BULL. AUSTRAL. MATH. SOC. 26D15, 33B15
VOL. 72 (2005) [455–459]

A FUNCTIONAL INEQUALITY FOR THE POLYGAMMA FUNCTIONS

HORST ALZER

Let

$$\Delta_n(x) = \frac{x^{n+1}}{n!}\left|\psi^{(n)}(x)\right| \quad (x > 0;\ n \in \mathbf{N}),$$

where ψ denotes the logarithmic derivative of Euler's gamma function. We prove that the functional inequality

$$\Delta_n(x) + \Delta_n(y) < 1 + \Delta_n(z), \quad x^r + y^r = z^r,$$

holds if and only if $0 < r \leqslant 1$. And, we show that the converse is valid if and only if $r < 0$ or $r \geqslant n + 1$.

1. INTRODUCTION

In 1973, Grünbaum [6] presented the following elegant inequality for the Bessel function J_0.

$$J_0(x) + J_0(y) \leqslant 1 + J_0(z), \quad x^2 + y^2 = z^2. \tag{1.1}$$

Askey [4] offered a new proof of (1.1) and showed that (1.1) can be extended to J_α with $\alpha > 0$.

$$J_\alpha^*(x) + J_\alpha^*(y) \leqslant 1 + J_\alpha^*(z), \quad x^2 + y^2 = z^2,$$

where

$$J_\alpha^*(x) = 2^\alpha \Gamma(\alpha + 1) x^{-\alpha} J_\alpha(x).$$

It is natural to ask whether there exist other special functions which satisfy inequalities of Grünbaum-type.

The logarithmic derivative of the gamma function, $\psi = \Gamma'/\Gamma$, is known in the literature as the digamma or psi function. Its derivatives

$$\psi', \psi'', \psi''', \ldots$$

Received 25th July, 2005
I thank the referee for helpful comments.

are called polygamma functions. We have the integral and series representations

$$\psi^{(n)}(x) = (-1)^{n+1} \int_0^\infty e^{-xt} \frac{t^n}{1-e^{-t}}\, dt = (-1)^{n+1} n! \sum_{k=0}^\infty \frac{1}{(x+k)^{n+1}} \quad (x > 0;\ n \in \mathbf{N}). \tag{1.2}$$

These functions have interesting applications in various fields. In particular, they play an important role in mathematical physics. Their main properties can be found, for instance, in [**1**, Chapter 6]. Inequalities for digamma and polygamma functions are discussed in [**3**]. We also refer to [**5**], where a survey on gamma function inequalities is given.

In this note, we show that the trigamma function ψ' satisfies

$$1 + z^2\psi'(z) < x^2\psi'(x) + y^2\psi'(y), \quad x^2 + y^2 = z^2. \tag{1.3}$$

Actually, (1.3) is a special case of a more general inequality involving the function

$$\Delta_n(x) = \frac{x^{n+1}}{n!}\left|\psi^{(n)}(x)\right| \quad (x > 0;\ n \in \mathbf{N}),$$

which we provide in the next section.

2. Main result

To prove our theorem we need properties of Δ_n and its derivative.

Lemma. *Let $n \geqslant 1$ be an integer. The functions Δ_n and Δ_n' are strictly increasing on $(0, \infty)$. Moreover,*

$$\lim_{x\to 0} \Delta_n(x) = 1 \quad \textit{and} \quad \lim_{x \to 0} \Delta_n'(x) = 0. \tag{2.1}$$

Proof: The monotonicity and the convexity of Δ_n are proved in [**2**] and [**3**], respectively. Using the recurrence formula

$$\left|\psi^{(n)}(x)\right| = \left|\psi^{(n)}(x+1)\right| + \frac{n!}{x^{n+1}}$$

(see [**1**, p. 260]), we obtain

$$\Delta_n(x) = 1 + \frac{x^{n+1}}{n!}\left|\psi^{(n)}(x+1)\right| \tag{2.2}$$

and

$$\Delta_n'(x) = \frac{n+1}{n!}x^n\left|\psi^{(n)}(x+1)\right| - \frac{x^{n+1}}{n!}\left|\psi^{(n+1)}(x+1)\right|. \tag{2.3}$$

From (2.2) and (2.3) we conclude that (2.1) holds. ∎

We are now in a position to prove (1.3) and its extension to higher derivatives.

THEOREM. *Let $n \geqslant 1$ be an integer and let $r \neq 0$ be a real number. The inequality*

$$\Delta_n(x) + \Delta_n(y) < 1 + \Delta_n(z) \tag{2.4}$$

holds for all positive real numbers x, y, z with $x^r + y^r = z^r$ if and only if $0 < r \leqslant 1$. And,

$$1 + \Delta_n(z) < \Delta_n(x) + \Delta_n(y) \tag{2.5}$$

is valid for all $x, y, z > 0$ with $x^r + y^r = z^r$ if and only if $r < 0$ or $r \geqslant n + 1$.

PROOF: We define for $x, y > 0$:

$$f_{n,r}(x,y) = 1 + \Delta_n\big((x^r + y^r)^{1/r}\big) - \Delta_n(x) - \Delta_n(y).$$

First, we assume that $f_{n,r}(x,y) > 0$ for all $x, y > 0$. Then we obtain

$$f_{n,r}(x,x) = 1 + \Delta_n(2^{1/r}x) - 2\Delta_n(x) > 0.$$

The asymptotic formula

$$\left|\psi^{(n)}(x)\right| \sim \frac{(n-1)!}{x^n} + \frac{n!}{2x^{n+1}} + \cdots \quad (x \to \infty)$$

(see [1, p. 260]), gives

$$\lim_{x\to\infty} \frac{\Delta_n(x)}{x} = \frac{1}{n}.$$

Thus,

$$0 \leqslant \lim_{x\to\infty} \frac{f_{n,r}(x,x)}{x} = \frac{1}{n}(2^{1/r} - 2).$$

This leads to $0 < r \leqslant 1$.

Next, we prove that if $0 < r \leqslant 1$, then

$$f_{n,r}(x,y) > 0 \quad \text{for all} \quad x, y > 0. \tag{2.6}$$

Since $r \mapsto (x^r + y^r)^{1/r}$ is decreasing on $(0, \infty)$, we conclude from the Lemma that $r \mapsto f_{n,r}(x,y)$ is also decreasing on $(0, \infty)$. Hence,

$$f_{n,r}(x,y) \geqslant f_{n,1}(x,y) = 1 + \Delta_n(x+y) - \Delta_n(x) - \Delta_n(y) = g_n(x,y), \quad \text{say}. \tag{2.7}$$

Applying the Lemma again we obtain

$$\frac{\partial}{\partial x} g_n(x,y) = \Delta_n'(x+y) - \Delta_n'(x) > 0.$$

This leads to

$$g_n(x,y) > g_n(0,y) = 0. \tag{2.8}$$

From (2.7) and (2.8) it follows that (2.6) holds.

Now, we consider (2.5). Let $r > 0$. We suppose that

$$f_{n,r}(x,y) < 0 = f_{n,r}(0,y) \quad (x,y>0). \tag{2.9}$$

Partial differentiation gives

$$\frac{1}{x^n}\frac{\partial}{\partial x} f_{n,r}(x,y) = x^{r-1-n}\Delta_n'\big((x^r+y^r)^{1/r}\big)(x^r+y^r)^{1/r-1} - \frac{\Delta_n'(x)}{x^n}. \tag{2.10}$$

Formula (2.3) yields

$$\lim_{x\to 0}\frac{\Delta_n'(x)}{x^n} = \frac{n+1}{n!}\left|\psi^{(n)}(1)\right| \tag{2.11}$$

and an application of the Lemma implies

$$\lim_{x\to 0}\Delta_n'\big((x^r+y^r)^{1/r}\big)(x^r+y^r)^{1/r-1} = \Delta_n'(y)y^{1-r} > 0. \tag{2.12}$$

From (2.9)–(2.12) we conclude that $r-1-n \geqslant 0$.

It remains to show that if $r<0$ or $r \geqslant n+1$, then

$$f_{n,r}(x,y) < 0 \quad \text{for all} \quad x,y>0. \tag{2.13}$$

Let $r<0$. We have

$$(x^r+y^r)^{1/r} < \min(x,y),$$

so that the Lemma implies

$$f_{n,r}(x,y) < 1 + \Delta_n\big(\min(x,y)\big) - \Delta_n(x) - \Delta_n(y) < 0.$$

Let $r \geqslant n+1$ and

$$s = s_n(x,y) = (x^{n+1}+y^{n+1})^{1/(n+1)}.$$

We obtain

$$f_{n,r}(x,y) \leqslant 1 + \Delta_n(s) - \Delta_n(x) - \Delta_n(y) = u_n(x,y), \quad \text{say.} \tag{2.14}$$

Differentiation yields

$$\frac{\partial}{\partial x}u_n(x,y) = x^n\big[v_n(s) - v_n(x)\big], \tag{2.15}$$

where

$$v_n(x) = \frac{\Delta_n'(x)}{x^n}.$$

Using

$$\frac{1}{x} = \int_0^\infty e^{-xt}\,dt \quad (x>0),$$

https://doi.org/10.1017/S0004972700035279 Published online by Cambridge University Press

the integral representation (1.2), and the convolution theorem for Laplace transforms, we obtain

$$(2.16) \qquad n!\frac{v_n'(x)}{x} = -\frac{n+2}{x}\left|\psi^{(n+1)}(x)\right| + \left|\psi^{(n+2)}(x)\right| = \int_0^\infty e^{-xt} Z_n(t)\,dt,$$

where

$$Z_n(t) = \frac{t^{n+2}}{1-e^{-t}} - (n+2)\int_0^t \frac{s^{n+1}}{1-e^{-s}}\,ds.$$

We have

$$Z_n(0) = 0 \quad \text{and} \quad Z_n'(t) = -\frac{t^{n+2}e^{-t}}{(1-e^{-t})^2}.$$

This implies that Z_n is negative on $(0,\infty)$. From (2.16) we find that v_n is strictly decreasing on $(0,\infty)$. Since $s > x$, we obtain from (2.15) that

$$(2.17) \qquad u_n(x,y) < u_n(0,y) = 0.$$

Combining (2.14) and (2.17) we conclude that (2.13) is valid. □

References

[1] M. Abramowitz and I.A. Stegun (eds.), *Handbook of mathematical functions with formulas and mathematical tables* (Dover Publications Inc., New York, 1965).

[2] H. Alzer, 'Mean-value inequalities for the polygamma functions', *Aequationes Math.* **61** (2001), 151–161.

[3] H. Alzer, 'Sharp inequalities for the digamma and polygamma functions', *Forum Math.* **16** (2004), 181–221.

[4] R. Askey, 'Grünbaum's inequality for Bessel functions', *J. Math. Anal. Appl.* **41** (1973), 122-124.

[5] W. Gautschi, 'The incomplete gamma function since Tricomi', in *Tricomi's ideas and contemporary applied mathematics*, Atti Convegni Lincei **147** (Accad. Naz. Lincei, Rome, 1998), pp. 203–237.

[6] F.A. Grünbaum, 'A new kind of inequality for Bessel functions', *J. Math. Anal. Appl.* **41** (1973), 115–121.

Morsbacher Str. 10
D-51545 Waldbröl
Germany
e-mail alzerhorst@freenet.de

https://doi.org/10.1017/S0004972700035279 Published online by Cambridge University Press