Zero and uniqueness sets for Fock spaces

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Abstract. We characterize zero sets for which every subset remains a zero set too in the Fock space $\mathcal{F}^p$, $1 \leq p < \infty$. We are also interested in the study of a stability problem for some examples of uniqueness set with zero excess in Fock spaces.

1 Introduction

Let β be a positive real number. The Gaussian measure on the complex plane $\mathbb{C}$ is defined as

$$(1) \qquad d\mu_\beta(z) := \frac{\beta}{2\pi} e^{-\frac{\beta}{2}|z|^2} dA(z), \qquad z \in \mathbb{C},$$

where dA is the Euclidean area measure. The Fock space $\mathcal{F}^p := \mathcal{F}^p_\pi$, where $1 \leq p < \infty$, is the collection of entire functions $f : \mathbb{C} \to \mathbb{C}$ such that

$$\|f\|_p := \left(\int_{\mathbb{C}} |f(z)|^p \, d\mu_{p\pi}(z) \right)^{\frac{1}{p}} < \infty.$$

The space $\mathcal{F}^p$ endowed with the norm $\|.\|_p$ is a vector Banach space, for every $p \geq 1$. For the particular case when $p = 2$, $\mathcal{F}^2$ is a reproducing kernel Hilbert space with the reproducing kernel given by

$$K(z, w) := e^{\pi \bar{z} w}, \quad z, w \in \mathbb{C}.$$

For instance, see the textbook [19, Chapitre 2] and the references therein.

A countable set $Z = \{z_n\}_{n \in \mathbb{N}} \subseteq \mathbb{C}$ is called a zero set for $\mathcal{F}^p$ if there exists a function $f \in \mathcal{F}^p \setminus \{0\}$ such that the zero set $\{z \in \mathbb{C} : f(z) = 0\}$ of f, counting multiplicities, coincides with Z. We say that Z is a uniqueness set for $\mathcal{F}^p$ if the unique function of $\mathcal{F}^p$ that vanishes on Z is the zero function. It is known that a complete characterization, of zero and uniqueness sets for the Fock spaces, still remains an open question, and we refer to [1, 13, 18, 19].

Due to the distinctiveness of Fock spaces among other spaces of analytic functions, there exist particular sets, uniqueness set with zero excess. These are those uniqueness sets that they become zero sets by removing just one point. Numerous examples of uniqueness sets with zero excess are known for $\mathcal{F}^p$. The first typical example is the square lattice for $\mathcal{F}^p$, when $2 < p < \infty$, and the square lattice without one point for

every $1 < p \leq 2$. More generally, for every $0 \leq \nu \leq 1$, the sequence Γ_ν is a uniqueness set with zero excess for $\mathcal{F}^p$, for every $\frac{2}{1+\nu} < p < \frac{2}{\nu}$, where

$$(2) \qquad \Gamma_\nu := \{\gamma_{m,n} := m + in \ : \ (m, n) \in \mathcal{I}_\nu\},$$

and where $\mathcal{I}_\nu := (\mathbb{Z} \times (\mathbb{Z}\backslash\{0\})) \cup (\mathbb{Z}_- \times \{0\}) \cup \{(m + \nu, 0)\}_{m \geq 0}$ (see [11, 18]). It is simple to see that Γ_ν is a separated sequence in the complex plan and of critical density for $\mathcal{F}^p$ in the study of interpolating and sampling problems (see, for example, [16, 18, 19]). Motivated by the results in [8, 14, 18], we are interested in the study of a stability problem of sequences Γ_ν, for every $0 \leq \nu \leq 1$.

The second kind of uniqueness sets with zero excess for $\mathcal{F}^p$ is an example of a nonseparated sequence localized at the real and imaginary lines. It constitutes the zero set of the sin-cardinal type function

$$(3) \qquad S(z) := (z^2 - 1) \frac{\sin\left(\frac{\pi}{2} z^2\right)}{\pi z^2}, \quad z \in \mathbb{C},$$

and it is given by

$$(4) \qquad \Gamma := \left\{\pm\sqrt{2n}, \ \pm i\sqrt{2n} \ : \ n \geq 1\right\} \cup \{\pm 1\}.$$

This sequence was constructed by Ascensi, Lyubarski, and Seip in [2] for the Hilbert case. It is still of the same kind for $\mathcal{F}^p$, for every $1 < p < \infty$.

As in the work of the second author in [14], we are interested in the study of a stability problem of the sequences Γ and Γ_ν, for every $0 \leq \nu \leq 1$, for the spaces $\mathcal{F}^p$, and for a fixed $1 < p < \infty$. Namely, if $\Lambda = \{\lambda_\sigma \ : \ \sigma \in \Sigma\}$ is a sequence of complex numbers, where $\Sigma = \Gamma$ or Γ_ν, we write $\lambda_\sigma = \sigma e^{\delta_\sigma} e^{i\theta_\sigma}$, where $\delta_\sigma, \theta_\sigma \in \mathbb{R}$ for every $\sigma \in \Sigma$. We are interesting in giving optimal conditions on (δ_σ) and (θ_σ) for which Λ fails to be a uniqueness set with zero excess for $\mathcal{F}^p$, for a fixed $1 < p < \infty$. Before stating our main results, we need some notations. For $\Lambda := \{\lambda_\gamma := \gamma e^{\delta_\gamma} e^{i\theta_\gamma} \ : \ \gamma \in \Gamma_\nu\}$, we denote

$$\hat{\delta}(\Lambda) := \liminf_{R \to \infty} \frac{1}{\log R} \sum_{|\gamma| \leq R} \delta_\gamma \ \text{ and } \ \delta(\Lambda) := \limsup_{R \to \infty} \frac{1}{\log R} \sum_{|\gamma| \leq R} \delta_\gamma.$$

Our first main result in this paper is the following theorem.

Theorem 1.1 *Let $0 \leq \nu \leq 1$, and let $\Lambda = \{\lambda_\gamma\}_{\gamma \in \Gamma_\nu}$ be a sequence of complex numbers. We write $\lambda_\gamma = \gamma e^{\delta_\gamma} e^{i\theta_\gamma}$, for every $\gamma \in \Gamma_\nu$, where $\delta_\gamma, \theta_\gamma \in \mathbb{R}$. If*

(1) Λ *is separated,*
(2)   *the sequences $(\gamma^2 \delta_\gamma)_{\gamma \in \Gamma_\nu}$ and $(\gamma^2 \theta_\gamma)_{\gamma \in \Gamma_\nu}$ are bounded, and*
(3)

$$\nu - \frac{2}{p} < \hat{\delta}(\Lambda) \leq \delta(\Lambda) < \nu + 1 - \frac{2}{p},$$

then Λ is a uniqueness set with zero excess for $\mathcal{F}^p$, whenever $\frac{2}{1+\nu} < p < \frac{2}{\nu}$.

Now, for the Ascensi–Lyubarskii–Seip sequence Γ, if $\Lambda := \{\lambda_\gamma := \gamma e^{\delta_\gamma} e^{i\theta_\gamma} \ : \ \gamma \in \Gamma\}$ is a sequence of $\mathbb{C}$, we will denote by Δ_n the quantity

$$\Delta_n := \sum_{|\gamma|=\sqrt{2n}} \delta_\gamma = \delta_{\sqrt{2n}} + \delta_{-\sqrt{2n}} + \delta_{i\sqrt{2n}} + \delta_{-i\sqrt{2n}}.$$

Our second main result is the following.

Theorem 1.2 *Let $\Gamma := \left\{\pm\sqrt{2n}, \ \pm i\sqrt{2n} \ : \ n \geq 1\right\} \cup \{\pm 1\}$, and let $\Lambda = \{\lambda_\gamma := \gamma e^{\delta_\gamma} e^{i\theta_\gamma} : \gamma \in \Gamma\}$ be a sequence of complex numbers. Suppose that:*

(1) *There exists $c > 0$ such that $\left|\lambda_\gamma - \lambda_{\gamma'}\right| \geq c / \min\{|\gamma|, |\gamma'|\}$, for every $\gamma, \gamma' \in \Gamma$.*

(2) *The sequences $(\gamma^2 \delta_\gamma)_{\gamma \in \Gamma}$ and $(\gamma^2 \theta_\gamma)_{\gamma \in \Gamma}$ are bounded.*

(3) $\Delta(\Lambda) := \limsup\limits_{n \to \infty} \dfrac{1}{\log n} \left| \sum\limits_{k=1}^{n} \Delta_k \right| < \dfrac{1}{2\max\{p,q\}}$, *where q is the Hölder conjugate number of p.*

Then, Λ is a uniqueness set with a zero excess for $\mathcal{F}^p$, where $1 < p < \infty$.

Another extreme case of sequences, we are interesting in, is motivated by the zero set of the sin-cardinal type function

$$s(z) = \frac{\sin\left(\frac{\pi}{2} z^2\right)}{z^2} \in \mathcal{F}^p.$$

The zero set of s, denoted by $Z(s)$, is a zero sequence for $\mathcal{F}^p$, for every $1 \leq p < \infty$. However, if we remove the subset which belongs to the imaginary axis from $Z(s)$, the remaining part is not a zero set anymore for $\mathcal{F}^p$. Such result can be viewed as a consequence of Lindelöf's theorem (see [5, Theorem 2.10.1]). Therefore, a natural question is: which zero set for $\mathcal{F}^p$ remains a zero set too for $\mathcal{F}^p$, even an infinite subset was removed?

Actually, the example above is a variant to the one given by Zhu in [17]. This phenomenon is one of the main deference between Fock spaces and Hardy spaces and even Bergman spaces of the unit disk where zero sets are well stable [6, 7].

In the following theorem, we give a complete description of zero sets for which all their subsequences are also zero sets for $\mathcal{F}^p$.

Theorem 1.3 *Let $Z = \{z_n\}_{n \in \mathbb{N}}$ be a zero set for $\mathcal{F}^p$, $1 \leq p < \infty$. The following statements are equivalents:*

(1) *Every subset of Z is a zero set for $\mathcal{F}^p$.*

(2) *Z satisfies*

$$(5) \qquad\qquad\qquad \sum_{n \in \mathbb{N}} \frac{1}{|z_n|^2} < \infty.$$

Before stating the proofs of our main results, we give first some remarks:

(1) In Theorem 1.1, if $\nu = 1$, we then get $1 < p < 2$. Actually, the result remains valid for $p = 2$, and this case was treated in [14]. Theorem 1.1 gives a result analogous to

those related to complete interpolating sequences for the Paley–Wiener spaces (see [3, 12]), and small Fock spaces [4, 15].

(2) Note that if there exists a positive integer N such that

$$(6) \qquad \sup_{n \geq 0} \frac{n+1}{N} \left| \sum_{k=n+1}^{n+N} \Delta_k \right| < \frac{1}{2 \max\{p, q\}},$$

then $\limsup\limits_{N \to \infty} \frac{1}{\log N} \left| \sum\limits_{k=0}^{N} \Delta_k \right| < \frac{1}{2 \max\{p,q\}}$ and the converse is not true (see [14, Lemma 5.4]). On the other hand, in Theorem 1.2, for $p = 2$, we have $q = 2$. This case was treated in [14]. Theorem 1.2 with the Avdonin-type condition (6) appears like the result proved by Lyubarskii and Seip in [12] concerning complete interpolating sequences for Paley–Wiener spaces. Such result generalizes those by Kadets and Avdonin for the Hilbert case (see [3, 9]).

(3) The conditions on the sequences (δ_γ) and (θ_γ) in Theorem 1.1 are optimal. The proof is similar to Theorem 1.5 and Proposition 5.3 in [14].

(4) An interesting fact appears in the proof of Theorem 1.2 (namely Lemma 2.2) is a confirmation of the result which confirms that $\mathcal{F}^p$ and $\mathcal{F}^q$, $p > q$, do not share the same zero sets, a result that we have already got in [1]. We provided a sequence with positive Beurling uniform density, while the example that we can construct here, by a precise choice of δ, is of null lower Beurling density.

We end this section with some words on notation: throughout this paper, the notation $A \lesssim B$ means that $A \leq cB$ for a certain positive constant c, and the notation $A \asymp B$ will be used to say that $A \lesssim B$ and $B \lesssim A$ hold in the same time. The paper is organized as follows: in the next section, we state some key lemmas containing estimates of some modified infinite products. Section 3 is devoted to prove Theorems 1.1 and 1.2. Theorem 1.3 will be proved in the last section.

2 Some lemmas

In this section, we introduce some modified Weierstrass products. These functions will play an important role in the proof of our main results. First, we recall that for every $0 \leq v \leq 1$, the sequence $\Gamma_v = \{\gamma_{m,n} : m, n \in \mathbb{Z}\}$ is given by

$$\{\gamma_{m,n} := m + in : (m, n) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})\} \cup \mathbb{Z}_- \cup \{m + v\}_{m \geq 0}.$$

If $\Lambda = \{\lambda_{m,n} : m, n \in \mathbb{Z}\}$ is a sequence of complex numbers, we will write $\lambda_{m,n} := \gamma_{m,n} e^{\delta_{m,n}} e^{i\theta_{m,n}}$, where $\delta_{m,n}, \theta_{m,n} \in \mathbb{R}$, for every $m, n \in \mathbb{Z}$. We associate with Λ the following infinite product

$$G_\Lambda(z) := (z - \lambda_{0,0}) \prod_{m,n \in \mathbb{Z}} {}' \left(1 - \frac{z}{\lambda_{m,n}}\right) \exp\left[\frac{z}{\gamma_{m,n}} + \frac{z^2}{2\gamma_{m,n}^2}\right], \quad z \in \mathbb{C}.$$

The product with the prime is taken over all integers m and n such that $(m, n) \neq (0, 0)$. The following lemma provides an estimates of the function G_Λ.

Lemma 2.1 [14, Lemma 3.2]   *If Λ satisfies the conditions of Theorem 1.1, G_Λ is an entire function vanishing exactly on Λ and verifying*

$$\frac{(1+|\operatorname{Im}(z)|)^M}{(1+|z|)^{\nu-\hat{\delta}+M}}\operatorname{dist}(z,\Lambda) \lesssim |G_\Lambda(z)|e^{-\frac{\pi}{2}|z|^2} \lesssim \frac{(1+|z|)^{-\nu+\delta+M}}{(1+|\operatorname{Im}(z)|)^M}\operatorname{dist}(z,\Lambda), \; z\in\mathbb{C},$$

for some positive constant M, where $\delta = \delta(\Lambda) + \varepsilon$ and $\hat{\delta} = \hat{\delta}(\Lambda) - \varepsilon$ for a positive ε small enough.

On the other hand, for the Ascensi–Lyubarskii–Seip sequence given in [2] by

$$\Gamma = \left\{\pm\sqrt{2n}, \pm i\sqrt{2n} \,:\, n \geq 1\right\} \cup \{\pm 1\},$$

we associate the modified sin-cardinal function $G_\Gamma(z) := \frac{z^2-1}{\pi z^2}\sin\left(\frac{\pi}{2}z^2\right)$. Now, if $\Lambda = \{\lambda_\gamma := \gamma e^{\delta_\gamma}e^{i\theta_\gamma} \,:\, \gamma\in\Gamma\}$ is a sequence of complex numbers, we associate with Λ the infinite product

$$G_\Lambda(z) := \lim_{r\to\infty}\prod_{\lambda\in\lambda,\,|\lambda|\leq r}\left(1-\frac{z}{\lambda}\right), \quad z\in\mathbb{C}.$$

According to the proof of [14, Theorem 1.10], we have the following lemma that gives an estimate of the function G_Λ.

Lemma 2.2   *If Λ satisfies the conditions of Theorem 1.2, the infinite product G_Λ converges uniformly on every compact set of $\mathbb{C}$ and verifies*

$$\frac{\operatorname{dist}(z,\Lambda)}{\operatorname{dist}(z,\Gamma)}\frac{(1+|\operatorname{Im} z^2|)^M}{(1+|z|)^{2\delta+2M}}|G_\Gamma(z)| \lesssim |G_\Lambda(z)|,$$

$$|G_\Lambda(z)| \lesssim \frac{\operatorname{dist}(z,\Lambda)}{\operatorname{dist}(z,\Gamma)}\frac{(1+|z|)^{2\delta+2M}}{(1+|\operatorname{Im} z^2|)^M}|G_\Gamma(z)|,$$

for every $z\in\mathbb{C}\backslash\Gamma$, where $\delta = \Delta(\Lambda) + \varepsilon$ for a small positive ε.

3   Proofs of Theorems 1.1 and 1.2

This section is devoted to the proofs of Theorem 1.1 and 1.2.

3.1   Proof of Theorem 1.1.

First, we show that $\Lambda\backslash\{\lambda\}$ is a zero set for $\mathcal{F}^p$, for some fixed (any) $\lambda\in\Lambda$. To this end, it suffices to prove that $\frac{G_\Lambda}{z-\lambda}$ belongs to $\mathcal{F}^p$. Indeed, by Lemma 2.1, we have

$$\int_\mathbb{C}\left|\frac{G_\Lambda(z)}{z-\lambda}e^{-\frac{\pi}{2}|z|^2}\right|^p dA(z) \lesssim \int_\mathbb{C}\frac{(1+|z|)^{p(-1-\nu+\delta+M)}}{(1+|\operatorname{Im}(z)|)^{pM}}dA(z)$$

$$\asymp \int_\mathbb{C}\frac{1}{(1+|z|)^{p(1+\nu-\delta)}}dA(z),$$

and the last integral converges if and only if $p(1 + v - \delta) > 2$. In view of the third assumption, the integral converges. Hence, $\frac{G_\Lambda}{z - \lambda} \in \mathcal{F}^p$ (obviously, $z \mapsto \frac{G_\Lambda(z)}{z - \lambda}$ is an entire function).

Second, we prove that Λ is a uniqueness set for $\mathcal{F}^p$. Let F be a function of $\mathcal{F}^p$ that vanishes on Λ. Then there exists an entire function h such that $F = hG_\Lambda$. According to the estimates of G_Λ in Lemma 2.1, we have

$$|h(z)| \frac{(1 + |\operatorname{Im}(z)|)^M}{(1 + |z|)^{v - \hat{\delta} + M}} \operatorname{dist}(z, \Lambda) \lesssim |h(z)G_\Lambda(z)| e^{-\frac{\pi}{2}|z|^2} = |F(z)| e^{-\frac{\pi}{2}|z|^2} \lesssim 1.$$

This implies that h is a polynomial of z, and we denote later by k its degree. Integrating the last inequality with respect to the measure $dA(z)$, we get

$$\int_{\mathbb{C}} \left| F(z) e^{-\frac{\pi}{2}|z|^2} \right|^p dA(z) \gtrsim \int_{\mathbb{C}} |h(z)|^p \frac{(1 + |\operatorname{Im}(z)|)^{pM}}{(1 + |z|)^{p(v - \hat{\delta} + M)}} \operatorname{dist}(z, \Lambda)^p dA(z)$$

$$\asymp \int_{\mathbb{C}} \frac{(1 + |\operatorname{Im}(z)|)^{pM}}{(1 + |z|)^{p(v - k - \hat{\delta} + M)}} dA(z)$$

$$\asymp \int_{\mathbb{C}} \frac{1}{(1 + |z|)^{p(v - k - \hat{\delta})}} dA(z).$$

The last integral converges if and only if $p(v - k - \hat{\delta}) > 2$, and this implies that $k + \hat{\delta} < v - 2/p$. This is in contradiction with the assumption $v - 2/p < \hat{\delta}(\Lambda)$. Thus, h is zero and F too. This completes the proof of Theorem 1.1.

3.2 Proof of Theorem 1.2.

First, we need the lemma below, which is analogous to Lemma 3.4 in [14], we include the proof for completeness. We denote

$$dv_{p,\alpha,\beta}(z) = \left(\frac{1 + |z|^2}{1 + |\operatorname{Im} z^2|} \right)^{\alpha p} \frac{1}{(1 + |z|)^{p\beta}} e^{-\frac{p\pi}{2}|z|^2} dA(z),$$

where $1 \le p < \infty$ and α and β are two real numbers.

Lemma 3.1 *Let α and β be two real numbers. The sin-cardinal type function G_Γ belongs to $L^p(\mathbb{C}, dv_{p,\alpha,\beta})$ if and only if $\beta > \frac{1}{p}$.*

Proof Let $1 \le p < \infty$. Recall that

$$|\sin(z)|^2 = \left(\sin\left(\operatorname{Re} z \right) \right)^2 + \left(\sinh(\operatorname{Im} z) \right)^2, \qquad z \in \mathbb{C}.$$

It follows that, G_Γ belongs to $L^p(\mathbb{C}, dv_{p,\alpha,\beta})$ if and only if $\sinh\left(\frac{\pi}{2} z^2 \right)$ does. This is equivalent to

$$\int_{\mathbb{C}} \left(\frac{e^{\frac{\pi}{2}|\operatorname{Im} z^2|}}{1 + |z|^\beta} \right)^p \left(\frac{1 + |z|^2}{1 + |\operatorname{Im} z^2|} \right)^{p\alpha} e^{-\frac{p\pi}{2}|z|^2} dA(z) < \infty.$$

Using the Tonelli theorem, we obtain

$$I := \int_{|z|>1} \frac{1}{(1+|z|)^{p\beta-2p\alpha}} \frac{1}{(1+|\operatorname{Im} z^2|)^{p\alpha}} e^{-\frac{p\pi}{2}(|z|^2-|\operatorname{Im} z^2|)} dA(z)$$

$$(7) \qquad \asymp 8 \int_1^\infty \int_0^x \frac{e^{-\frac{p\pi}{2}(x-y)^2}}{(x^2+y^2)^{p(\beta-2\alpha)/2}} \frac{1}{(1+xy)^{p\alpha}} dy dx.$$

On the other hand, we have

(8)
$$\int_0^x \frac{e^{-\frac{p\pi}{2}(x-y)^2}}{(x^2+y^2)^{p(\beta-2\alpha)/2}} \frac{1}{(1+xy)^{p\alpha}} dy \asymp \frac{1}{x^{p(\beta-2\alpha)}} \int_0^\infty \frac{e^{-\frac{p\pi}{2}y^2}}{(1+x^2-xy)^{p\alpha}} \chi_{[0,x]}(y) dy.$$

Combining (7) and (8), we obtain

$$I \asymp \int_1^\infty \int_0^\infty \frac{1}{x^{p(\beta-2\alpha)}} \frac{e^{-\frac{p\pi}{2}y^2}}{(1+x^2-xy)^{p\alpha}} \chi_{[0,x]}(y) dx dy$$

$$= \int_0^\infty e^{-\frac{p\pi}{2}y^2} \int_{\max\{1,y\}}^\infty \frac{1}{x^{p(\beta-2\alpha)}(1+x^2-xy)^{p\alpha}} dx dy.$$

Consequently, the integral needed converges if and only if

$$\int_{\max\{1,y\}}^\infty \frac{1}{x^{p(\beta-2\alpha)}(1+x^2-xy)^{p\alpha}} dx$$

converges too. That is, if and only if, $\beta > \frac{1}{p}$. ∎

Now, we can start the proof of Theorem 1.2. First, we will show that $\Lambda \setminus \{\lambda\}$ is a zero set for $\mathcal{F}^p$, for fixed $\lambda \in \Lambda$, and for every $1 < p < \infty$. To do this, it suffices to prove that $\frac{G_\Lambda}{z-\lambda}$ belongs to $\mathcal{F}^p$. According to Lemma 2.2 and by a subharmonicity argument, we have

$$\int_{\mathbb{C}} \left| \frac{G_\Lambda(z)}{z-\lambda} e^{-\frac{\pi}{2}|z|^2} \right|^p dA(z) \asymp \int_{\mathbb{C}} \left| \frac{G_\Lambda(z)}{z-\lambda} \frac{\operatorname{dist}(z,\Gamma)}{\operatorname{dist}(z,\Lambda)} e^{-\frac{\pi}{2}|z|^2} \right|^p dA(z)$$

$$\lesssim \int_{\mathbb{C}} |G_\Gamma(z)|^p \left(\frac{1+|z|^2}{1+|\operatorname{Im} z^2|} \right)^{pM} \frac{e^{-\frac{p\pi}{2}|z|^2}}{(1+|z|)^{p(1-2\delta)}} dA(z)$$

$$= \int_{\mathbb{C}} |G_\Gamma(z)|^p dv_{p,M,1-2\delta}(z).$$

By Lemma 3.1, the last integral converges since $2\delta < 1 - \frac{1}{p} = \frac{1}{q}$.

To prove that Λ is a uniqueness set for $\mathcal{F}^p$, $1 < p < \infty$, let F be a function of $\mathcal{F}^p$ that vanishes on Λ. Write $F(z) = h(z) G_\Lambda(z)$, for some entire function h. Again, by Lemma 2.2, we have

$$(9) \qquad |F(z)| \frac{\operatorname{dist}(z,\Gamma)}{\operatorname{dist}(z,\Lambda)} \gtrsim |h(z) G_\Gamma(z)| \frac{(1+|\operatorname{Im} z^2|)^M}{(1+|z|)^{2\delta+2M}}.$$

Integrating both sides of the last inequality over $\mathbb{C}$ with respect to the measure $d\mu_{p\pi}(z) = e^{-\frac{p\pi}{2}|z|^2} dA(z)$,

$$\int_{\mathbb{C}} |F(z)|^p \, d\mu_{p\pi}(z) \asymp \int_{\mathbb{C}} \left| F(z) \frac{\text{dist}(z, \Gamma)}{\text{dist}(z, \Lambda)} \right|^p d\mu_{p\pi}(z)$$

$$\gtrsim \int_{\mathbb{C}} \left| h(z) G_\Gamma(z) \frac{(1 + |\text{Im } z^2|)^M}{(1 + |z|)^{2\delta + 2M}} \right|^p d\mu_{p\pi}(z)$$

$$\gtrsim \int_{\mathbb{C}} \left| \frac{h(z) G_\Gamma(z)}{(1 + |z|)^{2\delta + 2M}} \right|^p d\mu_{p\pi}(z).$$

In the second line, we have used a subharmonicity argument. Since $2\delta < \frac{1}{\max\{p,q\}} \leq \frac{1}{2}$, then for a fixed $\gamma \in \Gamma$, we have

$$\int_{\mathbb{C}} \left| \frac{h(z) G_\Gamma(z)}{(1 + |z|)^{2\delta + 2M}} \right|^p d\mu_{p\pi}(z) \gtrsim \int_{\mathbb{C}} \left| \frac{h(z) G_\Gamma(z)}{(z - \gamma) P_{2M}(z)} \right|^p d\mu_{p\pi}(z),$$

where P_{2M} is a polynomial of degree $\lfloor 2M \rfloor + 1$, that vanishes on $\lfloor 2M \rfloor + 1$ points of $\Gamma \backslash \{\gamma\}$. This implies that the function $z \mapsto \frac{h(z) G_\Gamma(z)}{(z-\gamma) P_{2M}(z)}$ belongs to $\mathcal{F}^p$. Since the sequence $\Gamma \backslash \{\gamma\}$ is a maximal zero sequence for $\mathcal{F}^p$, then h must be a polynomial of degree less than $\lfloor 2M \rfloor + 1$ (see [18, 19]). Suppose that h is not zero and denote by k its degree. Now, return to (9) and integrate both sides over $\mathbb{C}$ with respect to the measure $d\mu_{p\pi}$ again, and we obtain

$$\int_{\mathbb{C}} |F(z)|^p d\mu_{p\pi}(z) \asymp \int_{C} \left| F(z) \frac{\text{dist}(z, \Gamma)}{\text{dist}(z, \Lambda)} \right|^p d\mu_{p\pi}(z)$$

$$\gtrsim \int_{\mathbb{C}} \left| h(z) G_\Gamma(z) \frac{(1 + |\text{Im } z^2|)^M}{(1 + |z|)^{2\delta + 2M}} \right|^p d\mu_{p\pi}(z)$$

$$\asymp \int_{\mathbb{C}} \left| G_\Gamma(z) \frac{(1 + |\text{Im } z^2|)^M}{(1 + |z|)^{2\delta + 2M - k}} \right|^p d\mu_{p\pi}(z)$$

$$= \int_{\mathbb{C}} |G_\Gamma(z)|^p d\nu_{p, -M, 2\delta - k}(z).$$

By Lemma 3.1, the latter integral converges if and only if $2\delta - k > 1/p$. Since $\delta < 1/(2p)$, we then get

$$1/p < 2\delta - k < 2\delta < 1/p.$$

This is a contradiction. Hence, h and F are zero. Therefore, Λ is a uniqueness set for $\mathcal{F}^p$. This completes the proof of Theorem 1.2.

4 Proof of Theorem 1.3

The proof of Theorem 1.3 is essentially based on Lindelöf's theorem below. First, we recall some main tools very useful for our proof, and we refer to [5, 10] for more details. If f is an entire function and r is a positive real number, we denote $M(r, f)$

the maximum modulus of f on the circle $|z| = r$:

$$M(r, f) = \max_{|z|=r} |f(z)|.$$

The order of f is given by the quantity

$$\rho_f := \limsup_{r \to \infty} \frac{\log \log M(r, f)}{\log(r)}.$$

Always, we have $0 \le \rho_f \le \infty$. In the case $0 < \rho_f < \infty$, the type of f is defined as

$$\tau_f := \limsup_{r \to \infty} \frac{\log M(r, f)}{r^{\rho_f}}.$$

Let $Z = \{z_n\}_{n \in \mathbb{N}}$ be the zero set of an entire function f. Following [5], the convergence exponent of the sequence $Z = \{z_n\}_{n \in \mathbb{N}}$ (excluding 0 if it belongs to Z) is defined as the infimum of all positive numbers s such that

$$(10) \qquad \sum_{n \in \mathbb{N}} \frac{1}{|z_n|^s} < \infty,$$

and it will be denoted by ρ_1 (for short, the convergence exponent of f). A consequence of Jensen's formula gives the following relations among the order ρ_f and the exponent of convergence ρ_1 of an entire function f (see [5] for complete proof):

$$(11) \qquad \rho_1 \le \rho_f.$$

The following theorem characterizes entire functions of integral order and of finite type.

Theorem 4.1 (Lindelöf [5])   *If ρ is a positive integer, the entire function f of order $\rho_f = \rho$ is of finite type if and only if:*

(1)   *$n(r) = O(r^{\rho_f})$, where $n(r)$ is the number of zeros of f in the disk $|z| \le r$, counting multiplicity, and*

(2)   *the sums*

$$(12) \qquad S(r) := \sum_{|z_n| \le r} \frac{1}{z_n^\rho}$$

are bounded, where $\{z_n\}_n$ is the zero sequence of f.

In the sequel, our constructions are based on dividing the complex plane into sectors with some defined opening aperture. To this end, for a given two angles $\beta \in (-\pi, \pi]$ and $\theta \in (0, \pi]$, define

$$\mathcal{S}(\beta, \theta) := \{z \in \mathbb{C} \,:\, |\arg(z) - \beta| \le \theta\} \cup \{z \in \mathbb{C} \,:\, |\arg(-z) - \beta| \le \theta\}.$$

The following lemma will be of prominent role in the proof of Theorem 1.3 later on.

Lemma 4.2 *Let $1 \leq p < \infty$. If $Z = \{z_n\}_{n \in \mathbb{N}}$ is a zero set for $\mathcal{F}^p$ such that $Z \subset \mathcal{S}(\beta, \theta)$ for some $0 \leq \theta < \frac{\pi}{4}$ and $\beta \in (-\pi, \pi]$, then Z satisfies*

$$(13) \qquad \sum_{n \in \mathbb{N}} \frac{1}{|z_n|^2} < \infty.$$

Proof Without loss of generality, we may suppose $\beta = 0$. Aiming to come to a contradiction, assume that

$$(14) \qquad \sum_{n \in \mathbb{N}} \frac{1}{|z_n|^2} = \infty.$$

If g is a function in $\mathcal{F}^p$ with $Z(g) = Z$, then by [19, Theorem 5.1], for every $\varepsilon > 0$, we have

$$(15) \qquad \sum_{n \in \mathbb{N}} \frac{1}{|z_n|^{2+\varepsilon}} < \infty.$$

Thus, $Z = \{z_n\}_{n \in \mathbb{N}}$ is of convergence exponent $\rho_1 = 2$. Combining (11), (14), and (15), we obtain

$$2 = \rho_1 \leq \rho_g \leq 2.$$

Hence, g is of order 2, and of type τ_g less than or equal to $\frac{\pi}{2}$. Since g is of integral order, Lindelöf's theorem applies. Writing $z_n = |z_n| e^{i\theta_n}$, $n \in \mathbb{N}$, a straightforward calculation gives

$$
\begin{aligned}
|S(r)|^2 &= \left| \sum_{|z_n| \leq r} \frac{1}{z_n^2} \right|^2 = \left| \sum_{|z_n| \leq r} \frac{e^{-2i\theta_n}}{|z_n|^2} \right|^2 \\
&= \left| \sum_{|z_n| \leq r} \frac{\cos(2\theta_n) - i \sin(2\theta_n)}{|z_n|^2} \right|^2 \\
&= \left(\sum_{|z_n| \leq r} \frac{\cos(2\theta_n)}{|z_n|^2} \right)^2 + \left(\sum_{|z_n| \leq r} \frac{\sin(2\theta_n)}{|z_n|^2} \right)^2 \\
&\geq \left(\sum_{|z_n| \leq r} \frac{\cos(2\theta_n)}{|z_n|^2} \right)^2 \\
&\geq \cos^2(2\theta) \left(\sum_{|z_n| \leq r} \frac{1}{|z_n|^2} \right)^2 \longrightarrow \infty, \qquad \text{as } r \to \infty.
\end{aligned}
$$

This contradicts Lindelöf's Theorem 4.1. As a conclusion,

$$\sum_{n=0}^{\infty} \frac{1}{|z_n|^2} < \infty. \qquad \blacksquare$$

Remark 1   We mention that the constant $\frac{\pi}{4}$, which appears in Lemma 4.2, is the best possible. A counterexample is given by the zero set of the sin-cardinal function

$$s(z) = \frac{\sin(\frac{\pi}{2}z^2)}{z^2} \in \mathcal{F}^p.$$

Now, we can prove Theorem 1.3.

4.1 Proof of Theorem 1.3.

Let $Z = \{z_n\}_{n \in \mathbb{N}}$ be a sequence of complex numbers which is a zero sequence for $\mathcal{F}^p$ and satisfies (5). If Z' is any subsequence of Z, then it is a zero sequence too for $\mathcal{F}^p$ by the sufficient condition [19, Theorem 5.3]. Therefore, Z belongs to the desired class.

Conversely, let Z be a zero sequence for $\mathcal{F}^p$ with the property: every subset of Z is also a zero set for $\mathcal{F}^p$. Now, we divide the sequence Z into eight subsets, by writing

$$Z = \bigcup_{k=0}^{7} Z_k,$$

where for each $0 \le k \le 7$,

$$Z_k := Z \cap \left\{z \in \mathbb{C} \ : \ -\frac{\pi}{8} \le \arg(z) - \frac{k\pi}{4} < \frac{\pi}{8}\right\} \subset \mathcal{S}\left(\frac{\pi}{8}, \frac{\pi}{8} + \varepsilon\right),$$

and ε is an arbitrary small positive number in $(0, \frac{\pi}{8})$. By the assumption, each Z_k, $k \in \{0, 1, \ldots, 7\}$ is a zero set for $\mathcal{F}^p$. We conclude by Lemma 4.2.

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