A SIMPLE BOUNDING FORMULA FOR INTEGRALS

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1. Introduction and summary. In this paper I establish the following bounding formula for the integral of a function of n variables:

$$\int_D F\,dD = \frac{\Delta}{n+1}\sum_i m_i F_i + E_1 + E_2, \tag{1.1}$$

the "error terms" E_1 and E_2 being bounded by

$$|E_1| \leqslant \bar{F}(D-T), \quad |E_2| \leqslant kMT. \tag{1.2}$$

Here F is the function, D the domain of integration in Euclidean n-space (and also the n-volume of that domain), and $\bar{F}$ an upper bound of $|F|$ in D. D is divided up into congruent cells with $n+1$ vertices (segments on a line, triangles in a plane, tetrahedra in space), the total n-volume occupied by these cells being T, so that $D-T$ is the n-volume of that part of D not covered by the cells. Δ is the n-volume of a cell. The summation runs over all the vertices of the cells, F_i being the value of F at the vertex i, and m_i the number of cells meeting at this vertex. M is an upper bound to $|d^2F/ds^2|$, for differentiation in all directions and at all positions in D, ds being an element of length in the direction of differentiation. The quantity k is a parameter with the dimensions [length]^2, depending only on the shape and size of the cell, and given by

$$k = \tfrac{1}{2}(r_0^2 - P/\Delta), \qquad P = \int_\Delta r^2\,dD, \tag{1.3}$$

where r_0 is the radius of the circumscribed sphere of a cell and P the polar moment of inertia of a cell relative to its circumcentre. In particular

$$\begin{cases} \text{for } n=1, \quad k = \tfrac{1}{12}h^2 & (h = \text{length of segment-cell}); \\ \text{for } n=2, \quad k = \tfrac{1}{24}(s_1^2+s_2^2+s_3^2) & (s_1, s_2, s_3 = \text{sides of triangular cell}); \\ \text{for } n=3, \quad k = \tfrac{2}{5}(r_0^2 - R^2) & \\ \quad (r_0 = \text{radius of circumscribed sphere,} & \\ \quad R = \text{distance between circumcentre and centroid of the cell}). & \end{cases} \tag{1.4}$$

For $n = 1$ we would naturally divide D into equal segments, these segments being the cells. Then $D = T$, $E_1 = 0$, and (1.1), (1.2) may be written

$$\int_a^b F\,dx = \tfrac{1}{2}h(F_0 + 2F_1 + \ldots + 2F_{r-1} + F_r) + E_2, \tag{1.5}$$
$$|E_2| \leqslant \tfrac{1}{12}h^2 M(b-a);$$

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here h is the length of a segment and $F_0, F_1, \ldots$ the values of F at the end points and points of division. This simple formula is very easy to prove, being in fact a by-product in the proof of the Euler-Maclaurin formula [**3**, p. **142** (6)]. Its significance here is as the prototype of the general multi-dimensional formula (1.1).

If we like we can write (1.1) in a more general form applicable to the case where the cells are not all the same. Or we can improve the bounds by using different upper bounds M in different parts of D. But these considerations complicate the formula and we shall confine ourselves to establishing (1.1).

Geometrical intuition is a great help and the plan we shall adopt is to prove (1.1) for $n = 2$ and then recognize that the same method is available for $n = 3$ and indeed for all values of n.

2. Bounds for a function and its integral when the function vanishes at the vertices of a triangle. Generalization to n dimensions. As a first step towards the proof of (1.1), we consider a triangle ABC and a function ϕ, continuous and with continuous first and second derivatives. We impose on ϕ the condition

$$\phi = 0 \quad \text{at } A, B \text{ and } C, \tag{2.1}$$

and we write as bound for the second derivative

$$|d^2\phi/ds^2| \leqslant M, \tag{2.2}$$

this inequality holding for differentiation in all directions and at all positions in the triangle.

Let G be the circumcentre of ABC and r_0 the radius of its circumscribed circle. Consider the two functions

$$\Phi_1 = \tfrac{1}{2} M(r_0^2 - r^2), \qquad \Phi_2 = \tfrac{1}{2} M(r^2 - r_0^2), \tag{2.3}$$

where r is distance from G. Note that in ABC, Φ_1 is positive and Φ_2 negative, both functions vanishing at A, B, and C. If we draw three-dimensional graphs of these two functions, they enclose a paraboloidal lens, with A, B, and C lying on its edge. We note that for all directions and positions

$$d^2\Phi_1/ds^2 = -M, \qquad d^2\Phi_2/ds^2 = M. \tag{2.4}$$

The essential theorem we require is this: *By virtue of* (2.1) *and* (2.2), *the three-dimensional graph of ϕ within the triangle ABC cannot pass outside the paraboloidal lens formed from Φ_1 and Φ_2; equivalently,*

$$\Phi_2 \leqslant \phi \leqslant \Phi_1 \tag{2.5}$$

in the triangle ABC.

To prove this, consider first the side AB. On it

$$\frac{d^2}{ds^2}(\Phi_1 - \phi) \leqslant 0, \qquad \Phi_1 - \phi = 0 \text{ at } A \text{ and } B. \tag{2.6}$$

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Hence $\Phi_1 - \phi$ is positive (or zero) on AB. Similarly $\phi - \Phi_2$ is positive (or zero). Thus on AB, and similarly on BC and CA, the inequalities (2.5) hold.

Now take any point E on AB and join it to C. On EC we have the inequality (2.6) and the boundary conditions

$$(2.7) \qquad \Phi_1 - \phi = 0 \text{ at } C, \qquad \Phi_1 - \phi \geqslant 0 \text{ at } E.$$

Hence $\Phi_1 - \phi$ is positive (or zero) on CE. Similarly $\phi - \Phi_2$ is positive (or zero). Since lines such as CE traverse the whole triangle, the inequalities (2.5) are established.

It follows then that, for integration over the triangle,

$$(2.8) \qquad \left| \int_\Delta \phi \, dD \right| \leqslant \int_\Delta \Phi_1 \, dD = \tfrac{1}{2} M(r_0^2 \Delta - P),$$

$$P = \int_\Delta r^2 \, dD,$$

where dD is an element of area, Δ the area of ABC and P the polar moment of inertia with respect to the circumcentre. We may also write this as

$$(2.9) \qquad \left| \int_\Delta \phi \, dD \right| \leqslant kM\Delta, \qquad k = \tfrac{1}{2}(r_0^2 - P/\Delta).$$

The extension of the above reasoning to three dimensions is very easy. We now take a tetrahedron $ABCD$, with $\phi = 0$ at the vertices and the bounding condition (2.2) as before. The functions Φ_1 and Φ_2 are defined as in (2.3), G being now the centre of the circumscribed sphere and r_0 its radius. The inequalities (2.5) are established in the same manner as above, and we get again the formula (2.9), wherein dD is now an element of volume and Δ the volume of the tetrahedron.

We can carry the same type of reasoning on into space of higher dimensionality or back to $n = 1$. In fact, (2.9) holds for $n = 1, 2, 3, \ldots$.

The factor k in (2.9) agrees with (1.3), and the special values shown in (1.4) are easily obtained from the following equimomental properties [**2**, pp. 23, 27]:

(i) A straight segment of mass m is equimomental to particles each of mass $m/6$ at its ends and a particle of mass $2m/3$ at its centre.

(ii) A triangle of mass m is equimomental to three particles each of mass $m/3$ at the middle points of its sides.

(iii) A tetrahedron of mass m is equimomental to four particles each of mass $m/20$ at its vertices and a particle of mass $4m/5$ at its centroid.

3. Proof of the bounding formulae (1.1), (1.2). Consider a domain D in the plane and a function F in D with the bounds

$$(3.1) \qquad |F| \leqslant \bar{F}, \qquad |d^2F/ds^2| \leqslant M.$$

Divide up D into triangles covering an area T, and so leaving an area $D - T$ between the triangulation and the boundary of D. Then

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$$\int_D F\,dD = \int_T F\,dD + E_1, \tag{3.2}$$

where

$$|E_1| \leqslant \bar{F}\,(D - T). \tag{3.3}$$

Now define a function F' throughout T by the conditions that $F' = F$ at every vertex and F' is a linear function of the coordinates in each triangle. Thus, in the triangle (x_1, y_1), (x_2, y_2), (x_3, y_3), F' is given by the determinantal equation

$$\begin{vmatrix} F' & x & y & 1 \\ F_1 & x_1 & y_1 & 1 \\ F_2 & x_2 & y_2 & 1 \\ F_3 & x_3 & y_3 & 1 \end{vmatrix} = 0, \tag{3.4}$$

where F_1, F_2, F_3 are the values of F at the vertices.

Write $\phi = F - F'$. Then $\phi = 0$ at the vertices, and

$$d^2\phi/ds^2 = d^2F/ds^2, \tag{3.5}$$

since F' is linear. Hence, by (3.1),

$$|d^2\phi/ds^2| \leqslant M. \tag{3.6}$$

Application of (2.9) gives, for integration over a triangle,

$$\left| \int_\Delta \phi dD \right| \leqslant kM\Delta, \tag{3.7}$$

where Δ is the area of the triangle and k as in (2.9).

So far we have not assumed the triangles to be congruent with one another. Let us now make this assumption, so that k and Δ are the same for all triangles. Then, for integration over the whole triangulation, we have from (3.7)

$$\left| \int_T \phi\,dD \right| \leqslant kMT. \tag{3.8}$$

Now

$$\int_T F\,dD = \int_T F'\,dD + \int_T \phi\,dD, \tag{3.9}$$

and so, by (3.2) and (3.8),

$$\int_D F\,dD = \int_T F'\,dD + E_1 + E_2, \tag{3.10}$$

where E_1 is bounded as in (3.3) and

$$|E_2| \leqslant kMT. \tag{3.11}$$

To complete the derivation of (1.1) from (3.10), we have to evaluate $\int_T F'\,dD$. To do this, we integrate (3.4) over the triangle, obtaining

(3.12)
$$\begin{vmatrix} \int_\Delta F'\,dD & \bar{x}\Delta & \bar{y}\Delta & \Delta \\ F_1 & x_1 & y_1 & 1 \\ F_2 & x_2 & y_2 & 1 \\ F_3 & x_3 & y_3 & 1 \end{vmatrix} = 0,$$

where $(\bar{x}, \bar{y})$ is the centroid of the triangle. This gives at once

(3.13)
$$\int_\Delta F'\,dD = \tfrac{1}{3}\Delta(F_1 + F_2 + F_3).$$

Summing for all the triangles, we get

(3.14)
$$\int_T F'\,dD = \tfrac{1}{3}\Delta \sum_i m_i F_i,$$

where i runs over all the vertices and m_i is the number of triangles meeting at the vertex i. When we substitute in (3.10), we get

(3.15)
$$\int_D F\,dD = \tfrac{1}{3}\Delta \sum_i m_i F_i + E_1 + E_2,$$

which is (1.1) for $n = 2$.

For $n \neq 2$, the initial factor must be changed from 1/3 to $1/(n + 1)$, because in using the analogue of (3.12) in n dimensions the centroid of the cell has coordinates $\bar{x}, \bar{y}, \ldots$, where

(3.16)
$$\bar{x} = \frac{1}{n+1}(x_1 + x_2 + \ldots + x_{n+1}).$$

But otherwise the reasoning is the same, and so we may regard the bounding formulae (1.1), (1.2) as established for all values of n.

4. Two special triangulations. In dealing with an integral over a plane domain, two triangulations are particularly simple: (i) equilateral triangles, (ii) isosceles right-angled triangles. For them we have:

(4.1) Equilateral triangle of side $2a$: $\Delta = 3^{\frac{1}{2}}a^2, \; k = \tfrac{1}{2}a^2.$

(4.2) Isosceles right-angled triangle with hypotenuse $2a$: $\Delta = a^2, \quad k = \tfrac{1}{3}a^2.$

If we use N triangles (all of one type or the other), (1.1) gives

(4.3) Equilateral triangles:

$$\int_D F\,dD = 3^{-\frac{1}{2}}a^2 \sum_i m_i F_i + E_1 + E_2,$$

$$| E_2 | \leqslant \tfrac{1}{2}\, 3^{\frac{1}{2}}\, a^4\, NM.$$

(4.4) Isosceles right-angled triangles:

$$\int_D F\,dD = \tfrac{1}{3}a^2 \sum_i m_i F_i + E_1 + E_2,$$
$$|E_2| \leqslant \tfrac{1}{3}a^4 NM.$$

At each vertex inside the triangulation, $m_i = 6$ in (4.3) and $m_i = 4$ or 8 in (4.4). On the boundary these values are reduced. If the domain D is such that it can be broken up into equal equilateral triangles or into equal isosceles right-angled triangles, then for such triangulations we have $E_1 = 0$.

5. The method of von Mises. Professor W. Prager has drawn my attention to a paper in which von Mises [**1**] develops a powerful method of approximating an integral over a plane domain by a weighted sum of the values of the integrand at selected stations. The error term depends on derivatives $d^\mu F/ds^\mu$ and by taking sufficient stations we can make μ as large as we like. Thus the method of von Mises is more elastic than that of the present paper, for which only $\mu = 2$ occurs. Moreover his method does not use triangulation.

Let us compare the results given by the two methods in the simple case where the domain of integration is a triangle with sides s_1, s_2, s_3. Then (1.1), (1.2), (1.4) of the present paper give for the approximation and the error

(5.1)
$$\int_D F\,dD = \tfrac{1}{3}D(F_1 + F_2 + F_3) + E_2,$$
$$|E_2| \leqslant \tfrac{1}{24} M\,D(s_1^2 + s_2^2 + s_3^2).$$

To apply the method of von Mises, let us take the stations at the vertices of the triangle. By formula (3) of his paper, weighting factors A_1, A_2, A_3 are to be found to satisfy

(5.2)
$$\begin{aligned} A_1 \quad + A_2 \quad + A_3 \quad &= D,\\ A_1 x_1 + A_2 x_2 + A_3 x_3 &= D\bar{x},\\ A_1 y_1 + A_2 y_2 + A_3 y_3 &= D\bar{y}, \end{aligned}$$

where $(x_1, y_1), \ldots$ are the vertices and $(\bar{x}, \bar{y})$ the centroid. The solution is

(5.3)
$$A_1 = A_2 = A_3 = \tfrac{1}{3}D,$$

and formulae (I), (II) of his paper give for the approximation and the error

(5.4)
$$\int_D F\,dD = \tfrac{1}{3}D(F_1 + F_2 + F_3) + E_2',$$
$$|E_2'| \leqslant \tfrac{1}{2}M\,[P_0 + \tfrac{1}{3}D(r_1^2 + r_2^2 + r_3^2)],$$

where P_0 is the polar moment of inertia of the triangle with respect to the origin O and r_1, r_2, r_3 the distances of the vertices from O. This bound for the error is minimized by choosing O at the centroid, and then we have

(5.5)
$$|E_2'| \leqslant \tfrac{1}{24} M\,D(s_1^2 + s_2^2 + s_3^2) + MP_c,$$

where P_c is the polar moment of inertia with respect to the centroid.

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Comparison with (5.1) shows that the method of the present paper gives closer bounds than the method of von Mises. But of course it must be remembered that this particular example favours the method of the present paper, and that in applying the method of von Mises the stations need not be taken at the vertices, nor need there be only three of them.

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