Bull. Aust. Math. Soc. **100** (2019), 428–433
doi:10.1017/S0004972719000315

ON THE REGULARITY OF CHARACTER DEGREE GRAPHS

Z. SAYANJALI, Z. AKHLAGHI✉ and B. KHOSRAVI

(Received 23 January 2019; accepted 11 February 2019; first published online 8 March 2019)

Abstract

Let G be a finite group and let $\mathrm{Irr}(G)$ be the set of all irreducible complex characters of G. Let $\rho(G)$ be the set of all prime divisors of character degrees of G. The character degree graph $\Delta(G)$ associated to G is a graph whose vertex set is $\rho(G)$, and there is an edge between two distinct primes p and q if and only if pq divides $\chi(1)$ for some $\chi \in \mathrm{Irr}(G)$. We prove that $\Delta(G)$ is k-regular for some natural number k if and only if $\overline{\Delta}(G)$ is a regular bipartite graph.

2010 *Mathematics subject classification*: primary 20C15; secondary 05C25.

Keywords and phrases: finite groups, character degree graph.

1. Introduction

Let G be a finite group, $\mathrm{Irr}(G)$ the set of all irreducible complex characters of G and $\mathrm{cd}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$ the set of character degrees of G. Denote by $\rho(G)$ the set of all prime divisors of character degrees of G. The character degree graph $\Delta(G)$ is a graph whose vertex set is $\rho(G)$, and there is an edge between two distinct primes p and q in $\rho(G)$ if and only if pq divides $\chi(1)$ for some $\chi \in \mathrm{Irr}(G)$. Let Γ be a graph and x be a vertex of Γ. We call x a complete vertex if its degree is $n-1$, where n is the order of Γ, and we call Γ a complete graph if all vertices of Γ are complete. A graph Γ is k-regular for some integer k if all vertices have the same degree k. We denote the complement of the graph $\Delta(G)$ by $\overline{\Delta}(G)$. This is the graph whose vertex set is the same as that of $\Delta(G)$, and two vertices are adjacent in $\overline{\Delta}(G)$ if and only if they are not adjacent in $\Delta(G)$. We denote the degrees of the vertex x in $\Delta(G)$ and $\overline{\Delta}(G)$ by $d_G(x)$ and $d_{\bar{G}}(x)$, respectively.

The character degree graph $\Delta(G)$ is a helpful tool for studying the character degree set $\mathrm{cd}(G)$. In [1], it is shown that $\overline{\Delta}(G)$ has an odd cycle if and only if $O^{\pi'}(G) \cong S \times A$, where A is abelian and $S \cong \mathrm{SL}_2(u^\alpha)$ or $S \cong \mathrm{PSL}_2(u^\alpha)$ for a prime $u \in \pi$. Zuccari [8, Theorem A] proved that if G is a solvable group and $\Delta(G)$ is regular, then $\Delta(G)$ is a complete graph or an $(n-2)$-regular graph, where $n = |\rho(G)|$. Tong-Viet [5] posed the following conjecture.

Conjecture 1.1. *If $\Delta(G)$ is k-regular for some integer $k \geq 2$, then $\Delta(G)$ is a complete graph of order $k+1$ or a k-regular graph of order $k+2$.*

Tong-Viet showed that his conjecture is true for $k = 3$ (see [5]). As the main result of this note we prove the following theorem.

THEOREM 1.2. *Let G be a finite group and suppose the character degree graph $\Delta(G)$ of G has more than two vertices. Then $\Delta(G)$ is k-regular for some integer k if and only if one of the following conditions holds:*

(a) $k = 0$ *and* $G \cong A_5 \times C$ *or* $\mathrm{PSL}_2(8) \times C$ *for an abelian group* C*; or*
(b) $\overline{\Delta}(G)$ *is a regular bipartite graph and in particular, if* $k \neq |\rho(G)| - 1$*, then* $|\rho(G)|$ *is even.*

By Theorem 1.2, if $\Delta(G)$ is a regular graph with odd order then $\Delta(G)$ is complete, providing some evidence for Tong-Viet's conjecture. If $\Delta(G)$ is a regular graph with even order, then either $\Delta(G)$ is complete (as predicted by Conjecture 1.1), or $\rho(G) = X \cup Y$ for some disjoint nonempty sets X and Y, such that $|X| = |Y|$ and the subgraphs induced in $\Delta(G)$ by X and Y are complete.

Throughout the paper, all groups are finite and all characters are complex characters. We denote by $\pi(n)$ the set of primes dividing n and by $\pi(G)$ the set of primes dividing the order of G. If $N \trianglelefteq G$ and $\theta \in \mathrm{Irr}(N)$, the inertia group of θ in G is denoted by $I_G(\theta)$. We write $\mathrm{Irr}(G|\theta)$ for the set of all irreducible constituents of θ^G. We write $\mathbf{Z}(K)$ for the centre of K in G. The rest of our notation follows [4].

2. Preliminaries

LEMMA 2.1 [7, Zsigmondy's theorem]. *Let p be a prime and n a positive integer. Then one of the following holds:*

(i) $p^n - 1$ *has a primitive prime divisor* p'*, that is,* $p' \mid (p^n - 1)$ *but* $p' \nmid (p^m - 1)$ *for* $1 \leq m < n$*;*
(ii) $p = 2$ *and* $n = 1$ *or* 6*;*
(iii) p *is a Mersenne prime and* $n = 2$*.*

LEMMA 2.2 [3, Itô–Michler theorem]. *Let $\rho(G)$ be the set of all prime divisors of the elements of $\mathrm{cd}(G)$. Then $p \notin \rho(G)$ if and only if G has a normal abelian Sylow p-subgroup.*

LEMMA 2.3 [4, Theorems 6.2, 6.8 and 11.29]. *Let $N \trianglelefteq G$ and $\chi \in \mathrm{Irr}(G)$. Let θ be an irreducible constituent of χ_N and suppose $\theta_1, \ldots, \theta_t$ are the distinct conjugates of θ in G. Then $\chi_N = e\sum_{i=1}^{t} \theta_i$, where $e = [\chi_N, \theta]$ and $t = |G : I_G(\theta)|$.*

Moreover, $\theta(1)$ divides $\chi(1)$ and $\chi(1)/\theta(1)$ is a divisor of $|G/N|$.

LEMMA 2.4 [1, Theorem A]. *Let G be a finite group and let π be a subset of the vertex set of $\Delta(G)$ such that $|\pi|$ is an odd number larger than 1. Then π is the set of vertices of a cycle in $\overline{\Delta}(G)$ if and only if $O^{\pi'}(G) = S \times A$, where A is abelian, $S \cong \mathrm{SL}_2(u^\alpha)$ or $S \cong \mathrm{PSL}_2(u^\alpha)$ for a prime $u \in \pi$ and a positive integer α, and the primes in $\pi \backslash \{u\}$ are alternately odd divisors of $u^\alpha + 1$ and $u^\alpha - 1$.*

https://doi.org/10.1017/S0004972719000315 Published online by Cambridge University Press

Lemma 2.5 [2, Proposition 2.4]. *Let G be a group.*

(a) *Suppose that there exists $\pi \subseteq \rho(G)$, with $|\pi|$ an odd number larger than 1, such that π is the set of vertices of a cycle in $\overline{\Delta}(G)$. Then there is a characteristic subgroup N of G with $\pi \subseteq \pi(N)$ such that N is a two-dimensional special or projective special linear group over a finite field $\mathbb{F}$ with $|\mathbb{F}| \geq 4$.*

(b) *Let $N \trianglelefteq G$ such that N is isomorphic to $\mathrm{PSL}_2(u^\alpha)$ or $\mathrm{SL}_2(u^\alpha)$, where $u^\alpha \geq 4$ is a prime power. Then $|C_G(N) \cap N| \leq 2$ and the prime divisors of $|G/NC_G(N)|$ are adjacent in $\Delta(G)$ to all primes in $\rho(N)\backslash\{u\}$.*

(c) *Let $\mathcal{K}$ be any (nonempty) set of normal subgroups of G as in (b) (possibly with different values of u^α) and define K as the product of all the subgroups in $\mathcal{K}$. Set $C = C_G(K)$. Then every prime t in $\rho(C)$ is adjacent in $\Delta(G)$ to all the primes q (different from t) in $|G/C|$, with the possible exception of $(t, q) = (2, u)$ when $|\mathcal{K}| = 1$, $K \cong SL_2(u^\alpha)$ for some $u \neq 2$ and $\mathbf{Z}(K) = P'$, $P \in \mathrm{Syl}_2(C)$. In any case, $\rho(G) = \rho(G/C) \cup \rho(C)$.*

Lemma 2.6 [6, Theorem 5.2]. *Let $G \cong \mathrm{PSL}_2(q)$, where $q \geq 4$ is a power of a prime p.*

(1) *If q is even then $\Delta(G)$ has three connected components, $\{2\}$, $\pi(q-1)$ and $\pi(q+1)$, and each component is a complete graph.*

(2) *If $q > 5$ is odd then $\Delta(G)$ has two connected components, and these are $\{p\}$ and $\pi((q-1)(q+1))$.*

   (a) *The connected component $\pi((q-1)(q+1))$ is a complete graph if and only if $q-1$ or $q+1$ is a power of 2.*

   (b) *If neither of $q-1$ or $q+1$ is a power of 2, then $\pi((q-1)(q+1))$ can be partitioned as $\{2\} \cup M \cup P$, where $M = \pi(q-1)\backslash\{2\}$ and $P = \pi(q+1)\backslash\{2\}$ are both nonempty sets. The subgraph of $\Delta(G)$ corresponding to each of the subsets M, P is complete, all primes are adjacent to 2 and no prime in M is adjacent to any prime in P.*

Lemma 2.7 [2, Lemma 2.2]. *Let G be an almost simple group with socle S isomorphic to $\mathrm{PSL}_2(u^\alpha)$, where u is a prime. Let $s \neq u$ be a prime divisor of $|G/S|$. Then the following conclusions hold:*

(a) *the prime s is adjacent in $\Delta(G)$ to every prime in $\pi(u^{2\alpha}-1)$;*

(b) *the prime s is adjacent in $\Delta(G)$ to every prime in $\pi(G)\backslash\pi(S)$;*

(c) *the set of vertices $\rho(G)\backslash\{u\}$ is covered by two complete subgraphs of $\Delta(G)$.*

Remark 2.8. Let p be a prime number and $(x, p) = 1$. Let $k \geq 1$ be the smallest positive integer such that $x^k \equiv 1 \pmod p$. Then k is called the order of x with respect to p and we denote it by $\mathrm{ord}_p(x)$. By Fermat's little theorem, $\mathrm{ord}_p(x) \mid (p-1)$. Also if $x^n \equiv 1 \pmod p$, then $\mathrm{ord}_p(x) \mid n$.

https://doi.org/10.1017/S0004972719000315 Published online by Cambridge University Press

3. Proof of Theorem 1.2

Since the 'if' part of the theorem is straightforward, we concentrate on the 'only if' part. If $\overline{\Delta}(G)$ is bipartite, then the rest of the statement of part (b) of the theorem follows. So we assume that $\overline{\Delta}(G)$ is not a bipartite graph and we show that G is isomorphic to $A_5 \times C$ or $\mathrm{PSL}_2(8) \times C$ for some abelian group C.

Since $\overline{\Delta}(G)$ is not a bipartite graph, there is an odd cycle in $\overline{\Delta}(G)$ and we denote the vertices of this cycle by π. Obviously, $|\pi| > 1$ and $|\pi|$ is odd. By Lemma 2.4, $O^{\pi'}(G) \cong S \times A$, where A is abelian and $S \cong \mathrm{SL}_2(q)$ or $S \cong \mathrm{PSL}_2(q)$ for a prime power $q \geq 4$. Let $\mathcal{K}$ be the set of all normal subgroups of G isomorphic to $\mathrm{SL}_2(q)$ or $\mathrm{PSL}_2(q)$ for some prime power $q \geq 4$. Note that $\mathcal{K}$ is nonempty and we can define K as the product of all subgroups in $\mathcal{K}$. We fix a subset $\{N_1, \ldots, N_\ell\}$ of $\mathcal{K}$ such that $K/\mathbf{Z}(K) \cong N_1/\mathbf{Z}(N_1) \times \cdots \times N_\ell/\mathbf{Z}(N_\ell)$. For $i \in \{1, \ldots \ell\}$, let $q_i = p_i^{\alpha_i}$ be a prime power such that $N_i/\mathbf{Z}(N_i) \cong \mathrm{PSL}_2(q_i)$. Set $C = C_G(K)$. Note that $\rho(C) \cap \rho(G/C) \subseteq \{2\}$. (Otherwise, if $2 \neq r \in \rho(C) \cap \rho(G/C)$, then, by Lemma 2.5(c), r is adjacent to all vertices in $\rho(C)$ and $\rho(G/C)$. Since $\rho(G) = \rho(C) \cup \rho(G/C)$ and $\Delta(G)$ is regular, $\Delta(G)$ is a complete graph and so $\overline{\Delta}(G)$ is bipartite, which is a contradiction.)

First, we assume that $\ell \neq 1$. By Lemma 2.5(c), every prime $t \in \rho(C)$ is adjacent in $\Delta(G)$ to all primes q (different from t) in $\pi(G/C)$. Hence $\rho(C) \cap \rho(G/C) = \emptyset$, as otherwise $\Delta(G)$ is complete. No vertex in $\rho(C)$ in $\overline{\Delta}(G)$ is adjacent to any vertex in $\rho(G/C)$. Obviously, C does not have any characteristic subgroup isomorphic to a two-dimensional special or projective special linear group over a finite field of order at least four. Thus Lemma 2.5(a) guarantees that $\overline{\Delta}(C)$ is a bipartite graph. Since $\overline{\Delta}(G)$ is regular, $\overline{\Delta}(C)$ is a regular bipartite graph. As $K/\mathbf{Z}(K) \cong N_1/\mathbf{Z}(N_1) \times \cdots \times N_\ell/\mathbf{Z}(N_\ell)$ and $\ell \neq 1$, by Lemmas 2.5(c) and 2.6, 2 is adjacent to all vertices in $\Delta(G)$ and so $\Delta(G)$ is a complete graph which is impossible by our assumption. It follows that $\ell = 1$. In this case $G/C \leqslant \mathrm{Aut}(\mathrm{PSL}_2(p_1^{\alpha_1}))$ and $K \cong \mathrm{SL}_2(p_1^{\alpha_1})$ or $K \cong \mathrm{PSL}_2(p_1^{\alpha_1})$. We suppose $p_1 \neq 2$ and return to the case $p_1 = 2$ later.

First, assume that 2 is not adjacent to p_1 in $\Delta(G)$. Then, using Lemmas 2.6, 2.5(c) and 2.7, 2 is adjacent to all vertices except for p_1. So $d_G(2) = |\rho(G)| - 2$ and hence $d_G(p_1) = |\rho(G)| - 2$, which implies that p_1 is adjacent to all vertices, apart from 2. If $2 \neq s \in \pi(G/CK)$, then, by Lemmas 2.5 and 2.7, s must be adjacent to all vertices of $\Delta(G)$, which is a contradiction, as $d_G(s) = |\rho(G)| - 1 \neq d_G(2)$. So we deduce that $\pi(G/CK) \subseteq \{2\}$. By the above argument, for every odd prime $s \in \pi(p_1^{2\alpha_1} - 1)$ there exists $\chi \in \mathrm{Irr}(G)$ such that sp_1 divides $\chi(1)$ and hence there exists $\theta \in \mathrm{Irr}(CK)$ such that $\chi(1)/\theta(1)$ divides $|G/CK|$ by Lemma 2.3. Since $\pi(G/CK) \subseteq \{2\}$, we see that sp_1 divides $\theta(1)$. As $\rho(C) \cap \rho(G/C) \subseteq \{2\}$, it follows that $s, p_1 \notin \rho(C)$ and so, by Lemma 2.2, C has a normal abelian Hall $\{s, p_1\}$-subgroup, say H. Set

$$\overline{CK} = CK/(\mathbf{Z}(K)H) \cong \mathrm{PSL}_2(q_1) \times C/(\mathbf{Z}(K)H).$$

Obviously, there is no irreducible character in $\mathrm{Irr}(\overline{CK})$ whose degree is divisible by sp_1, since $(|C/H|, sp_1) = 1$ and sp_1 does not divide any character degree of $\mathrm{PSL}_2(q_1)$. So there exists $\xi \in \mathrm{Irr}(\mathbf{Z}(K)H)$, $\xi \neq 1$, such that $\theta \in \mathrm{Irr}(CK \mid \xi)$. Set $I = I_{CK}(\xi)$. By

https://doi.org/10.1017/S0004972719000315 Published online by Cambridge University Press

Clifford's theorem [4, Theorem 6.11], there exists a character $\psi \in \mathrm{Irr}(I)$ such that $\theta(1) = |CK : I|\psi(1)$. As $HK \trianglelefteq I$, sp_1 does not divide $|CK : I|$ and hence sp_1 divides $\psi(1)$. Let $\psi \in \mathrm{Irr}(I \mid \nu)$ for some $\nu \in \mathrm{Irr}(HK)$. Since $HK \cong H \times K$, $\nu(1)$ is not divisible by both s and p_1. Therefore $\psi(1)/\nu(1)$ is divisible by either s or p_1. As $\psi(1)/\nu(1)$ divides $|I/HK|$, it follows that $\psi(1)/\nu(1)$ divides $|C/H|$, a contradiction. So we may assume that 2 is adjacent to p_1 in $\Delta(G)$. Then 2 is a complete vertex, which implies that $\Delta(G)$ is complete, a contradiction.

Now we may assume that p_1 is even. In this case $\rho(C) \cap \rho(G/C) = \emptyset$, since otherwise $\Delta(G)$ is complete by Lemma 2.5. If 2 divides $|G/CK|$, then, by Lemma 2.7, 2 is adjacent to all vertices in $\Delta(G/C)$. Therefore 2 is a complete vertex in $\Delta(G)$ and hence $\Delta(G)$ is complete, which is impossible. So 2 does not divide $|G/CK|$. Assume that $|G/CK| \neq 1$ and r is an odd prime divisor of $|G/CK|$. By Lemmas 2.7 and 2.5, r is adjacent to all elements in $\pi(2^{2\alpha_1} - 1) \cup \rho(C)$. If r is adjacent to 2, then r is a complete vertex, which implies that $\Delta(G)$ is complete, a contradiction. Hence $d_G(r) = |\rho(G)| - 2$. Then by the regularity of $\Delta(G)$, we have $d_G(s) = |\rho(G)| - 2$ for all $s \in \rho(G)$. If there is an odd prime $t \in \rho(G)\backslash(\pi(G/CK) \cup \rho(C))$, then there exists $\chi \in \mathrm{Irr}(CK)$ such that $2t$ divides $\chi(1)$, which gives a contradiction because $C \times K \cong CK$ and $2 \notin \rho(C)$. So we may assume that $\rho(G) \subseteq \pi(G/CK) \cup \rho(C) \cup \{2\}$. Hence $\pi(2^{2\alpha_1} - 1) \subseteq \pi(G/CK) \subseteq \pi(\mathrm{Out}(\mathrm{PSL}_2(2^{\alpha_1})))$ and so $\pi(2^{2\alpha_1} - 1) \subseteq \pi(\alpha_1)$. If a is a primitive prime divisor of $2^{2\alpha_1} - 1$, then, by Remark 2.8, $2\alpha_1$ divides $a - 1$ and also a divides α_1, a contradiction. Thus $2^{2\alpha_1} - 1$ has no primitive prime divisor. By Lemma 2.1, $\alpha_1 = 3$, which is impossible, as $7 \in \pi(2^6 - 1) \nsubseteq \pi(3)$. Hence $|G/CK| = 1$ and thus $G \cong C \times K$. So, by the structure of G, $\Delta(K)$ is regular. Therefore $|\pi(2^{\alpha_1} + 1)| = |\pi(2^{\alpha_1} - 1)| = 1$ and $K \cong A_5$ or $\mathrm{PSL}_2(8)$. We now invoke the Feit–Thompson odd-order theorem that every finite group of odd order is solvable. Since $2 \notin \rho(C)$, Lemma 2.2 and the odd-order theorem together show that C is solvable. Suppose C is nonabelian and $|\rho(C)| = m$. By [8, Theorem A], $\Delta(C)$ is a complete graph or an $(m-2)$-regular graph. So $d_{\bar{C}}(s) = 0$ or $d_{\bar{C}}(s) = 1$, for every $s \in \rho(C)$, and hence $d_{\bar{G}}(s) = 0$ or $d_{\bar{G}}(s) = 1$. But that is impossible as $d_{\bar{G}}(b) = 2$ for every vertex $b \in \rho(K)$. So C is abelian, which is our desired result.

Acknowledgement

The authors would like to thank the referee for valuable comments and suggestions.

References

[1] Z. Akhlaghi, C. Casolo, S. Dolfi, E. Pacifici and L. Sanus, 'On the character degree graph of finite groups', *Annali di Mat. Pura Appl.*, to appear.
[2] Z. Akhlaghi, S. Dolfi, E. Pacifici and L. Sanus, 'Bounding the number of vertices in the degree graph of a finite group', Preprint, 2018, arXiv:1811.01674.
[3] B. Huppert, *Character Theory of Finite Groups* (de Gruyter, Berlin, 2011).
[4] I. M. Isaacs, *Character Theory of Finite Groups* (Academic Press, New York, 1976).
[5] H. P. Tong-Viet, 'Finite groups whose prime graphs are regular', *J. Algebra* **397** (2014), 18–31.
[6] D. L. White, 'Degree graphs of simple groups', *Rocky Mountain J. Math.* **39** (2009), 1713–1739.

[7] K. Zsigmondy, 'Zur Theorie der Potenzreste', *Monatsh. Math. Phys.* **3** (1892), 265–284.
[8] C. P. M. Zuccari, 'Regular character degree graphs', *J. Algebra* **353** (2014), 215–224.

Z. SAYANJALI, Faculty of Mathematics and Computer Science,
Amirkabir University of Technology (Tehran Polytechnic),
15914 Tehran, Iran
e-mail: zohrehsayanjali@aut.ac.ir

Z. AKHLAGHI, Faculty of Mathematics and Computer Science,
Amirkabir University of Technology (Tehran Polytechnic),
15914 Tehran, Iran
e-mail: z_akhlaghi@aut.ac.ir

B. KHOSRAVI, Faculty of Mathematics and Computer Science,
Amirkabir University of Technology (Tehran Polytechnic),
15914 Tehran, Iran
e-mail: khosravibbb@yahoo.com