The Focal and Bi-Tangent Properties of Bi-Circular Quartics.

By WILLIAM P. MILNE, M.A., D.Sc.

(*Read 9th June 1911. Received 17th June 1911*).

1. The following direct method of attack establishes the fundamental properties of the Foci and Bi-Tangents of a Bi-Circular Quartic very simply, using only the properties of a homographic correspondence. The method was suggested to me by my previous paper on the focal properties of Circular Cubics.

I am indebted to Mr Peter Fraser, University of Bristol, for his criticism.

2. The following theorem will be required in what follows:—

All Bi-Nodal Quartics, which pass through seven given points and have their Nodes at two given points, pass through an eighth fixed point.

For a curve to have a Node at a fixed point is tantamount to three linear conditions.

Hence for a Quartic to have a Node at each of two given points is tantamount to six linear conditions.

But a Quartic can in general be made to satisfy fourteen linear conditions, hence one Quartic can in general be found to have Nodes at each of two given points I, J, and to pass through eight given points; while a pencil of Quartics $S_1 + \lambda S_2 = 0$ can in general be found to pass through seven given points and to have Nodes at each of two given points I, J. But each member of the system $S_1 + \lambda S_2$ passes through all the intersections of S_1 and S_2. Now S_1 and S_2, being Quartics, intersect in sixteen points. Also S_1 and S_2 having each a Node at I, have four of their intersections coinciding at I. Similarly other four of their intersections coincide at J. Thus eight further points of section remain. But seven of these points must be the seven given points through which all the members of the pencil $S_1 + \lambda S_2$ are to pass. Hence since S_1 and S_2 intersect in an eighth fixed point, therefore all the members of the pencil $S_1 + \lambda S_2$ must pass through that eighth fixed point.

3. *By a Bi-Circular Quartic is meant a Bi-Nodal Quartic having a Node at each of the Circular points at Infinity, I and J.*

If a Bi-Circular Quartic is touched at both H and K by the same Circle, and if IPQ be a line through one Circular Point I meeting the Quartic in P and Q, the Circles HKP and HKQ cut the Quartic again in P′ and Q′ respectively where P′Q′ passes through J.

Consider the points H, H, K, K, P, Q, P′, Q′.

Two Quartics passing through these eight points and having Nodes at I and J are :—

I. The given Quartic ;

II. The Quartic consisting of the Circles HKPP′ and HKQQ′.

Hence by Article 2, any other Bi-Circular Quartic passing through seven of these points will pass also through the eighth.

Consider the Bi-Circular Quartic consisting of the Circle touching the given Quartic at H and K, the line IPQ, and the line JP′. This latter degenerate Bi-Circular Quartic has Nodes at I and J, and passes through H, H, K, K, P, Q, P′. It must therefore pass through Q′.

Now the line IPQ cannot pass through Q′, or else it would meet the given Quartic in more than four points.

Also, the Circle touching at H and K cannot pass through Q, or else it would meet the given Quartic in more than eight points.

Hence IP′ passes through Q′.

4. The theorem of last article can now be quoted in a form more suitable for our purpose :—

If H and K be two points on a Bi-Circular Quartic, such that the same Circle touches at both H and K, and if a line be drawn through I meeting the Quartic in P and Q, then the circles passing through H, K and either P or Q cut the quartic again in two points P′, Q′, such that the lines JP′ and JQ′ coincide.

Thus a one-to-one algebraic correspondence exists between the lines IP and JP′, where a HKPP′ are concyclic.

5. *The Cross-Ratio of the four tangents from I to the Bi-Circular Quartic is equal to the cross-ratio of the four tangents from J.*

That there are four tangents from I is plain from the fact that a Bi-Nodal Quartic is of Class 8, and two tangents are coincident with the tangent to each branch of the Node. Thus other four tangents remain.

Now consider a tangent from I meeting the Quartic again in the two coincident points P, P′.

It is obvious that the Circles HKP and HKP′ are coincident, and therefore cut the Quartic again in Q, Q′, which are coincident. Thus JQQ′ is also a tangent by Art. 4.

Thus in the one-to-one algebraic correspondence established in Article 4, the tangents from I correspond to the tangents from J each to each, and consequently the cross-ratio of the tangents from I is equal to the cross-ratio of the tangents from J.

Corollary.—It is also plain that if P_1, P_2, P_3, P_4 be the points of contact of the tangents from I and Q_1, Q_2, Q_3, Q_4, the points of contact of the corresponding tangents from J, then in the above one-to-one system of correspondence, IQ_1 corresponds to JP_1, etc.

We thus get the following :—

The cross-ratio of the pencil of rays joining I to the points of contact of the tangents from J is equal to the cross-ratio of the pencil of rays joining J to the points of contact of the tangents from I.

6. Let us now take a point H′ near to H. It is plain that there will be a point K′ near to K, such that one and the same circle cuts the Quartic at both H′ and K′. Proceeding as in Articles 3 and 4, we see that, using H′ and K′, we can again establish a one-to-one algebraic correspondence between rays passing through I with those passing through J. Now the four tangents from I correspond respectively to the four tangents through J, precisely as they did when we used H and K. Thus the one-to-one algebraic correspondence established when using H and K is exactly the same as that established when using H′ and K′.

We thus have the following theorem :—

Let the four tangents from I be IP_1, IP_2, IP_3, IP_4; and the four tangents from J be JQ_1, JQ_2, JQ_3, JQ_4; and let IP_1 correspond to JQ_1, IP_2 to JQ_2, IP_3 to JQ_3, IP_4 to JQ_4. Let IPP′ and JQQ′

be corresponding rays in the algebraic one-to-one correspondence, thus defined, and let them meet the Quartic in PP' and QQ' respectively. Then, being given P (say), either Q (or Q') will be such that any Circle through P and Q cuts the Quartic again in two points H and K, such that one and the same Circle touches the Quartic at both H and K.

If this property hold between P and Q, it will not *in general hold between P and Q'.*

If this property hold between P and Q, it will also hold between P' and Q'.

If IP_0P_0' and JQ_0Q_0' be another pair of corresponding rays belonging to the above homographic relationship, then we may find which of the two points P_0 or P_0' correspond to each of the two points Q_0 or Q_0' by a process of continuous deflection from a position in which this relationship can be detected.

7. *Cyclo-Points and Conjugate Cyclo-Points.*

There are cases in which H and K, as defined above, coincide. The points on the Quartic thus defined may be called "Cyclic Points." They have the property that the Osculating Circles thereat cut the Quartic in four coincident points.

We shall call H and K, when distinct, "Conjugate Cyclo-Points." They possess the property that one and the same Circle touches the Quartic at both H and K.

Two Conjugate Cyclo-Points, if coincident, constitute a Cyclic Point.

8. *The Four Systems.*

We have shown that the four tangents from I, viz., IP_1, etc., have the same cross-ratio as the four tangents from J. We have exhibited this correspondence in the form

$$I[P_1P_2P_3P_4] = J[Q_1Q_2Q_3Q_4] \tag{1}$$

But we know from the elementary theory of cross-ratios that if (1) is true, the following are also true:—

$$I[P_1P_2P_3P_4] = J[Q_2Q_1Q_4Q_3] \tag{2}$$

$$I[P_1P_2P_3P_4] = J[Q_3Q_4Q_1Q_2] \tag{3}$$

$$I[P_1P_2P_3P_4] = J[Q_4Q_3Q_2Q_1] \tag{4}$$

Thus we can get four systems of algebraic correspondences of the kind treated above.

Taking each of these four systems of homographic correspondences and reasoning as above, we get the following theorem :—

If H be a given point on the Quartic, there are four points K on the Quartic, such that the same Circle touches at both H and K.

Thus there are four systems of Conjugate Cyclo-Points and (in particular) of Cyclic Points.

9. We shall now devote ourselves to the development of the characteristic properties of *one* of the above systems.

The line joining one pair of Conjugate Cyclo-Points cuts the Quartic in another pair of Conjugate Cyclo-Points.

Any Circle passing through a pair of Conjugate Cyclo-Points cuts the Quartic again in another pair of Conjugate Cyclo-Points by Art. 6.

Now one Circle through a pair of Conjugate Cyclo-Points consists of the line joining the pair of Conjugate Cyclo-Points coupled with the Line at Infinity—whence the theorem is evident.

10. *The Focal Circles of one System.*

Consider a pair of corresponding rays IPQ and JP′Q′ of the First System (say). Let them meet in R. Then the locus of R is a Conic passing through I and J, *i.e.* a Circle.

Now if we extend the ordinary signification of a Focus S in Conics to mean that SI and SJ each touch the Bi-Circular Quartic, we see that the Circle traced out by R passes through the four Foci defined by the four intersections of the four tangents from I with the four tangents from J that respectively correspond according to the first system.

We shall therefore refer to the locus of R as the "Focal Circle of the First System."

11. *To prove that a Focal Circle intersects the Quartic in Cyclic Points.*

Let U be a Cyclic Point. Then the Osculating Circle at U to the Quartic cuts the curve in four coincident points.

Let a Circle touching the Quartic at U cut the curve again in P and Q.

Then by Art. 4 IP and JQ are corresponding rays of the same system.

Let us find the ray corresponding to IU in this system. Let it be JV.

Then U, U, U, V are Concyclic by Art. 6. Hence V coincides with U since U is a Cyclic Point.

Thus the Focal Circle of that particular system passes through that Cyclic Point.

We thus have the following results :—

The Focal Circle of the First System cuts the Bi-Circular Quartic in four Cyclic Points.

Since there are four Focal Circles, there must be sixteen Cyclic Points corresponding, four by four, to the four given systems of homography.

12. *The lines joining Conjugate Cyclo-Points of the First System pass through the same point, viz., the centre of the First Focal Circle.*

Let H and K be Conjugate Cyclo-Points of the First System. Then, if we consider the Coaxal System of Circles having H and K as their common points, we see that if any circle of this Coaxal System cut the Quartic again in P and Q that IP and JQ are corresponding rays of the First System. Hence it is plain that IH and JK are corresponding rays of the First System. Let them meet in L. Then L is a Limiting Point of the Coaxal System defined by HK, and lies on the first Focal Circle. Let M be the other Limiting Point, obtained as the intersection of IK and JH.

Then, by the known properties of Coaxal Circles, HK bisects LM perpendicularly. But LM has been shown to be a chord of the First Focal Circle, and therefore HK must pass through the centre of the First Focal Circle.

Corollary I. The Tangents to the Quartic at the four Cyclic Points of the First System are concurrent, the point of concurrency being the centre of the First Focal Circle.

Corollary II. The Focal Circles cut the Bi-Circular Quartic orthogonally.

Corollary III. The Limiting Points of all systems of Coaxal Circles defined by pairs of Conjugate Cyclo-Points of the First System lie on the first Focal Circle.

13. *Double-Tangents.*

Let C_1 be the centre of the First Focal Circle. Since a Bi-Nodal Quartic is of Class 8, eight tangents can be drawn from C_1 to the Quartic. Of these, four have been shown to be the Tangents at the *four* Cyclic Points of the First System. We wish to discover what the remaining tangents from C_1 are.

To show that the remaining tangents from C_1 are two Bi-Tangents. Let a tangent from C_1 meet the Quartic in the points X, X, Y, Z.

Then by Art. 9 these four points are capable of being conjoined in two pairs of Conjugate Cyclo-Points.

If X and X are a coincident pair of Conjugate Cyclo-Points, we obtain the Four Cyclic Points of the First System.

In the remaining cases X and X must not form a pair of coincident Conjugate Cyclo-Points, otherwise there would be more than four Cyclic Points of the First System, which is impossible by Art. 11.

Let then X, Y and X, Z be each a pair of Conjugate Cyclo-Points. Let a Tangent from I and its corresponding Tangent from J according to the First System touch at P and Q respectively.

Since X and Y are Conjugate Cyclo-Points, therefore XYPQ are concyclic by Art. 6.

Similarly XZPQ are concyclic.

But the Circle XPQ meets the Quartic in one other point (other than I or J). Hence Y and Z are coincident. Thus XY is a Bi-Tangent.

We must therefore reckon XY as two coincident Tangents from from C_1.

The following theorem will now be plain :—

There are two Bi-Tangents defined by the First System which intersect in the centre of the First Focal Circle.

Corollary.—The eight Bi-Tangents of a Bi-Circular Quartic intersect in pairs in the centres of the Focal Circles.

14. *The Sixteen Directrices.*

Let the definition of a Directrix used in the case of Conics be extended to Higher Curves to mean that, *if S be a Focus, and if SI touch the Curve in P, and similarly SJ touch the Curve in Q, PQ is a Directrix.*

Let IP be a Tangent from I touching at P, and JQ be the corresponding Tangent from J according to the First System. Then IP and JQ may be regarded as together forming a degenerate Circle having double contact with the Bi-Circular Quartic. Hence by Art. 12 the Directrix PQ passes through the centre of the First Focal Circle.

We thus have :—

The Four Directrices of the First System pass through the centre of the First Focal Circle.

We thus have a remarkable system of two Tetrads of Points (viz., the points of contact of the Tangents from I and J) such that the points of the first group can be joined to the points of the second so as to be quadruply concurrent. This is usually known as a Hessian Configuration of Lines.

15. The same methods can be applied very successfully to deal with Quartics whose Double Points are larger in number or are Cusps. We shall take only one case by way of illustration, viz., the case of a Tri-Nodal Quartic, one of whose Nodes is at I and the other at J. Let the third Node be O.

The following results will be evident if we apply our former methods of reasoning in detail.

Since two tangents can be drawn from I and two from J, there will be Two *Systems of Circles having double contact with the given Quartic.*

There will be Four Foci, viz., Two of the First System and Two of the Second.

16. *Both Focal Circles pass through the Node O.*

For if H and K be a pair of Conjugate Cyclo-Points, then the Circle HKO cuts the Quartic again in O. Hence IO corresponds to JO. Now the locus of corresponding rays is a Focal Circle by Art. 10. Hence a Focal Circle passes through O.

17. *There are two Cyclic Points of each system.*

For each Focal Circle cuts the Quartic in two points besides the Nodes.

18. *There are Four Bi-Tangents.*

This can be shown as in Art. 13.

19. *The lines joining the Node O to the centres of the two Focal Circles are the bisectors of the angles between the nodal tangents at O.*

We shall first show that *if H be a fixed point on the Quartic, and if the tangent circles at H to the Quartic, cutting the curve again in K and K,' be drawn, then will the tangents at O to the series of circles OHK and OHK' form an involution whose self-corresponding rays are the tangents to these two circles obtained when K and K' coincide.*

Let OT be any straight line drawn from O. Draw the circle touching OT at O and passing through H. Let this circle cut the Quartic again at K. Draw the circle H, H, K, cutting the curve again at K'. Let OT' be the tangent to the circle OHK'. It is thus plain that a one-to-one algebraic correspondence exists between OT and OT', and from the symmetry of their relationship it is evidently an involution.

The double rays of this involution are evidently the tangents to the two circles got in the two cases when K and K' coincide, *i.e. the double rays of this involution are the two lines joining O to the two Focal Centres,* since a circle passing through two Conjugate Cyclo-Points H and K cuts the Quartic again in two Conjugate Cyclo-Points (this time at O), and the join of a pair of Conjugate Cyclo-Points passes through one or other of the Focal Centres.

Now consider the line that corresponds to OI in this involution. The circle touching OI at O and passing through H is evidently the degenerate circle OI, HJ. Let HJ cut the Quartic again in K. Next draw the circle H, H, K (*i.e.* HJK, HI), cutting the Quartic again in K'. Finally, draw the circle OHK', *i.e.* HI, OJ. Thus the line corresponding in the above involution to OI is OJ.

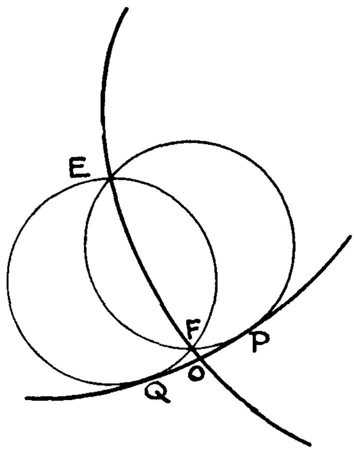
Now, any two corresponding rays in an involution harmonically separate the two self-corresponding rays, *i.e.* OI and OJ harmonically separate the two lines joining O to the Focal Centres. *Thus the lines joining O to the two Focal Centres are perpendicular.* (1)

Next, we shall show that *if E be a fixed point and F a variable point, both on the Quartic, and if the two circles through E and F that touch the Quartic have as their points of contact P and Q respectively, then the tangents at O to the circles OEP and OEQ are corresponding rays in an involution.*

https://doi.org/10.1017/S0013091500033848 Published online by Cambridge University Press

If we take any line through O and take it as a tangent to define a circle through O and E, and trace the connexion between the two lines through O above defined, we shall easily see that an involution exists between such pairs.

To find the self-corresponding rays, let us find the ray that corresponds to one of the nodal tangents at O.



If we draw the two circles through E and F (where F is very near the node) touching the Quartic at P and Q respectively, we see that the tangents to the circles OEP and OEQ in the limit coincide and become one of the nodal tangents at O.

Thus the self-corresponding rays in the above involution are the nodal tangents.

Now take the case when E and F coincide. E and P and E and Q will then be Conjugate Cyclo-Points of one or other of the two systems of the given Quartic. In that case the tangents at O to the circles OEP and OEQ become the lines joining O to the Focal Centres.

Hence the lines joining O to the Focal Centres harmonically separate the nodal tangents. (2)

We thus see by (1) and (2) that *the two lines joining O to the Focal Centres are the bisectors of the angles between the nodal tangents.*