_k > 0$ and $h_k \to 0$ so that $\pi(\mathcal{R}f(x), \mathcal{R}f(x + h_k e_i)) \to 0$ as $k \to \infty$ for almost all $x \in K$ (see Remark 2.4). Then we have

(i) $\|D_i M_\Omega f - (M_\Omega f)^i_{h_k}\|_{p,K} \to 0$ as $k \to \infty$,

(ii) $\|D_i f - f^i_{h_k}\|_{p,K} \to 0$ as $k \to \infty$,

(iii) $\|M_\Omega(D_i f - f^i_{h_k})\|_{p,K} \to 0$ as $k \to \infty$.

Now, by moving to a subsequence if needed, we may assume that the convergences above also hold pointwise almost everywhere. Observe also that

$$\{x \in \Omega : \exists k \in \mathbb{N} \text{ such that } 0 \in \mathcal{R}f(x + h_k e_i) \text{ with } M_\Omega f(x + h_k e_i) \neq f(x + h_k e_i)\}$$

has measure zero as a countable union of sets of measure zero. Let $x \in K$ be such that pointwise analogies of (i)–(iii) hold at x and let $r \in \mathcal{R}f(x)$, $r < \delta(x)$.

Since $\pi(\mathcal{R}f(x), \mathcal{R}f(x + h_k e_i)) \to 0$, there exist radii $r_k \in \mathcal{R}f(x + h_k e_i)$ such that $r_k \to r$ when $k \to \infty$. Suppose first that $r > 0$. We have $r < \delta(x)$, and thus $r_k < \delta(x)$ when k is large, and we can estimate

$$\begin{aligned}
D_i M_\Omega f(x) &= \lim_{k\to\infty} \frac{1}{h_k}(M_\Omega f(x + h_k e_i) - M_\Omega f(x)) \\
&\leqslant \lim_{k\to\infty} \frac{1}{h_k}\left(\fint_{B(x+h_k e_i, r_k)} f(y)\,\mathrm{d}y - \fint_{B(x,r_k)} f(y)\,\mathrm{d}y\right) \\
&= \lim_{k\to\infty} \frac{1}{m(B(x,r_k))} \int_{B(x,r_k)} \frac{f(y + h_k e_i) - f(y)}{h_k}\,\mathrm{d}y \\
&= \fint_{B(x,r)} D_i f(y)\,\mathrm{d}y.
\end{aligned}$$

https://doi.org/10.1017/S0013091507000867 Published online by Cambridge University Press

The last equation holds, because $m(B_{r_k}) \to m(B_r)$ and $\chi_{B(x,r_k)} f^i_{h_k} \to \chi_{B(x,r)} D_i f$ in $L^1(\Omega)$ when $k \to \infty$. On the other hand, we obtain that

$$\begin{aligned} D_i M_\Omega f(x) &\geqslant \lim_{k\to\infty} \frac{1}{h_k}\biggl(\fint_{B(x+h_k e_i, r)} f(y)\,\mathrm{d}y - \fint_{B(x,r)} f(y)\,\mathrm{d}y \biggr) \\ &= \lim_{k\to\infty} \frac{1}{m(B(x,r))} \int_{B(x,r)} \frac{f(y+h_k e_i) - f(y)}{h_k}\,\mathrm{d}y \\ &= \fint_{B(x,r)} D_i f(y)\,\mathrm{d}y. \end{aligned}$$

Now suppose that $r = 0$. The same argument as above applies in this case to show that $D_i M_\Omega f(x) \geqslant D_i f(x)$. If we have $r_k = 0$ for infinitely many k, then it follows that $D_i M_\Omega f(x) = D_i f(x)$. If $r_k > 0$ starting from some k_0, we obtain in the same way as above that

$$D_i M_\Omega f(x) \leqslant \lim_{k\to\infty} \fint_{B(x,r_k)} f^i_{h_k}(y)\,\mathrm{d}y = D_i f(x),$$

since

$$\lim_{k\to\infty} \biggl| \fint_{B(x,r_k)} f^i_{h_k}(y)\,\mathrm{d}y - D_i f(x) \biggr| \leqslant \lim_{k\to\infty} M_\Omega (f^i_{h_k} - D_i f)(x) = 0.$$

Now we have proved the claim for $x \in K$. Since $K \subset\subset \Omega$ was arbitrary, this gives the claim in Ω when f is in $W^{1,p}(\Omega)$ for some $1 < p < \infty$. The claim for general $f \in W^{1,\psi}(\Omega)$ follows easily from this by (2.3). More precisely, suppose that (Q) holds and $f \in W^{1,\psi}(\Omega)$. Let $r > 0$ and observe that

$$R := \sup\{r' : r' \in \mathcal{R}f(x),\ x \in \Omega \cap B(0,r)\} < \infty.$$

Then we choose a function $\omega \in C_0^\infty(\mathbb{R}^n)$ such that $\omega \leqslant 1$ everywhere and $\omega \equiv 1$ in $B(0, R + r)$. Using (2.3) we get that $f\omega \in W^{1,p}(\Omega)$ for some $p > 1$. Also we have $M_\Omega(f\omega) = M_\Omega f$ in $B(0,r) \cap \Omega$. Then, what we proved above applies for $f\omega$ and this yields the desired formula for $D_i M_\Omega f(x)$ a.e. in $B(0,r) \cap \Omega$. Since r was arbitrary, the proof is complete. □

To prove our main result, which is the continuity of the maximal operator in $W^{1,\psi}(\Omega)$, we have to deal with the difficult case where $\delta(x) \in \mathcal{R}f(x)$. At this type of point we do not have the formula for $D_i M f(x)$ which otherwise is successfully used (in the same way as in [**12**, § 3.1]) to estimate the difference of the derivatives of two maximal functions. Because of this, we need several technical lemmas which help us control the behaviour of the derivative of the maximal function at the points where the maximal average is achieved in the largest ball contained in Ω.

The first one of these lemmas is simple, but it has an important role in our argument (see Remark 2.13). In the following, write $f_j \rightharpoonup 0$ if the sequence of functions $f_j \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ converges to zero in the sense of distributions that is, if

$$\int_\Omega f_j \varphi \xrightarrow{j\to\infty} 0$$

for every $\varphi \in C_0^\infty(\Omega)$.

Lemma 2.6. *Assume that $A \subset \Omega$ is measurable. Let f_j be a sequence in $W^{1,1}_{\mathrm{loc}}(\Omega)$ so that $f_j \rightharpoonup 0$ and $|\nabla f_j(x)| \leqslant F(x)$ a.e. in Ω, where $\|F\|_{\psi,\Omega} < \infty$. If, for all $\varepsilon > 0$,*

$$m(\{x \in A : D_i f_j(x) > \varepsilon\}) \to 0 \quad \text{when } j \to \infty \tag{2.12}$$

or

$$m(\{x \in A : D_i f_j(x) < -\varepsilon\}) \to 0 \quad \text{when } j \to \infty, \tag{2.13}$$

then we have $\|D_i f_j\|_{\psi,A} \to 0$.

Proof. Because of the pointwise majorant of $D_i f_j$, it is sufficient to prove that $\|D_i f_j\|_{\psi,K} \to 0$ for all compact $K \subset A$. Moreover, because of symmetry, we may assume that (2.12) holds. Let $\varepsilon > 0$ and suppose that there exists $\lambda > 0$, so that, for all j,

$$m(\{x \in K : D_i f_j(x) < -\varepsilon\}) > \lambda. \tag{2.14}$$

Let us fix an open set $V \subset\subset \Omega$ so that $K \subset V$ and $m(V \setminus K) < \delta(\varepsilon)$. Since K lies strictly inside the set V, it is possible to choose a function $\varphi \in C_0^\infty(\Omega)$ so that $\varphi \equiv 1$ in K, $\varphi \equiv 0$ in $\Omega \setminus V$ and $\varphi \leqslant 1$ everywhere. Now

$$\begin{aligned}\int_\Omega f_j(x) D_i\varphi(x)\,\mathrm{d}x &= -\int_\Omega D_i f_j(x)\varphi(x)\,\mathrm{d}x \\ &= -\int_{V\setminus K} D_i f_j(x)\varphi(x)\,\mathrm{d}x - \int_K D_i f_j(x)\,\mathrm{d}x.\end{aligned}$$

By the pointwise estimate F and the fact that $\varphi \leqslant 1$ we get

$$\biggl|\int_{V\setminus K} D_i f_j(x)\varphi(x)\,\mathrm{d}x\biggr| \leqslant \int_{V\setminus K} F(x)\,\mathrm{d}x < \frac{\varepsilon\lambda}{10}$$

by the absolute continuity if we choose $\delta(\varepsilon)$ properly. Furthermore,

$$-\int_K D_i f_j(x)\,\mathrm{d}x = -\int_{\{x\in K : D_i f_j \leqslant 0\}} D_i f_j(x)\,\mathrm{d}x - \int_{\{x\in K : D_i f_j > 0\}} D_i f_j(x)\,\mathrm{d}x. \tag{2.15}$$

It follows from (2.14) that the absolute value of the first term on the right-hand side of (2.15) is greater than $\varepsilon\lambda$ for all j. For the latter term, write

$$\begin{aligned}-\int_{\{x\in K : D_i f_j > 0\}} D_i f_j(x)\,\mathrm{d}x = \int_{\{x\in K : 0 < D_i f_j < \varepsilon\lambda/10m(K)\}} & D_i f_j(x)\,\mathrm{d}x \\ &+ \int_{\{x\in K : D_i f_j \geqslant \varepsilon\lambda/10m(K)\}} D_i f_j(x)\,\mathrm{d}x.\end{aligned}$$

By the assumption,

$$m\biggl(\biggl\{x \in K : D_i f_j \geqslant \frac{\varepsilon\lambda}{10m(K)}\biggr\}\biggr) \to 0 \quad \text{as } j \to \infty.$$

Again, using the absolute continuity of integral and the pointwise estimate $D_i f_j \leqslant F$, this guarantees that the latter term on the right-hand side above converges to 0 as $j \to \infty$. The absolute value of the first term is clearly less than $\varepsilon\lambda/10$.

Putting the above estimates together, we deduce that

$$\left| \int_\Omega f_j D_i \varphi(x)\,\mathrm{d}x \right| > \frac{\varepsilon\lambda}{2}$$

when j is large. Clearly, this contradicts our assumption that $f_j \rightharpoonup 0$. From this we conclude that our assumption (2.14) cannot hold. Combining this with (2.12) we get that

$$m(\{x \in K : |D_i f_j(x)| > \varepsilon\}) \to 0 \quad \text{when } j \to 0.$$

This is true for every $\varepsilon > 0$ since ε was chosen arbitrarily. Now the claim follows by absolute continuity and the pointwise majorant. □

Corollary 2.7. *Suppose that in Lemma 2.6 we have, instead of the uniform pointwise bound, that* $|\nabla f_j(x)| \leqslant F(x) + F_j(x)$ *a.e. in* Ω *and* $\|F_j\|_{\psi,\Omega} \to 0$ *as* $j \to 0$*. Then, the result of Lemma 2.6 still holds.*

Proof. Suppose, on the contrary, that for some subsequence we have $\|D_i f_j\|_{1,A} > \lambda > 0$. Again, by extracting a subsequence we may assume that $\sum_{j=1}^{\infty} \|F_j\|_{1,\Omega} < \infty$; hence, we get that $|\nabla f_j(x)| \leqslant F(x) + G(x)$, where $G(x) = \sum_{j=1}^{\infty} F_j(x)$ and $\|G\|_{1,\Omega} < \infty$. Using Lemma 2.6 we get a contradiction. □

Recall that the domain $\Omega \subset \mathbb{R}^n$ is uniform with a constant $c > 0$, if for every x and y we find a rectifiable curve $\gamma : [0, l(\gamma)] \mapsto \Omega$, parametrized by the arc length $l(\gamma)$, such that $\gamma(0) = x$ and $\gamma(l(\gamma)) = y$ and

$$d(\gamma(t), \partial\Omega) \geqslant c \min\{t, l(\gamma) - t\} \quad \text{and} \quad l(\gamma) \leqslant \frac{1}{c} d(x, y).$$

We will use the following lemma in the proof of Lemma 2.9 in a very simple case, where $\Omega = B(x, r) \cup B(x + h, r)$.

Lemma 2.8. *Let* Ω *be uniform with a constant* c *and let* $f \in W^{1,1}_{\mathrm{loc}}(\Omega)$*. Suppose that* x *and* y *are Lebesgue points of* f*. Then it holds that*

$$|f(x) - f(y)| \leqslant C(c, n)|x - y|(M(\chi_\Omega|\nabla f|)(x) + M(\chi_\Omega|\nabla f|(y))).$$

Proof. The case when $\Omega = \mathbb{R}^n$ is proved (for example) in [**4**]. By using the Poincaré inequality combined with a standard chaining argument on a suitable ('cigar') path γ joining x and y, we obtain the result in the general case. □

In the following two lemmas we establish some basic convergence properties which we need in computations in the case where $\delta(x) \in \mathcal{R}f(x)$.

https://doi.org/10.1017/S0013091507000867 Published online by Cambridge University Press

Lemma 2.9. *Suppose that $f \in W^{1,\psi}(\Omega)$, where ψ is such that (Q) holds, and $x \in \Omega$. Let r_k and h_k be positive real numbers so that $h_k \to 0$, $r_k \leqslant \delta(x)$ for every k and $r_k \to \delta(x)$ as $k \to \infty$. Moreover, assume that $r_k \leqslant \delta(x + h_k e_i)$ for all k. Then*

$$\lim_{k\to\infty} \fint_{B(x,r_k)} \frac{f(y + h_k e_i) - f(y)}{h_k} \, \mathrm{d}y = \fint_{B(x,\delta(x))} D_i f(y) \, \mathrm{d}y.$$

Proof. To simplify notation, we denote $M(\nabla f) := M(\chi_\Omega |\nabla f|)$. Let k_0 be such that $r_k > h_k$ for all $k > k_0$. The domain $B(x, r_k) \cup B(x + h_k e_i, r_k) \subset \Omega$ is clearly uniform with a constant which does not depend on x or k. Moreover, for almost every $y \in \Omega$ it holds that $y + h_k$ is a Lebesgue point of f for all $k \in \mathbb{N}$. Therefore, by Lemma 2.8, we get that

$$|f^i_{h_k}(y)| = \frac{|f(y + h_k e_i) - f(y)|}{h_k} \leqslant C(M(\nabla f)(y + h_k e_i) + M(\nabla f)(y))$$

for almost every $y \in B(x, r_k)$. From this we conclude that, for small $t > 0$,

$$\begin{aligned} &\left| \int_{B(x,r_k)\setminus B(x,r_k-t)} f^i_{h_k}(y) \, \mathrm{d}y \right| \\ &\qquad\qquad \leqslant \left| \int_{B(x,r_k)\setminus B(x,r_k-t)} C(M(\nabla f)(y + h_k e_i) + M(\nabla f)(y)) \, \mathrm{d}y \right|. \end{aligned}$$

Now, $M(\nabla f) \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and by using the absolute continuity of the integral, we see that for every $\varepsilon > 0$ there exists $t_0 > 0$ such that

$$\left| \int_{B(x,r_k)\setminus B(x,r_k-t)} f^i_{h_k}(y) \, \mathrm{d}y \right| < \varepsilon$$

for all $t \leqslant t_0$. Also we know that

$$\lim_{t\to 0} \int_{B(x,\delta(x)-t)} D_i f(y) \, \mathrm{d}y = \int_{B(x,\delta(x))} D_i f(y) \, \mathrm{d}y$$

and, after that, the claim follows when we use the fact that $B(x, r - t)$ is compactly in Ω, implying that

$$\lim_{k\to\infty} \int_{B(x,r_k-t)} f^i_{h_k}(y) \, \mathrm{d}y = \int_{B(x,\delta(x)-t)} D_i f(y) \, \mathrm{d}y.$$

The above equality holds because $f^i_{h_k} \to D_i f$ in $L^1_{\mathrm{loc}}(\Omega)$. $\square$

Lemma 2.10. *Let $f \in W^{1,\psi}(\Omega)$ and define, for every $h > 0$,*

$$a_h(x) := \frac{1}{h} \left(\fint_{B(x+he_i,\delta(x+he_i))} f(y) \, \mathrm{d}y - \fint_{B(x,\delta(x))} f(y) \, \mathrm{d}y \right).$$

Moreover, assume that $h_k > 0$ and $h_k \to 0$ as $k \to \infty$. Then we have

$$\liminf_{k\to\infty} |a_{h_k}(x)| \leqslant 5 M_\Omega \nabla f(x)$$

almost everywhere.

https://doi.org/10.1017/S0013091507000867 Published online by Cambridge University Press

Proof. We observe that

$$\begin{aligned}|a_h(x)| &\leqslant \frac{1}{h}\biggl|\biggl(⨍_{B(x+he_i,\delta(x+he_i))} f(y)\,\mathrm{d}y - ⨍_{B(x+he_i,\delta(x)-h)} f(y)\,\mathrm{d}y\biggr)\biggr| \\ &\quad + \frac{1}{h}\biggl|\biggl(⨍_{B(x+he_i,\delta(x)-h)} f(y)\,\mathrm{d}y - ⨍_{B(x,\delta(x)-h)} f(y)\,\mathrm{d}y\biggr)\biggr| \\ &\quad + \frac{1}{h}\biggl|\biggl(⨍_{B(x,\delta(x)-h)} f(y)\,\mathrm{d}y - ⨍_{B(x,\delta(x))} f(y)\,\mathrm{d}y\biggr)\biggr| \\ &=: a_1(x,h) + a_2(x,h) + a_3(x,h).\end{aligned}$$

We know that

$$0 \leqslant \frac{1}{h}(\delta(x+he_i) - (\delta(x)-h)) \leqslant 2$$

and, if we assume that $f \in C^1(\Omega)$, we can use the scaling argument to get

$$\begin{aligned}a_1(x,h) &= \frac{1}{h}\biggl|⨍_{B(x+he_i,\delta(x)-h)} \biggl(f\biggl(z + (z-(x+he_i))\biggl(\frac{\delta(x+he_i)}{\delta(x)-h} - 1\biggr)\biggr) - f(z)\biggr)\,\mathrm{d}z\biggr| \\ &\leqslant ⨍_{B(x+he_i,\delta(x)-h)} \biggl(\int_0^1 \biggl|\nabla f\biggl(z + t(z-(x+he_i))\biggl(\frac{\delta(x+he_i)}{\delta(x)-h} - 1\biggr)\biggr)\biggr|\,\mathrm{d}t\biggr)\,\mathrm{d}z \\ &= 2\int_0^1 ⨍_{B(x+he_i,t(\delta(x+he_i)))} |\nabla f(z)|\,\mathrm{d}z\,\mathrm{d}t.\end{aligned}$$

By approximation, we also obtain this estimate for every function $f \in W^{1,\psi}(\Omega)$ and using this inequality we directly obtain that

$$\liminf_{k\to\infty} a_1(x,h_k) \leqslant \liminf_{k\to\infty} 2M_\Omega|\nabla f|(x+h_k e_i) = 2M_\Omega|\nabla f|(x) \quad \text{a.e.}$$

Exactly the same computation also shows that $a_3(x,h_k) \leqslant M_\Omega|\nabla f|(x)$. For $a_2(x,h)$ we get that

$$\begin{aligned}a_2(x,h) &= \biggl|\frac{1}{h}⨍_{B(x,\delta(x)-h)} f(y+h) - f(y)\,\mathrm{d}y\biggr| \\ &\leqslant \frac{1}{h}⨍_{B(x,\delta(x)-h)} \biggl(\int_0^h |D_i f(y+te_i)|\,\mathrm{d}t\biggr)\,\mathrm{d}y \\ &= \frac{1}{h}\int_0^h \biggl(⨍_{B(x,\delta(x)-h)} |D_i f(y+te_i)|\,\mathrm{d}y\biggr)\,\mathrm{d}t \\ &\leqslant \frac{1}{h}\int_0^h M_\Omega|\nabla f|(x+te_i)\,\mathrm{d}t \\ &\leqslant 2⨍ [-h,h] M_\Omega|\nabla f|(x+te_i)\,\mathrm{d}t,\end{aligned}$$

which converges to $2M_\Omega\nabla f(x)$ almost everywhere on almost every line parallel to e_i. Combining the estimates for a_1, a_2 and a_3, we obtain the wanted result. □

The following lemma is needed when we use the difference quotients to give the partial derivatives.

Lemma 2.11. *Let $A_j \subset \mathbb{R}^n$ be measurable sets and let $h_k \in \mathbb{R}^n$ such that $|h_k| \to 0$ when $k \to \infty$. Then we can find a subsequence of (h_k) such that for every j and for almost every $x \in A_j$ we have $x + h_{k_i} \in A_j$ when i is large enough.*

Proof. Let $R > 0$. From basic measure theory we get that $m((A_j + h_k) \setminus A_j) \to 0$ as $k \to \infty$ if $m(A_j) < \infty$. In particular, this means that

$$\int_{\mathbb{R}^n} |\chi_{A_1 \cap B_R}(y) - \chi_{((A_1 \cap B_R)+h_k)}(y)| \, \mathrm{d}y \to 0 \quad \text{as } k \to \infty.$$

By extracting a subsequence, we get that $\chi_{((A_1 \cap B_R)+h_k)} \to \chi_{A_1 \cap B_R}$ a.e. in $\mathbb{R}^n$. By the standard Cantor diagonal argument we now find a subsequence (h_{k_i}) for which $\chi_{A_1 + h_{k_i}} \to \chi_{A_1}$ a.e. as $i \to \infty$. Again, using a diagonal argument we find a subsequence (h_{k_i}) such that $\chi_{A_j + h_{k_i}} \xrightarrow{i \to \infty} \chi_{A_j}$ a.e. for every j. This proves the claim. □

Next we will prove the main theorem of this section, the continuity of M_Ω in $W^{1,\psi}(\Omega)$.

Theorem 2.12. *Let $\Omega \subset \mathbb{R}^n$ be a subdomain and $1 < p < \infty$. Then M_Ω is continuous on $W^{1,p}(\Omega)$. More generally, if M_Ω is bounded on $W^{1,\psi}(\Omega)$, then it is also continuous.*

Proof. We know that M_Ω is bounded on $W^{1,p}(\Omega)$, so it is clearly enough to prove the general claim. Let $f_j \to f$ in $W^{1,\psi}(\Omega)$ when $j \to \infty$. As before, we may assume that the functions f_j and f are non-negative. We have to show that $\|M_\Omega f_j - M_\Omega f\|_{1,\psi} \to 0$. Because we know the continuity of M_Ω in $L^\psi(\Omega)$, it suffices to prove that $\|D_i M_\Omega f_j - D_i M_\Omega f\|_\psi \to 0$ for all i, $1 \leqslant i \leqslant n$.

Because $|D_i M_\Omega f(x)| \leqslant 2M_\Omega|\nabla f|(x)$ almost everywhere (see (2.4)), we observe that

$$\begin{aligned} |D_i(M_\Omega f_j - M_\Omega f)(x)| &\leqslant 2M_\Omega|\nabla f_j|(x) + 2M_\Omega|\nabla f|(x) \\ &\leqslant 2(M_\Omega(|\nabla f_j - \nabla f|)(x) + M_\Omega|\nabla f|(x)) + 2M_\Omega|\nabla f|(x) \\ &= 4M_\Omega|\nabla f|(x) + 2M_\Omega|\nabla f_j - \nabla f|(x) \end{aligned} \tag{2.16}$$

for almost every $x \in \Omega$. We start by choosing $K \subset\subset \Omega$ so that

$$\|4M_\Omega|\nabla f|\|_{\psi,\Omega\setminus K} < \varepsilon.$$

By the absolute continuity of the integral we find an $\alpha > 0$ such that

$$\|4M_\Omega|\nabla f|\|_{\psi,A} < \varepsilon, \quad \forall A \subset K,\ A \text{ measurable},\ m(A) < \alpha. \tag{2.17}$$

Then we divide the proof into two parts. In the first part, the same argument as in the case of the global maximal operator can be applied.

https://doi.org/10.1017/S0013091507000867 Published online by Cambridge University Press

(i) $\delta(x) \notin \mathcal{R}f(x)$. Let us define

$$B = \{x \in K : \delta(x) \notin \mathcal{R}f(x)\}. \tag{2.18}$$

In this part we show that

$$\|D_i M_\Omega f_j - D_i M_\Omega f\|_{\psi,B} \to 0 \quad \text{as } j \to \infty.$$

It turns out that outside a certain small exceptional set we can use Theorem 2.5 in the estimation. Accordingly, let us first define this exceptional set.

Because the sets $\mathcal{R}f(x)$ are compact, we can find $\gamma > 0$ such that

$$m(\{x \in B : \mathcal{R}f(x) \not\subset [0, \delta(x) - \gamma]\}) =: m(B_\gamma) < \tfrac{1}{3}\alpha. \tag{2.19}$$

Let (see (2.5)) $u_x(r)$ denote the average of $D_i f$ in the ball $B(x, r)$ and $u_x(0) = D_i f(x)$. As already observed, for almost every $x \in \mathbb{R}^n$, the functions u_x are continuous on $[0, \delta(x)]$. Therefore, there exists $\beta > 0$ such that $\beta < \gamma$ and

$$m(\{x \in K : |u_x(r_1) - u_x(r_2)| > \varepsilon \text{ for some } r_1, r_2, |r_1 - r_2| < \beta\}) =: m(C_{\gamma,\beta}) < \tfrac{1}{3}\alpha. \tag{2.20}$$

The set $C_{\gamma,\beta}$ is easily shown to be measurable. For the last exceptional set, recall Lemma 2.2, which says that we can find j_0 so that

$$m(\{x \in K : \mathcal{R}f_j(x) \not\subset \mathcal{R}f(x)_{(\beta)}\}) =: m(U_j) < \tfrac{1}{3}\alpha \quad \text{when } j \geqslant j_0. \tag{2.21}$$

Then, let $j \geqslant j_0$ be fixed. It follows from Theorem 2.5 that, for almost every $x \in K$, if there exist r_1, r_2 such that $r_1 \in \mathcal{R}f_j(x), r_2 \in \mathcal{R}f(x)$ and $r_1, r_2 < \delta(x)$, one has

$$\begin{aligned}
&|D_i M_\Omega f_j(x) - D_i M_\Omega f(x)| \\
&\quad = \left| \fint_{B(x,r_1)} D_i f_j(y)\,\mathrm{d}y - \fint_{B(x,r_2)} D_i f(y)\,\mathrm{d}y \right| \\
&\quad \leqslant \left| \fint_{B(x,r_1)} D_i f_j(y)\,\mathrm{d}y - \fint_{B(x,r_1)} D_i f(y)\,\mathrm{d}y \right| \\
&\qquad\quad + \left| \fint_{B(x,r_1)} D_i f(y)\,\mathrm{d}y - \fint_{B(x,r_2)} D_i f(y)\,\mathrm{d}y \right| \\
&\quad \leqslant M_\Omega(D_i f_j - D_i f)(x) + \left| \fint_{B(x,r_1)} D_i f(y)\,\mathrm{d}y - \fint_{B(x,r_2)} D_i f(y)\,\mathrm{d}y \right|.
\end{aligned} \tag{2.22}$$

This inequality also applies to the cases $r_1 = 0$ or $r_2 = 0$ when we agree that

$$\fint_{B(x,0)} D_i f(y)\,\mathrm{d}y := D_i f(x).$$

This is obvious because for almost every x we have $M_\Omega f(x) \geqslant f(x)$, and by Theorem 2.5 we have $D_i M_\Omega f(x) = D_i f(x)$ if $0 \in \mathcal{R}f(x)$.

https://doi.org/10.1017/S0013091507000867 Published online by Cambridge University Press

Now, if x is a point outside the exceptional sets, i.e $x \in B \setminus (B_\gamma \cup C_{\gamma,\beta} \cup U_j)$, we can pick $r_1 \in \mathcal{R}f_j(x)$ and $r_2 \in \mathcal{R}f(x)$ so that $|r_1 - r_2| < \beta$ and $r_1, r_2 < \delta(x)$. Then our choice of β implies that

$$\left| \fint_{B(x,r_1)} D_i f(y)\, \mathrm{d}y - \fint_{B(x,r_2)} D_i f(y)\, \mathrm{d}y \right| < \varepsilon.$$

Combining this with (2.22) we get that

$$|D_i M_\Omega f_j(x) - D_i M_\Omega f(x)| \leqslant M_\Omega(D_i f_j - D_i f)(x) + \varepsilon, \tag{2.23}$$

when $x \in B \setminus (B_\gamma \cup C_{\gamma,\beta} \cup U_j)$. Since $D_i f_j \to D_i f$ in $L^\psi(\Omega)$ and M_Ω was bounded on $L^\psi(\Omega)$, we get that

$$\|D_i M_\Omega f_j(x) - D_i M_\Omega f(x)\|_{\psi, B\setminus(B_\gamma \cup C_{\gamma,\beta} \cup U_j)} \leqslant \|\varepsilon\|_{\psi, B\setminus(B_\gamma \cup C_{\gamma,\beta} \cup U_j)} + \varepsilon \tag{2.24}$$

when j is large. On the other hand, if $x \in B \cap (B_\gamma \cup C_{\gamma,\beta} \cup U_j)$, we use the estimate given in (2.16) (note that $m(B_\gamma \cup C_{\gamma,\beta} \cup U_j) < \alpha$). We get that

$$\begin{aligned} &\|D_i M_\Omega f_j - D_i M_\Omega f\|_{\psi, B\cap(B_\gamma \cup C_{\gamma,\beta} \cup U_j)} \\ &\qquad \leqslant \|4M_\Omega |\nabla f|\|_{\psi, B_\gamma \cup C_{\gamma,\beta} \cup U_j} + \|2M_\Omega(|\nabla f_j - \nabla f|)\|_{\psi, B_\gamma \cup C_{\gamma,\beta} \cup U_j} \\ &\qquad \leqslant \varepsilon + \|2M_\Omega(|\nabla f_j - \nabla f|)\|_{\psi, B}. \end{aligned}$$

In the above sum the last term converges to zero when $j \to \infty$. As ε was arbitrary we conclude that $\|D_i M_\Omega f_j - D_i M_\Omega f\|_{\psi,B} \to 0$ as $j \to \infty$.

(ii) $\delta(x) \in \mathcal{R}f(x)$. Yet we have to prove that $\|D_i(M_\Omega f_j - M_\Omega f)\|_{\psi, K\setminus B} \to 0$. Let us first choose a sequence $(h_k)_{k=1}^\infty$, $h_k \to 0_+$, so that for all j and $1 \leqslant i \leqslant n$ we have that

$$(M_\Omega f_j - M_\Omega f)^i_{h_k}(x) \to D_i(M_\Omega f_j - M_\Omega f)(x) \quad \text{as } k \to \infty$$

for almost every $x \in K$. Then we divide the problem into subcases. To that end let us define

$$\begin{aligned} A^j &:= \{x \in K \setminus B : \delta(x) \in \mathcal{R}f_j(x)\}, \\ A_+ &:= \{x \in K \setminus B : \delta(x + h_k e_i) \geqslant \delta(x) \text{ for infinitely many } k\}, \\ A_- &:= \{x \in K \setminus B : \delta(x + h_k e_i) \leqslant \delta(x) \text{ for infinitely many } k\}. \end{aligned}$$

The above sets need not be disjoint, but what we need is that $K \setminus B \subset A_+ \cup A_-$. Depending on which of the above sets x lies within, we will use the different arguments to estimate $|D_i(M_\Omega f_j - M_\Omega f)(x)|$.

Let us first treat the case when $x \in A^j$. For this, let us recall Lemma 2.11, to see that by extracting a subsequence, if needed, we may assume that for almost every $x \in K \setminus B$ we have $x + h_k e_i \in K \setminus B$, when k is large enough, and we have $x + h_k e_i \in A^j$, for all j for almost every $x \in A^j$ if k is large. For all $x \in A^j$ we have

$$M_\Omega f_j(x) = \fint_{B(x,\delta(x))} f_j(y)\, \mathrm{d}y \quad \text{and} \quad M_\Omega f(x) = \fint_{B(x,\delta(x))} f(y)\, \mathrm{d}y. \tag{2.25}$$

Let us then set $f_j - f =: F_j$. Using (2.25) and Lemma 2.10 we get that

$$\begin{aligned}|D_i(M_\Omega f_j - M_\Omega f)(x)| &= \Big|\lim_{k\to\infty}(M_\Omega f_j - M_\Omega f)^i_{h_k}(x)\Big| \\ &= \bigg|\lim_{k\to\infty}\frac{1}{h_k}\bigg(\fint_{B(x+h_ke_i,\delta(x+h_ke_i))}F_j(y)\,\mathrm{d}y - \fint_{B(x,\delta(x))}F_j(y)\,\mathrm{d}y\bigg)\bigg| \\ &\leqslant 5M_\Omega|\nabla F_j|(x)\end{aligned}$$

a.e. in A^j. Since $|\nabla F_j| \to 0$ in $L^\psi(\Omega)$, this guarantees that

$$\|D_i(M_\Omega f_j - M_\Omega f)\|_{\psi,A^j} \to 0 \quad \text{as } j\to\infty. \tag{2.26}$$

Let us then consider the case of A_+. By the definition, for every $x \in A_+$ it holds that $\delta(x+h_ke_i) \geqslant \delta(x)$ for large k. Moreover, for almost every $x \in A_+$ we have $x + h_k \in A_+$ for k large enough. This implies that, for almost every $x \in A_+$,

$$\begin{aligned}M_\Omega f(x+h_ke_i) &= \fint_{B(x+h_ke_i,\delta(x+h_ke_i))} f(y)\,\mathrm{d}y \\ &\geqslant \fint_{B(x+h_ke_i,\delta(x))} f(y)\,\mathrm{d}y\end{aligned}$$

for k large enough. We get the following inequality:

$$\begin{aligned}D_iM_\Omega f(x) &= \lim_{k\to\infty}\frac{1}{h_k}\bigg(\fint_{B(x+h_ke_i,\delta(x+h_ke_i))} f(y)\,\mathrm{d}y - \fint_{B(x,\delta(x))} f(y)\,\mathrm{d}y\bigg) \\ &\geqslant \limsup_{k\to\infty}\frac{1}{h_k}\bigg(\fint_{B(x+h_ke_i,\delta(x))} f(y)\,\mathrm{d}y - \fint_{B(x,\delta(x))} f(y)\,\mathrm{d}y\bigg) \\ &= \limsup_{k\to\infty}\fint_{B(x,\delta(x))}\frac{f(y+h_ke_i)-f(y)}{h_k} \\ &= \fint_{B(x,\delta(x))} D_if(y)\,\mathrm{d}y\end{aligned}$$

for almost every $x \in A_+$. The last equation above follows from Lemma 2.9 when we set $r_k = \delta(x)$ for every k. Combining the above inequality with Theorem 2.5, we get that, for almost every $x \in A_+$, it holds that

$$D_iM_\Omega f(x) \geqslant \fint_{B(x,r)} D_if(y)\,\mathrm{d}y \quad \text{for all } r\in\mathcal{R}f(x) \quad (\text{equality if } r<\delta(x)). \tag{2.27}$$

We apply the above observation in the case when $x \in A_+ \setminus A^j$. For this, recall the definitions of β and U_j from part (i) (see (2.20) and (2.21)). For j_0 sufficiently large we had $\mathcal{R}f_j(x) \subset \mathcal{R}f(x)_{(\beta)}$, when $j > j_0$ and $x \in U_j^{\mathrm{c}}$. That is to say, for every $x \in U_j^{\mathrm{c}}$ there exists $r_j \in \mathcal{R}f_j(x)$ such that $|r - r_j| \leqslant \beta$ for some $r \in \mathcal{R}f(x)$ (it may be that $r = \delta(x)$).

https://doi.org/10.1017/S0013091507000867 Published online by Cambridge University Press

Since $x \notin A^j$, $r_j < \delta(x)$ holds. This implies, by Theorem 2.5 and (2.27), that

$$\begin{aligned} D_i M_\Omega f(x) - D_i M_\Omega f_j(x) &\\ &\geqslant \fint_{B(x,r)} D_i f(y)\,\mathrm{d}y - \fint_{B(x,r_j)} D_i f_j(y)\,\mathrm{d}y \\ &= \biggl(\fint_{B(x,r)} D_i f(y)\,\mathrm{d}y - \fint_{B(x,r_j)} D_i f(y)\,\mathrm{d}y\biggr) + \biggl(\fint_{B(x,r_j)} D_i (f-f_j)(y)\,\mathrm{d}y\biggr) \\ &=: s_1(j,x) + s_2(j,x). \end{aligned}$$

With the same notation as earlier, $s_1(j,x) = u_x(r) - u_x(r_j)$. As earlier in this proof, using the continuity of the functions u_x on $[0,\delta(x)]$ we get

$$m(\{x \in K : |s_1(j,x)| > \varepsilon\}) \to 0 \quad \text{as } j \to \infty.$$

Also we observe that the corresponding property as above for $|s_2(j,x)|$ holds since $|s_2(j,x)| \leqslant M_\Omega(D_i(f-f_j))(x)$ almost everywhere. These facts guarantee that

$$m(\{x \in A_+ \setminus A^j : D_i(M_\Omega f - M_\Omega f_j)(x) \leqslant -\varepsilon\}) \to 0 \quad \text{as } j \to \infty,$$

which together with (2.26) implies that

$$m(\{x \in A_+ : D_i(M_\Omega f - M_\Omega f_j)(x) \leqslant -\varepsilon\}) \to 0 \quad \text{as } j \to \infty.$$

Applying Corollary 2.7 (this is permissible because of (2.16)), we get that

$$\|D_i(M_\Omega f - M_\Omega f_j)\|_{\psi,A_+} \to 0 \quad \text{as } j \to \infty. \tag{2.28}$$

In exactly the same way, it is also possible to prove that

$$\|D_i(M_\Omega f - M_\Omega f_j)\|_{\psi,A_-} \to 0 \quad \text{as } j \to \infty. \tag{2.29}$$

In the set A_- we only obtain the inequality (2.27) in the reverse direction (we use Lemma 2.9 in the case where $r_k = \delta(x + h_k e_i)$) to get that

$$m(\{x \in A_- : D_i(M_\Omega f - M_\Omega f_j)(x) \geqslant \varepsilon\}) \to 0 \quad \text{as } j \to \infty.$$

Then we again use Corollary 2.7 to verify (2.29). This completes the proof. □

Remark 2.13. We briefly sketch the additional difficulties brought in by the boundary effects in the proof of Theorem 2.12. To that end, assume that $f_j \to f$ in $W^{1,p}(\Omega)$. One obviously needs to estimate the $L^p(\Omega)$-norm of $|D_i(M_\Omega f) - D_i(M_\Omega f_j)|$. In the case where the maxima in the definitions of the various maximal operators involved are obtained over balls strictly inside the domain, the argument runs pretty much along the lines of the global case. This is shown by our Lemmas 2.2 and 2.3 and Theorem 2.5, which are modifications of the corresponding results in the global case [**12**]. More serious new difficulties are met when some of the maxima are attained over balls touching the boundary. There arise various possibilities according to exactly which of the quantities $Mf(x)$ and $Mf_j(x)$ are achieved over balls touching the boundary.

The easiest case, where $Mf(x)$ is achieved strictly inside the domain, is pretty much similar to the global case. If both $Mf(x)$ and $Mf_j(x)$ correspond to balls touching the boundary, some technical work (especially Lemma 2.10) is required. The hardest part is the case where only $Mf(x)$ is achieved over a ball touching the boundary. In this case we have a formal expression for the difference of derivatives (assuming that $Mf_j(x)$ is achieved over $B(x,r)$):

$$|D_i(M_\Omega f)(x) - D_i(M_\Omega f_j)(x)| = \left| D_i\left(\fint_{B(x,\delta(x))} |f(y)|\,\mathrm{d}y \right) - \fint_{B(x,r)} D_i f_j(y) \right|.$$

A priori one cannot expect this quantity to be small. However, a suitable trick to treat this case is found. Roughly speaking, we divide the set of these points into two parts and observe the inequality in (2.27) in the first part and the inequality corresponding to the reverse direction in the other part. By combining these observations with Lemma 2.6 (here we also need Lemma 2.9) we can verify the desired convergence.

Remark 2.14. In this paper we have not treated the endpoint cases $p = 1$ and $p = \infty$. When $p = 1$, the natural counterpart to Theorem 2.12 would be the continuity of M_Ω from $W^{1,1}(\Omega)$ to the weak Sobolev space $W^{1,1}_w(\Omega)$. We regard these cases as worthy of study and remark that our argument can be partly applied to both cases but some more specific inspection is needed for the complete answer, especially as the case when $p = \infty$ is somewhat different.

Remark 2.15. It is of interest to note that the proof of Theorem 2.12 also applies to certain other maximal operators, most notably to the Hardy–Littlewood maximal operator defined through averages over cubes (instead of balls). We also remark that in Theorem 2.12 one does not use any smoothness properties for Ω.

3. Boundedness in $F^{\boldsymbol{p}}_{s,q}(\Omega)$

Korry [**6**] proved that M is bounded on the spaces $F^p_{s,q}(\mathbb{R}^n)$ assuming that $s \in (0,1)$ and $1 < p, q < \infty$. In this section we provide a counterpart to results of Kinnunen and Lindqvist [**7**] by extending the result of Korry to the local maximal operator.

We start with appropriate definitions and some auxiliary results. Let $0 < s < 1$, $1 < p, q < \infty$, $1 \leqslant r < \min(p,q)$ and $\boldsymbol{p} = (p,q,r)$. Denote the n-dimensional unit ball by B_n. We will use the following notion when characterizing $F^p_{s,q}(\mathbb{R}^n)$ by differences: let $I \subset \mathbb{R}$. For a measurable function $g : \mathbb{R}^n \times I \times B_n \mapsto \mathbb{R}$ we define

$$\|g\|_{\boldsymbol{p},I} := \left(\int_{\mathbb{R}^n} \left(\int_I \left(\int_{B_n} |g(x,t,h)|^r \,\mathrm{d}h \right)^{q/r} \mathrm{d}t \right)^{p/q} \mathrm{d}x \right)^{1/p}$$

and for a measurable function $f : \mathbb{R}^n \mapsto \mathbb{R}$ define $\|f\|_{F^p_{s,q}} = \|Sf\|_{\boldsymbol{p},(0,1)} + \|f\|_{L^p}$, where Sf is defined by setting

$$Sf(x,t,h) = \frac{|f(x+th) - f(x)|}{t^{s+(1/q)}}.$$

The result of Triebel [**14**, p. 194] tells us that the norm defined in this way is equivalent to the usual norm in $F^p_{s,q}(\mathbb{R}^n)$ and, moreover, if $1 \leqslant r_1, r_2 < \min(p, q)$, then the norms corresponding to these numbers are equivalent. In particular, this means that if $1 < r < \min(p, q)$, there is $C(n, p, q, s)$ such that

$$\|f\|_p + \|Sf\|_{(p,q,1),(0,1)} \leqslant C(\|f\|_p + \|Sf\|_{(p,q,r),(0,1)}). \tag{3.1}$$

When $\Omega \subset \mathbb{R}^n$ is open we define the space $F^p_{s,q}(\Omega)$ and the norm in that space by setting

$$F^p_{s,q}(\Omega) = \{f|_\Omega : f \in F^p_{s,q}(\mathbb{R}^n)\}$$

and

$$\|f\|_{F^p_{s,q}(\Omega)} = \inf\{\|g\|_{F^p_{s,q}(\mathbb{R}^n)} : g|_\Omega = f\}.$$

This definition is due to Triebel. It is important to note that this is not the only way to define these spaces. As we know, in the case of Sobolev spaces $F^p_{s,2}(\Omega)$, our definition in certain cases differs from the usual 'inner' definition. Unfortunately, there seems not to exist a satisfying way to give an inner description of all of these spaces at the same time. In standard Sobolev spaces we know that our definition is always equivalent to the inner description when Ω is (for instance) a Lipschitz domain. For more about the definitions, we refer the reader to [**13**, § 3.1].

In [**10**, § 4.3], Korry proved the following result, which implies the boundedness of M in $F^p_{s,q}(\mathbb{R}^n)$: assume that $f : \mathbb{R}^n \times I \times B_n \mapsto \mathbb{R}$, set $f^{t,h}(x) = f(x, t, h)$ and define $M_x f(x, t, h) = M f^{t,h}(x)$. Then there exists $C = C(n, \boldsymbol{p})$ such that

$$\|M_x f\|_{\boldsymbol{p},(0,1)} \leqslant C\|f\|_{\boldsymbol{p},(0,1)}. \tag{3.2}$$

In fact, Korry proved this result for a certain general class of operators. Because of this result we are able to say that $\|M_x Sf\|_{\boldsymbol{p},(0,1)} \leqslant C_{\boldsymbol{p},s}\|Sf\|_{\boldsymbol{p},(0,1)}$. This implies the boundedness by the fundamental fact that

$$S(Mf) \leqslant M_x(Sf) \quad \text{a.e.}, \tag{3.3}$$

which follows by the commutativity (with translations) and sublinearity of M.

As the most important auxiliary tool in his work, Korry used the fundamental result of Benedek *et al.* [**2**]. Let F and E be Banach spaces and denote by $\mathcal{L}(E, F)$ the space of all bounded linear operators from E to F. An operator T is called a Benedek–Calderón–Panzone (BCP) operator if T is bounded from $L^p(\mathbb{R}^n, E) \mapsto L^p(\mathbb{R}^n, F)$ for some $p \in (1, \infty)$, and if there exists a strongly measurable $\mathcal{L}(E, F)$-valued kernel K defined on $\mathbb{R}^n$, locally integrable outside the origin such that for every compactly supported continuous function $f : \mathbb{R}^n \mapsto E$ we have

$$T(f)(x) = \int_{\mathbb{R}^n} K(x - y)(f(y))\,\mathrm{d}y \quad \text{for almost every } x \notin \operatorname{supp} f$$

https://doi.org/10.1017/S0013091507000867 Published online by Cambridge University Press

and, moreover, K satisfies Hörmander's condition: there exists a constant M such that for every $y \in \mathbb{R}^n$ we have

$$\int_{|x|>2|y|} \|K(x-y) - K(x)\|_{\mathcal{L}(E,F)} \, \mathrm{d}x \leqslant M.$$

The result of [**2**] states that the operator T can be extended to a bounded operator from $L^q(\mathbb{R}^n, E)$ to $L^q(\mathbb{R}^n, F)$ for all $q \in (1, \infty)$. Let us call this result a BCP result.

In the following statements we assume that $\Omega \subset \mathbb{R}^n$, $\Omega \neq \mathbb{R}^n$ is open and for all $x \in \mathbb{R}^n$ we define $\delta(x) = d(x, \Omega^{\mathrm{c}})$. To avoid difficulties with notation, we also define

$$M_\Omega f(x) := \fint_{B(x,0)} |f(y)| \, \mathrm{d}y := |f(x)| \tag{3.4}$$

at points $x \in \Omega^{\mathrm{c}}$ where $f(x)$ is defined.

As mentioned previously, in the case of M_Ω the problem is that M_Ω does not commute with translations. We solve this problem (roughly speaking) by using the simple observation (3.8) combined with the following lemma and then applying the BCP result.

Lemma 3.1. *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $t > 0$, and define the function f_t by*

$$f_t(y) = \fint B(y, 2t) |f(z) - f(y)| \, \mathrm{d}z.$$

Then for almost every $x \in \mathbb{R}^n$ we have that

$$\left| \fint_{B(x,r_1)} |f(y)| \, \mathrm{d}y - \fint_{B(x,r_2)} |f(y)| \, \mathrm{d}y \right| \leqslant 2^{n+1} M f_t(x)$$

if $r_1 \geqslant 0$, $r_2 \geqslant 0$, such that $|r_1 - r_2| \leqslant t$.

Proof. If $|x - y| \leqslant t$, we get that $B(y, t) \subset B(x, 2t)$ and

$$\begin{aligned} |f(y) - f(x)| &\leqslant \fint B(y,t) |f(z) - f(y)| \, \mathrm{d}z + \fint B(y,t) |f(z) - f(x)| \, \mathrm{d}z \\ &\leqslant 2^n \fint B(y, 2t) |f(z) - f(y)| \, \mathrm{d}z + 2^n \fint_{B(x,2t)} |f(z) - f(x)| \, \mathrm{d}z \\ &= 2^n (f_t(y) + f_t(x)). \end{aligned} \tag{3.5}$$

Assume that $r_2, r_1 > 0$, $|r_2 - r_1| \leqslant t$. By scaling we get that

$$\fint_{B(x,r_2)} |f(z)| \, \mathrm{d}z = \fint_{B(x,r_1)} \left| f\left(x + \frac{r_2}{r_1}(z - x) \right) \right| \mathrm{d}z, \tag{3.6}$$

https://doi.org/10.1017/S0013091507000867 Published online by Cambridge University Press

and combining this with (3.5) we get that

$$\begin{aligned}\left| \fint_{B(x,r_1)} |f(z)|\,\mathrm{d}z - \fint_{B(x,r_2)} |f(z)|\,\mathrm{d}z \right| &\leqslant \fint_{B(x,r_1)} \left| f\left(x + \frac{r_2}{r_1}(z-x)\right) - f(z) \right| \mathrm{d}z \\ &\leqslant \fint_{B(x,r_1)} 2^n \left(f_t\left(x + \frac{r_2}{r_1}(z-x)\right) + f_t(z) \right) \mathrm{d}z \\ &= 2^n \left(\fint_{B(x,r_2)} f_t(z)\,\mathrm{d}z + \fint_{B(x,r_1)} f_t(z)\,\mathrm{d}z \right) \\ &\leqslant 2^{n+1} M f_t(x).\end{aligned}$$

The case where $r_1 = 0$ or $r_2 = 0$ can be obtained at Lebesgue points of f by taking a limit when $r_1^k \to r_1$ and $r_2^k \to r_2$, where $r_1^k > 0$ and $r_2^k > 0$. □

Now we are ready to prove the boundedness.

Theorem 3.2. *M_Ω is bounded in $F^p_{s,q}(\Omega)$, when $1 < p < \infty$, $1 < q < \infty$ and $0 < s < 1$.*

Proof. Let g be in $F^p_{s,q}(\Omega)$ and $f \in F^p_{s,q}(\mathbb{R}^n)$ satisfy $f|_\Omega = g$. We define a function $M_\Omega f$ on $\mathbb{R}^n$ according to the definition (3.4), thus $M_\Omega f(x) = M_\Omega g(x)$ if $x \in \Omega$ and $M_\Omega f(x) = |f(x)|$ if $x \in \Omega^{\mathrm{c}}$. We observe that the theorem follows by showing that

$$\|M_\Omega f\|_{F^p_{s,q}(\mathbb{R}^n)} \leqslant C(n,p,s,q)\|f\|_{F^p_{s,q}(\mathbb{R}^n)}. \tag{3.7}$$

This holds, because then every $M_\Omega f$ defined as above is in $F^p_{s,q}(\mathbb{R}^n)$, is the extension of $M_\Omega g$ to $\mathbb{R}^n$ and by choosing $\|f\|_{F^p_{s,q}(\mathbb{R}^n)}$ to be arbitrarily close to $\|g\|_{F^p_{s,q}(\Omega)}$ we get what we want. So, let us establish (3.7).

We may suppose that f is non-negative, because $M_\Omega f = M_\Omega |f|$ and, moreover, $\||f|\|_{F^p_{s,q}(\mathbb{R}^n)} \leqslant \|f\|_{F^p_{s,q}(\mathbb{R}^n)}$. Then we start with an easy observation which makes Lemma 3.1 useful. Let x and y be Lebesgue points of f in $\mathbb{R}^n$. We claim that one can choose $r_1 \geqslant 0$ and $r_2 \geqslant 0$ such that $|r_1 - r_2| \leqslant |x - y|$ and

$$|M_\Omega f(x) - M_\Omega f(y)| \leqslant \left| \fint_{B(x,r_1)} f(z)\,\mathrm{d}z - \fint B(y,r_2) f(z)\,\mathrm{d}z \right|. \tag{3.8}$$

Because of symmetry, we can assume that $M_\Omega f(y) > M_\Omega f(x)$. Then we choose r_1 and r_2 so that $r_2 \in \mathcal{R}f(y)$ and $r_1 = \max\{0, r_2 - |x-y|\}$. Now, trivially $|r_1 - r_2| \leqslant |x - y|$. When $r_1 = 0$, we see that equation (3.8) holds since $M_\Omega f(x) \geqslant f(x)$. If $r_1 > 0$, we get that $r_1 = r_2 - |x - y| \leqslant \delta(y) - |x - y| \leqslant \delta(x)$, guaranteeing that

$$M_\Omega f(x) \geqslant \fint_{B(x,r_1)} |f|$$

and implying (3.8).

https://doi.org/10.1017/S0013091507000867 Published online by Cambridge University Press

Assume then that $(x,t,h) \in \mathbb{R}^n \times (0,1) \times B_n$ so that $x+th$ and x are Lebesgue points of f. Let the function f_t be defined as in Lemma 3.1. By (3.8) we can find $r_1, r_2 \geqslant 0$ such that $|r_1 - r_2| \leqslant t$ and

$$\begin{aligned} |M_\Omega f(x+th) - M_\Omega f(x)| &\leqslant \left| \fint_{B(x+th,r_1)} |f(z)|\,\mathrm{d}z - \fint_{B(x,r_2)} |f(z)|\,\mathrm{d}z \right| \\ &\leqslant \left| \fint_{B(x,r_1)} f(z+th) - f(z)\,\mathrm{d}z \right| \\ &\quad + \left| \fint_{B(x,r_1)} |f(z)|\,\mathrm{d}z - \fint_{B(x,r_2)} |f(z)|\,\mathrm{d}z \right| \\ &\leqslant Mf_{t,h}(x) + 2^{n+1} Mf_t(x), \end{aligned} \tag{3.9}$$

where the function $f_{t,h}$ is defined by

$$f_{t,h}(z) = |f(z+th) - f(z)|. \tag{3.10}$$

Then, set $\phi(t) = 1/t^{s+(1/q)}$, whence we get from above that

$$\begin{aligned} \|M_\Omega f\|_{F^p_{s,q}(\mathbb{R}^n)} &= \|S(M_\Omega f)\|_{\boldsymbol{p},(0,1)} + \|M_\Omega f\|_{L^p} \\ &\leqslant \|\phi(t)(Mf_{t,h} + 2^{n+1} Mf_t)\|_{\boldsymbol{p},(0,1)} + C\|f\|_{L^p}. \end{aligned} \tag{3.11}$$

As we have already mentioned (see (3.2)), Korry proved that

$$\|\phi(t) Mf_{t,h}\|_{\boldsymbol{p},(0,1)} = \|M(\phi(t) f_{t,h})\|_{\boldsymbol{p},(0,1)} \leqslant C\|\phi(t) f_{t,h}\|_{\boldsymbol{p},(0,1)} = C\|Sf\|_{\boldsymbol{p},(0,1)}.$$

Thus, to complete the proof it is sufficient to show that

$$\|\phi(t) Mf_t\|_{\boldsymbol{p},(0,1)} \leqslant C(n,p,q,s)\|f\|_{F^p_{s,q}(\mathbb{R}^n)}.$$

It turns out that this can be done in exactly the same way as Korry proved his result. Korry needed to iterate the BCP result three times; we have to do the iteration twice.

Let $\omega \in C_0^\infty$ be radial and supported in $B(0,1)$. We recall [**10**, § 4.3] that if we define

$$Uf(x,u) = f * (\gamma(\cdot\,,u))(x), \quad \text{where } \gamma(x,u) = \frac{1}{u^n}\omega\left(\frac{x}{u}\right),$$

and $f : \mathbb{R}^n \mapsto \mathbb{R}$, $u \in (0,\infty)$, then U defines the BCP operator from $L^q(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n, L^\infty((0,\infty),\mathrm{d}u))$ and $Mf(x) \leqslant C(n)\|Uf(x,\cdot)\|_\infty$ for all $x \in \mathbb{R}^n$, $f \in L^q(\mathbb{R}^n)$ (recall that we have assumed $f \geqslant 0$). Let us denote the kernel of U by K and $L^\infty((0,\infty),\mathrm{d}u)$ by F (it is to easy to see that K equals the mapping $x \mapsto \gamma(x,\cdot)$). Then we define an operator U_1 by setting, for $f : \mathbb{R}^n \times (0,1) \mapsto \mathbb{R}$, $x \in \mathbb{R}^n$, $t \in (0,1)$ and $u \in (0,\infty)$,

$$(U_1 f)(x,t,u) = U(f(\cdot\,,t))(x,u).$$

By Fubini's theorem and after an easy computation we see that U_1 is a bounded operator from $L^q(\mathbb{R}^n, L^q((0,1)))$ to $L^q(\mathbb{R}^n, L^q((0,1),F))$. It is also a BCP operator: if we define for $x \in \mathbb{R}^n$, $g \in L^q((0,1))$ and $t \in (0,1)$ that

$$((K_1(x))(g))(t) = K(x)g(t),$$

https://doi.org/10.1017/S0013091507000867 Published online by Cambridge University Press

we obtain that $K_1 : \mathbb{R}^n \mapsto \mathcal{L}(L^q((0,1)), L^q((0,1),F))$ and K_1 is the kernel of U_1, the operator norm of $K_1(x-y) - K_1(x)$ equals $\|K(x-y) - K(x)\|_\infty$ and therefore K_1 satisfies the Hörmander condition because K satisfies it. Now we are able to use the BCP result to get that U_1 is also bounded from $L^p(\mathbb{R}^n, L^q((0,1)))$ to $L^p(\mathbb{R}^n, L^q((0,1),F))$. But writing this down reveals that this is exactly what we need. When verifying this, we use the notation $f^\star(x,t) = \phi(t)f_t(x)$. Because $\phi(t)Mf_t$ does not depend on h and $\phi(t)Mf_t(x) = M(\phi(t)f_t)(x)$, we get that

$$\begin{aligned}
\|\phi(t)Mf_t\|_{\boldsymbol{p},(0,1)} &= m(B_n)\biggl(\int_{\mathbb{R}^n}\biggl[\int_0^1 |\phi(t)Mf_t(x)|^q \,\mathrm{d}t\biggr]^{p/q}\mathrm{d}x\biggr)^{1/p} \\
&\leqslant C\biggl(\int_{\mathbb{R}^n}\biggl[\int_0^1 \|(U(\phi(t)f_t)(x,\cdot))\|_\infty^q \,\mathrm{d}t\biggr]^{p/q}\mathrm{d}x\biggr)^{1/p} \\
&= C\biggl(\int_{\mathbb{R}^n}\biggl[\int_0^1 \|(U(f^\star(\cdot\,,t))(x,\cdot))\|_\infty^q \,\mathrm{d}t\biggr]^{p/q}\mathrm{d}x\biggr)^{1/p} \\
&= C\biggl(\int_{\mathbb{R}^n} \|U_1 f^\star(x,\cdot\,,\cdot)\|_{L^q((0,1),F)}^p \,\mathrm{d}x\biggr)^{1/p} \\
&= C\|U_1 f^\star\|_{L^p(\mathbb{R}^n, L^q((0,1),F))} \\
&\leqslant C'\|f^\star\|_{L^p(\mathbb{R}^n, L^q((0,1)))} \\
&= C'\biggl(\int_{\mathbb{R}^n}\biggl[\int_0^1 \phi(t)|f_t(x)|^q\biggr]^{p/q}\mathrm{d}x\biggr)^{1/p} \\
&=: s.
\end{aligned}$$

Now, by a change of variables, we see that

$$\phi(t)f_t(x) = \int_{B_n} |f(x+2th) - f(x)|\phi(t)\,\mathrm{d}h,$$

which implies that $s = C(n)\|Sf\|_{(p,q,1),(0,2)}$. Now it is easy to observe that

$$\|Sf\|_{(p,q,1),(0,2)} \leqslant C\|f\|_p + \|Sf\|_{(p,q,1),(0,1)}.$$

Finally, we use (3.1) (or just Jensen's inequality) to obtain that $\|Sf\|_{(p,q,1),(0,1)} \leqslant C\|f\|_{F^p_{s,q}(\mathbb{R}^n)}$. This completes the proof. □

4. Continuity on $F^p_{s,q}(\Omega)$

We finally verify the continuity of the local maximal operator on Triebel–Lizorkin spaces defined on subdomains, thus providing a counterpart to [**12**]. The continuity will be deduced from the pointwise estimates for $SM_\Omega f$, which were obtained in the previous section, and the following lemma, which, in a sense, is an extension of the Lebesgue dominated convergence theorem.

https://doi.org/10.1017/S0013091507000867 Published online by Cambridge University Press

Lemma 4.1. *Let $\boldsymbol{p} = (p, q, r)$ $(1 \leqslant p, q, r < \infty)$ and F_j, G, G_j be mappings from $\mathbb{R}^n \times I \times B_n$ to $\mathbb{R}$ such that $\|G_j\|_{\boldsymbol{p},I} \to 0$ as $j \to \infty$, $\|G\|_{\boldsymbol{p},I} < \infty$, $|F_j| \leqslant G + G_j$ a.e. for all j. Moreover, assume that, for almost every x and t,*

$$m\{h \in B_n : |F_j(x,t,h)| > \varepsilon\} \xrightarrow{j\to\infty} 0 \tag{4.1}$$

for every $\varepsilon > 0$. Then $\|F_j\|_{\boldsymbol{p},I} \xrightarrow{j\to\infty} 0$.

Proof. We start by assuming to the contrary that $\|F_j\|_{\boldsymbol{p},(0,1)} > \lambda$ for some subsequence and $\lambda > 0$. Observe then that we may assume the functions F_j to have a uniform bound. This holds, since, again by extracting a subsequence, we have $\sum_{j=1}^{\infty} \|G_j\|_{\boldsymbol{p},(0,1)} < \infty$. Define $G_0 =: \sum_{j=1}^{\infty} G_j$, and obviously $\|G_0\|_{\boldsymbol{p},(0,1)} < \infty$ and, moreover, $|F_j| < G + G_0$ for every j. So, we may assume that

$$|F_j(x,t,h)| < G(x,t,h) \quad \text{for almost every } (x,t,h).$$

Then it holds for almost every (x,t) that

$$|F_j(x,t,h)| < G(x,t,h) \quad \text{for almost every } h \in B_n \qquad \text{and} \qquad \int_{B_n} G(x,t,h)^r \,\mathrm{d}h < \infty.$$

From this and assumption (4.1) we easily deduce that

$$\int_{B_n} |F_j(x,t,h)|^r \,\mathrm{d}h \to 0$$

for almost every (x,t).

We continue by observing that, for almost every x,

$$\biggl(\int_{B_n} |F_j(x,t,h)|^r \,\mathrm{d}h\biggr)^{q/r} < \biggl(\int_{B_n} G(x,t,h)^r \,\mathrm{d}h\biggr)^{q/r} \quad \text{for almost every } t.$$

Combined with the fact that, for almost every x,

$$\int_I \biggl(\int_{B_n} G(x,t,h)^r \,\mathrm{d}h\biggr)^{q/r} \mathrm{d}t < \infty,$$

this enables us to use the dominated convergence theorem to get that, for almost every x,

$$\int_I \biggl(\int_{B_n} F_j(x,t,h)^r \,\mathrm{d}h\biggr)^{q/r} \mathrm{d}t \to 0 \quad \text{as } j \to \infty.$$

Using the above simple method once more, we obtain that $\|F_j\|_{\boldsymbol{p},(0,1)} \to 0$ and we reach a contradiction. □

Finally, we verify the continuity of M_Ω on Triebel–Lizorkin spaces.

Theorem 4.2. *The maximal operator M_Ω is continuous in $F^p_{s,q}(\Omega)$ when $1 < p, q < \infty$ and $0 < s < 1$.*

https://doi.org/10.1017/S0013091507000867 Published online by Cambridge University Press

Proof. Let $f_j \to f$ in $F^p_{s,q}(\Omega)$, $1 < r < \min(p,q)$ and $\boldsymbol{p} = (p,q,r)$.

Let us first assume that $\Omega = \mathbb{R}^n$, $M_\Omega = M$. Suppose, on the contrary, that by extracting a subsequence, if needed, there exists $c > 0$ such that

$$\|S(Mf - Mf_j)\|_{\boldsymbol{p},(0,1)} > c \tag{4.2}$$

for every j. Again, since we know that $(Mf - Mf_j) \to 0$ in L^p, by extracting a subsequence we may assume that $|Mf_j(x) - Mf(x)| \to 0$ a.e. Then it is easy to check that for almost every $x \in \mathbb{R}^n$ and every $t \in (0,1)$ it holds that

$$m\{h \in B_n : S(Mf - Mf_j)(x,t,h) > \lambda\} \xrightarrow{j\to\infty} 0$$

for all $\lambda > 0$. By using the triangle inequality and sublinearity of M we get that

$$\begin{aligned} S(Mf - Mf_j) &\leqslant S(Mf) + S(Mf_j) \\ &\leqslant M_x(Sf) + M_x(Sf_j) \\ &\leqslant M_x(Sf) + M_x(Sf) + M_x(Sf_j - Sf). \end{aligned}$$

Furthermore, $\|M_x(Sf)\|_{\boldsymbol{p},(0,1)} < \infty$ (Korry's result) and

$$\|M_x(Sf - Sf_j)\|_{\boldsymbol{p},(0,1)} \leqslant C\|Sf_j - Sf\|_{\boldsymbol{p},(0,1)} \leqslant C\|S(f_j - f)\|_{\boldsymbol{p},(0,1)} \xrightarrow{j\to\infty} 0.$$

Therefore, the assumptions of Lemma 4.1 hold and we get that $\|S(Mf - Mf_j)\|_{\boldsymbol{p},(0,1)} \to 0$. This contradicts (4.2) and proves the case when $\Omega = \mathbb{R}^n$.

Assume then that $\Omega \neq \mathbb{R}^n$. Let g be an extension of f to the whole space $\mathbb{R}^n$ and let functions $\tilde{g}_j$ be the extensions of the functions $f - f_j$ with the property $\|\tilde{g}_j\|_{F^p_{s,q}(\mathbb{R}^n)} \to 0$. Define $g_j = g - \tilde{g}_j$. Now g_j extends f_j and, moreover, $g_j - g = \tilde{g}_j$, which implies that $g_j - g \to 0$ in $F^p_{s,q}(\mathbb{R}^n)$. Furthermore, we see that

$$\|M_\Omega f_j - M_\Omega f\|_{F^p_{s,q}(\Omega)} \leqslant \|M_\Omega g_j - M_\Omega g\|_{F^p_{s,q}(\mathbb{R}^n)},$$

because $(M_\Omega g_j - M_\Omega g)|_\Omega = M_\Omega f_j - M_\Omega f$. We must show that the right-hand side above vanishes when $j \to \infty$. For this we use Lemma 4.1, but now the task is not as easy as in the case when $\Omega = \mathbb{R}^n$ because in the present situation M_Ω does not commutate with translations.

As above, first suppose on the contrary that (4.2) holds. Again, by choosing a subsequence, we have for almost every $x \in \mathbb{R}^n$ and every $t \in (0,1)$ that

$$m\{h \in B_n : S(M_\Omega g - M_\Omega g_j)(x,t,h) > \lambda\} \xrightarrow{j\to\infty} 0 \tag{4.3}$$

for all $\lambda > 0$.

In the proof of Theorem 3.2 we established the inequality (3.9), which we use next to get (recalling the notation of (3.10)) that

$$\begin{aligned} &|M_\Omega g_j(x + th) - M_\Omega g_j(x)| \\ &\qquad \leqslant M((g_j)_{t,h})(x) + 2^{n+1}M((g_j)_t)(x) \\ &\qquad \leqslant M((g_j - g)_{t,h} + g_{t,h})(x) + 2^{n+1}M((g_j - g)_t + g_t)(x) \\ &\qquad \leqslant M(g_j - g)_{t,h}(x) + Mg_{t,h}(x) + 2^{n+1}(M(g_j - g)_t(x) + Mg_t(x)). \end{aligned}$$

https://doi.org/10.1017/S0013091507000867 Published online by Cambridge University Press

Then, define functions F_j and F by

$$F_j(x,t,h) = \phi(t)(M(g_j - g)_{t,h}(x) + 2^{n+1}M(g_j - g)_t(x)),$$
$$F(x,t,h) = \phi(t)(Mg_{t,h}(x) + Mg_t(x)).$$

From the above we get that

$$S(M_\Omega g_j - M_\Omega g) \leqslant S(M_\Omega g_j) + S(M_\Omega g) \leqslant S(M_\Omega g) + F + F_j.$$

Furthermore, Theorem 3.2 implies that $\|F\|_{\boldsymbol{p},(0,1)} < \infty$ and $\|F_j\|_{\boldsymbol{p},(0,1)} \to 0$ as $j \to \infty$. These facts, combined with (4.3), guarantee that we can use Lemma 4.1 to obtain

$$\|S(M_\Omega g_j - M_\Omega g)\|_{\boldsymbol{p},(0,1)} \xrightarrow{j\to\infty} 0.$$

This completes the proof. □

Acknowledgements. The author has been supported by the Academy of Finland, Project 201392. He thanks He thanks Eero Saksman, Tuomas Hytönen and Juha Kinnunen for valuable comments on the manuscript.

References

1. F. J. Almgren and E. H. Lieb, Symmetric decreasing rearrangement is sometimes continuous, *J. Am. Math. Soc.* **2** (1989), 683–773.
2. A. Benedek, A. P. Calderón and R. Panzone, Convolution operators on Banach space valued functions, *Proc. Natl Acad. Sci. USA* **48** (1962), 356–365.
3. D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, 2nd edn (Springer, 1983).
4. P. Hajłasz, Sobolev spaces on an arbitrary metric space, *Potent. Analysis* **5** (1996), 403–415.
5. P. Hajłasz and J. Onninen, On boundedness of maximal functions in Sobolev spaces, *Annales Acad. Sci. Fenn. Math.* **29** (2004), 167–176.
6. J. Kinnunen, The Hardy–Littlewood maximal function of a Sobolev function, *Israel J. Math.* **100** (1997), 117–124.
7. J. Kinnunen and P. Lindqvist, The derivative of the maximal function, *J. Reine Angew. Math.* **503** (1998), 161–167.
8. J. Kinnunen and E. Saksman, Regularity of the fractional maximal function, *Bull. Lond. Math. Soc.* **35**(4) (2003), 529–535.
9. V. Kokilashvili and M. Krbec, *Weighted inequalities in Lorenz and Orlicz spaces* (World Scientific, 1991).
10. S. Korry, Boundedness of Hardy–Littlewood maximal operator in the framework of Lizorkin–Triebel spaces, *Rev. Mat. Complut.* **15** (2002), 401–416.
11. A. Kufner *et al.*, *Function spaces* (Noordhoff, Leyden, 1977).
12. H. Luiro, Continuity of the maximal operator in Sobolev spaces, *Proc. Am. Math. Soc.* **135**(1) (2007), 243–251.
13. H. Triebel, *Theory of function spaces*, Monographs in Mathematics, Volume 78 (Birkhäuser, 1983).
14. H. Triebel, *Theory of function spaces, II*, Monographs in Mathematics, Volume 84 (Birkhäuser, 1992).