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Hopfological algebra

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ABSTRACT

We develop some basic homological theory of hopfological algebra as defined by Khovanov [*Hopfological algebra and categorification at a root of unity: the first steps*, Preprint (2006), arXiv:math/0509083v2]. Several properties in hopfological algebra analogous to those of usual homological theory of DG algebras are obtained.

1. Introduction

Since its birth, homological algebra has commonly been regarded as being centered around the equation $d^2 = 0$. Such a view can be best seen through the famous quote of Henri Cartan:

If I could only understand the beautiful consequence following from the concise proposition $d^2 = 0$.

Henri Cartan¹

Thus it is a natural question to ask whether and how we could deform this equation while maintaining an equally beautiful and useful theory. Indeed, in [May42a, May42b], Mayer defined a ‘new simplicial homology’ theory over a field of characteristic $p > 0$ by forgetting the usual alternating signs in the definition of boundary maps. The boundary maps satisfy $\partial^p = 0$, and associated with this kind of ‘ p -chain complex’ one obtains the ‘ p -cohomology groups’ $\text{Ker}(\partial^q)/\text{Im}(\partial^{p-q})$, for any $1 \leq q \leq p - 1$. Furthermore, when applied to singular chains on topological spaces, this construction results in a ‘new homology theory’ which is a topological invariant of the underlying space! Exciting as it might seem, however, Spanier [Spa49] soon found out that these homology groups can be recovered from the usual singular homology groups, due to the restrictions placed on any topological homology theory by the Eilenberg–Steenrod axioms. This immediately extinguished most of the interest in Mayer’s invariant, and people paid little attention to these pioneering works on p -complexes; they remained buried among historical documents until several decades later. In 1996, Kapranov [Kap96], and independently Sarkaria [Sar95], studied a ‘quantum’ analogue of the equation $d^p = 0$, working over a field of characteristic zero with n th roots of unity (e.g. the n th cyclotomic field $\mathbb{Q}[\zeta_n]$). The analogous construction yields n -complexes where the boundary maps satisfy $d^n = 0$ for some $n \in \mathbb{N}$. Similar homology groups of these complexes as in [May42a, May42b] are defined. This construction, as a purely algebraic object, rekindled more interest this time and found applications in theoretic physics. Nowadays there is a vast collection of literature on the subject. See, for instance, Angel and Díaz [AD07], Bichon [Bic03], Cibils *et al.* [CSW07], Dubois-Violette [Dub96], Sitarz [Sit98], Kassel and Wambst [KW98], and many of the references therein. It is worth

¹ See the foreword of [GM03].

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mentioning that [KW98] put both $d^p = 0$ and $d^n = 0$ on equal footing, and developed some general homological theory for both cases.

Meanwhile, Pareigis [Par81] reinterpreted the usual homological algebra over a base ring K as (co)modules over a non-commutative, non-cocommutative Hopf algebra. In fact, using Majid's 'bosonisation process' [Maj97], one can understand this Hopf algebra as a graded Hopf-algebra object $K[d]/(d^2)$ in the category of graded super modules over the ground ring K . Similar reformulations for the deformations $d^n = 0$ were given by Bichon [Bic03]. One crucial feature of such Hopf algebras used by these authors is that their (co)module categories are Frobenius. Indeed, finite-dimensional Hopf algebras or objects bearing enough similar properties are well known to have a left (co)integral, which in turn can be used to define non-degenerate associative bilinear forms on the algebras. See for instance [Kup96] for an arrow-diagrammatic proof of this result.

To this end, the work of Khovanov [Kho06] can be regarded as a general framework to unify both points of view about the homological algebra of $d^n = 0$. There he considers (co)module algebras over any finite-dimensional Hopf algebra (or a finite-dimensional Hopf-algebra object in some category). In this framework, Mayer's original p -complexes can be identified with (co)modules over the \mathbb{Z} -graded finite-dimensional Hopf algebra $\mathbb{k}[\partial]/(\partial^p)$, where \mathbb{k} is a field of characteristic $p > 0$. Moreover the usual notion of a differential graded algebra (DGA) can be reinterpreted as a module algebra over the graded Hopf super algebra $K[d]/(d^2)$, and therefore affords a generalization to arbitrary module algebras over finite-dimensional Hopf algebras, among which the Hopf algebra $\mathbb{k}[\partial^p]/(\partial^p)$ over a field of characteristic $p > 0$ is the simplest example. Nonetheless, one question dating back to the work of Mayer and Spanier should still be addressed: why should we care about this construction if its homology gives us nothing new?

One answer to this question was given by Khovanov in [Kho06]. Instead of homology, the Grothendieck groups K_0 of the triangulated (stable) categories $H\text{-gmod}$ are isomorphic to the p th (equivalently the $2p$ th) cyclotomic integers $\mathbb{Z}[\zeta]/(1 + \zeta + \cdots + \zeta^{p-1}) \cong \mathbb{Z}[\zeta_p]$. Furthermore, the (triangulated) module category over such a Hopf module algebra inherits a (triangulated) module category structure. Therefore the Grothendieck group of such a module category will be a module over the ring of cyclotomic integers. Finding interesting such module algebras could potentially realize the dreams dating back to Crane and Frenkel's work on categorification of quantum three-manifold invariants at certain roots of unity and extend them into four-dimensional topological quantum field theories [CF94]. With this motivation, Khovanov coined the terminology 'hopfological algebra' since this new framework is a mixture of the ring theory of Hopf algebras and homological algebra. We follow his suggestion and use this term vaguely to refer to the general homological theory of Hopf (co)module algebras and their module categories.

In the present work, we develop some general homological properties of hopfological algebra (or following [Kho06], we should say 'hopfological properties') in analogy with the usual homological theory of DG algebras. The strategy is rather straightforward since there are now beautiful structural expositions on DG algebras to mimic, such as the book by Bernstein and Lunts [BL94, § 10], the less formal and very readable online lecture notes by Kaledin [Kal09], or the papers of Keller [Kel94, Kel07], whose approach we follow.

Now we give a rough summary of the content of this paper. We start by briefly reviewing Khovanov's original constructions in the first three sections and giving ways to construct distinguished triangles in the 'homotopy' and 'derived' categories of hopfological modules, in analogy with DG algebras. Then we analyze more closely the morphism spaces in the homotopy category, which is needed to define the notion of cofibrant hopfological modules. As in the DG

case, we show that any hopfological module has a cofibrant replacement (Theorem 6.6), and the morphism spaces between cofibrant objects in the derived category coincide with their morphism spaces in the homotopy category. Such cofibrant replacements are also needed to define derived functors and to construct derived equivalences of different hopfological module categories. Next, we show that the derived categories of hopfological modules are compactly generated, and this allows us to use the formidable machinery of Ravenel and Neeman [Nee92, Nee96, Rav84] to give a characterization of compact objects in the derived category (Corollary 7.15), as well as to make precise the definition of Grothendieck groups of hopfological module categories. Finally, a restrictive version of Morita equivalence between derived categories is given (Corollary 8.18). Throughout, the general theory is illustrated by three specific examples in parallel comparison, namely the usual DG algebra, Kapranov and Sarkaria’s n -DG algebra, and Mayer’s p -DG algebra.

As this paper will mainly serve as a tool kit for our work in progress on categorification at roots of unity, there are some important caveats we have to make clear. The first remark to make is that we do not attempt to develop hopfological theory for Kapranov’s characteristic zero ‘ n -differential graded algebra’ in full generality. In §8, we need to assume that the underlying Hopf algebra is *(co)commutative*. One reason is that, given a left H -module algebra A , we could not find a natural way to define a left H -module algebra structure on A^{op} for arbitrary H . Another problem is that, given two module algebras equipped with n -differentials (i.e., $d(ab) = d(a)b + \zeta^{\deg(a)}ad(b)$, and $d^n = 0$ for any elements $a, b \in A$), there does not seem to be a natural way to define a module algebra structure on the tensor product algebra. This problem was already pointed out in [Sit98]. Such a monoidal structure plays a very important role in many existing examples of categorification, for instance [KL09]. Secondly, we will not develop in this paper the full analogue of DG Morita theory (as in Keller [Kel94]), as we wish to control the length of the paper. Such a theory might be better treated in a more categorical setting than the one we use here. In subsequent works we will investigate this question in parallel with Toën’s framework [Toë07] on DG categories, as well as more related K -theoretical questions.

2. Module categories

In this and the next two sections we review the basic constructions of hopfological algebra, following [Kho06, §§ 2.1–2.3]. Then we will develop some basic properties of hopfological algebra, adapting the framework for DG-categories (algebras) in [Kel94]. Our goal is to show that, as predicted in [Kho06], a fair amount of the general theory of DG-algebra generalizes to hopfological algebra.

2.1 The base category

Let H be a finite-dimensional Hopf algebra over a field \mathbb{k} . We denote by Δ the comultiplication, by ϵ the counit, and by S the antipode of H . It is well known that S is an invertible algebra anti-automorphism. We will fix a non-zero left integral Λ of H once and for all, which is uniquely determined (see, for instance, [Kup96, §3, Corollary 3.5]), up to a non-zero constant in the ground field \mathbb{k} by the property that, for any $h \in H$,

$$h\Lambda = \epsilon(h)\Lambda.$$

The category $H\text{-mod}$ of left H -modules is monoidal, with H acting on the tensor product $M \otimes N$ of two H -modules M and N via the comultiplication Δ . In what follows, we will constantly use the Sweedler notation: for any $h \in H$, $\Delta(h) = \sum_{(h)} h_{(1)} \otimes h_{(2)} \in H \otimes H$, and we will omit the summation symbol if no confusion can arise. Moreover, we will freely use the fact that,

for any $h \in H$, $h_{(2)}S^{-1}(h_{(1)}) = \epsilon(h) = S^{-1}(h_{(2)})h_{(1)}$, which follows by applying the anti-automorphism S^{-1} to the axiom $h_{(1)}S(h_{(2)}) = \epsilon(h) = S(h_{(1)})h_{(2)}$.

By convention, when a tensor product sign \otimes is undecorated, we always mean that it is over the base field \mathbb{k} . Moreover, when tensor products ‘ \otimes ’ and direct sums ‘ \oplus ’ appear together without brackets, tensor products always take precedence over direct sums. By modules over an algebra we will always mean left modules over the algebra unless otherwise stated.

PROPOSITION 2.1. (1) For any H -module M , we have a canonical isomorphism of H -modules $M \otimes H \cong M_0 \otimes H$, where M_0 denotes M as a \mathbb{k} -vector space equipped with the trivial H -module structure.

(2) The Hopf algebra H is Frobenius so that it is self-injective. The associated stable module category $H\text{-}\underline{\text{mod}}$ is triangulated monoidal.

(3) The shift functor T on $H\text{-}\underline{\text{mod}}$ is given as follows: for any H -module M , let $M \subset I$ be the inclusion of M into the injective H -module $I = M \otimes H$, given by $\text{Id}_M \otimes \Lambda : M \rightarrow M \otimes H$. Then $T(M)$ is defined to be the cokernel of this inclusion:

$$T : H\text{-}\underline{\text{mod}} \rightarrow H\text{-}\underline{\text{mod}}, \quad M \mapsto M \otimes (H/\mathbb{k}\Lambda).$$

(4) The tensor product of H -modules descends to an exact bifunctor on $H\text{-}\underline{\text{mod}}$

$$\otimes : H\text{-}\underline{\text{mod}} \times H\text{-}\underline{\text{mod}} \rightarrow H\text{-}\underline{\text{mod}},$$

which is compatible with the shift functor above. $H\text{-}\underline{\text{mod}}$ is symmetric monoidal if H is cocommutative. Here compatibility means that, for any $M, N \in H\text{-mod}$,

$$T(M) \otimes N \cong T(M \otimes N) \cong M \otimes T(N).$$

Proof. We give the proof of part (1) here. The rest of the statements are proved in [Kho06, § 1]. We define a map of H -modules: $f_M : M \otimes H \rightarrow M_0 \otimes H$ by sending $m \otimes l \mapsto S^{-1}(l_{(1)})m \otimes l_{(2)}$, for any $l \in H, m \in M$. Then we check that it is an H -module map: for any $h \in H$,

$$\begin{aligned} f_M(h(m \otimes l)) &= f_M(h_{(1)}m \otimes h_{(2)}l) = S^{-1}((h_{(2)}l)_{(1)})h_{(1)}m \otimes (h_{(2)}l)_{(2)} \\ &= S^{-1}(l_{(1)})S^{-1}(h_{(2)})h_{(1)}m \otimes h_{(3)}l_{(2)} = S^{-1}(l_{(1)})\epsilon(h_{(1)})m \otimes h_{(2)}l_{(2)} \\ &= S^{-1}(l_{(1)})m \otimes hl_{(2)} = hf_M(m \otimes l), \end{aligned}$$

where we have used the fact that $S^{-1}(h_{(2)})h_{(1)} = \epsilon(h)$ and $h_{(1)}\epsilon(h_{(2)}) = h$. Notice that in the second to the last equality, h only acts on the second factor. Finally, f_M is invertible whose two-sided inverse is given by $f_M^{-1} : M_0 \otimes H \rightarrow M \otimes H, m \otimes h \mapsto h_{(1)}m \otimes h_{(2)}$. We leave this verification to the reader. \square

We briefly remind the reader of the notion of a stable category associated with a Frobenius category (e.g. modules over a Frobenius algebra), and this will explain some of the notations we used in the above proposition. For more details, see [Hap88, § 2, ch. 1]. An abelian category \mathcal{C} (e.g. $H\text{-mod}$) is called Frobenius if it has enough injectives and enough projectives, and moreover the class of injectives coincides with that of the projectives. If \mathcal{C} is such a category, we denote by $\underline{\mathcal{C}}$ the stable category associated with it, whose objects are the same as that of \mathcal{C} , and the morphism space between any two objects $X, Y \in \text{Ob}(\mathcal{C})$ are constructed as the quotient

$$\text{Hom}_{\underline{\mathcal{C}}}(X, Y) := \text{Hom}_{\mathcal{C}}(X, Y)/I(X, Y),$$

where $I(X, Y)$ stands for the space of morphisms between X and Y in \mathcal{C} that factor through an injective (equivalently projective) object in \mathcal{C} . Theorem 2.6 of [Hap88, § 2, ch. 1] shows that $\underline{\mathcal{C}}$ is triangulated. The translation endo-functor of $T : \underline{\mathcal{C}} \rightarrow \underline{\mathcal{C}}$ is given as follows.

For any $X \in \text{Ob}(\mathcal{C})$, choose a monomorphism $\lambda_X : X \rightarrow I(X)$ of X into an injective object $I(X)$. We define $T(X) := I(X)/\text{Im}(\lambda_X)$, considered as an object of $\underline{\mathcal{C}}$. It can be checked that the isomorphism class of $T(X)$ in $\underline{\mathcal{C}}$ is independent of choices of $I(X)$, and this leads to a well-defined functor on $\underline{\mathcal{C}}$. Happel also shows that T is an automorphism of $\underline{\mathcal{C}}$ [Hap88, ch. 1, Proposition 2.2], and it is readily checked that its inverse is given as follows: for any $X \in \text{Ob}(\mathcal{C})$, take an epimorphism from a projective object $\mu_X : P(X) \rightarrow X$, then $T^{-1}(X) := \ker(\mu_X)$, regarded as an object in $\underline{\mathcal{C}}$. Finally, every short exact sequence of objects in \mathcal{C} descends to a distinguished triangle in $\underline{\mathcal{C}}$, and conversely any distinguished triangle in $\underline{\mathcal{C}}$ is isomorphic to one that arises in this way [Hap88, ch. 1, Lemma 2.7].

Example 2.2. We give some simple examples of finite-dimensional (graded, super) Hopf algebras and their left integrals.

- Let $H := \mathbb{k}[d]/(d^2)$ be the graded Hopf super algebra of ‘dual numbers’, where $\deg(d) = 1$. Then d is a left and right integral in H .
- Let \mathbb{k} be a field of positive characteristic p . Let $H = \mathbb{k}[\partial]/(\partial^p)$, with $\Delta(\partial) = \partial \otimes 1 + 1 \otimes \partial$, $S(\partial) = -\partial$, and $\epsilon(\partial) = 0$. The algebra H is graded if we fix a degree for ∂ . The space of (left and right) integrals in H is spanned by ∂^{p-1} .
- Let H_n be the Taft algebra (see [Bic03] or [Kho06, §4, Characteristic 0 case]) over the n th cyclotomic field $\mathbb{k} = \mathbb{Q}[\zeta]$, where ζ is a primitive n th root of unity. As a \mathbb{k} -algebra, H_n is generated by K, K^{-1} and d , subject to the relations $K^{-1}K = KK^{-1} = 1$, $K^n = 1$, $Kd = \zeta dK$, and $d^n = 0$. One can readily see that H_n is an n^2 -dimensional Hopf algebra with $\Delta(K) = K \otimes K$, $\Delta(d) = d \otimes 1 + K \otimes d$, $S(K) = K^{-1}$, $S(d) = -K^{-1}d$, $\epsilon(K) = 1$, $\epsilon(d) = 0$. It is easily checked using the commutator relations that a non-zero left integral is given by $\Lambda_l = (\sum_{i=0}^{n-1} K^i)d^{n-1}$, while a non-zero right integral is given by $\Lambda_r = d^{n-1}(\sum_{i=0}^{n-1} K^i)$.

The following lemma is a slight generalization of [Kho06, §1, Proposition 2], which will be needed for technical reasons later.

LEMMA 2.3. *Let M be an arbitrary H -module and N be a projective H -module. Then $M \otimes_{\mathbb{k}} N$, $\text{Hom}_{\mathbb{k}}(M, N)$ and $\text{Hom}_{\mathbb{k}}(N, M)$ are projective as H -modules. The H -module structures are defined in the usual way: for any $h \in H$, $m \in M$, $n \in N$, $f \in \text{Hom}_{\mathbb{k}}(M, N)$,*

$$\begin{aligned}
 h \cdot (m \otimes n) &:= \sum h_{(1)} \cdot m \otimes h_{(2)} \cdot n, \\
 (h \cdot f)(m) &:= \sum h_{(2)} \cdot f(S^{-1}(h_{(1)}) \cdot m).
 \end{aligned}$$

Proof. The case when either one of M or N is finite dimensional follows from [Kho06, Proposition 2]. When both M and N are infinite dimensional, we can write M as a union of its finite-dimensional submodules $M = \bigcup_{i \in I} M_i$ where I is some filtered partially ordered set, with $i \leq j$ in I if and only if $M_i \subset M_j$. In other words, we regard I as a small filtered category in which there is an arrow $i \rightarrow j$ if and only if $M_i \subset M_j$, and then M is the colimit of I . We also write N as a direct sum of finite-dimensional injective and projective modules $N = \bigoplus_{j \in J} P_j$. Now the tensor product is injective since we can write it as

$$M \otimes N \cong M \otimes \left(\bigoplus_{j \in J} P_j \right) \cong \bigoplus_{j \in J} M \otimes P_j$$

which is a direct sum of injectives,² where each term $M \otimes P_j$ is injective by [Kho06, Proposition 2].

Next, $\text{Hom}_{\mathbb{k}}(N, M)$ can be rewritten as

$$\text{Hom}_{\mathbb{k}}\left(\bigoplus_{j \in J} P_j, M\right) \cong \prod_{j \in J} \text{Hom}_{\mathbb{k}}(P_j, M) \cong \prod_{j \in J} P_j^* \otimes M.$$

Each P_j^* is injective since P_j is also finite-dimensional projective, and we are again reduced to the case of [Kho06, Proposition 2].

Finally, for $\text{Hom}_{\mathbb{k}}(M, N)$, we use the short exact sequence of vector spaces

$$0 \longrightarrow \bigoplus_{(i \rightarrow j) \in I} M_i \xrightarrow{\Psi} \bigoplus_{k \in I} M_k \longrightarrow M \longrightarrow 0,$$

where the first direct sum is over all arrows in I , the second direct sum is over all objects of I , and Ψ restricted on each summand M_i labeled by $i \rightarrow j$ is given by composing the map $M_i \rightarrow M_i \oplus M_j; m_i \mapsto (m_i, -m_i)$ with the natural inclusion $M_i \oplus M_j \hookrightarrow \bigoplus_{i \in I} M_i$. Applying $\text{Hom}_{\mathbb{k}}(-, N)$ to this exact sequence, we get a short exact sequence of H -modules:

$$0 \longrightarrow \text{Hom}_{\mathbb{k}}(M, N) \longrightarrow \prod_{k \in I} \text{Hom}_{\mathbb{k}}(M_k, N) \xrightarrow{\Psi^*} \prod_{(i \rightarrow j) \in I} \text{Hom}_{\mathbb{k}}(M_i, N) \longrightarrow 0.$$

Notice that

$$\prod_{k \in I} \text{Hom}_{\mathbb{k}}(M_k, N) \cong \prod_{k \in I} \left(\bigoplus_{j \in J} \text{Hom}_{\mathbb{k}}(M_i, P_j) \right) \cong \prod_{i \in I} \left(\bigoplus_{j \in J} P_j \otimes M_i^* \right),$$

so that it is injective once again by the finite-dimensional case [Kho06, Proposition 2]. Likewise for the last term in the short exact sequence. It follows that the above sequence of H -modules splits, and $\text{Hom}_{\mathbb{k}}(M, N)$ is injective. \square

2.2 Comodule algebras and stable module categories

Now we recall the notion of a (right) comodule algebra over H . We slightly modify the convention used in [Kho06] to better suit the special case of DG-algebras over the base field \mathbb{k} . In particular we will be mainly using the notion of right H -comodule algebras as opposed to left comodule algebras. The proofs of [Kho06] go through almost unchanged with appropriate ‘left’ notions switched to the ‘right’ ones.

DEFINITION 2.4. A right H -comodule algebra B is a unital, associative \mathbb{k} -algebra equipped with a map

$$\Delta_B : B \longrightarrow B \otimes H$$

making B into a right H -comodule and such that Δ_B is a map of algebras. Equivalently, we have the following identities:

$$\begin{aligned} (\text{Id}_B \otimes \epsilon)\Delta_B &= \text{Id}_B, & (\text{Id}_B \otimes \Delta)\Delta_B &= (\Delta_B \otimes \text{Id}_H)\Delta_B, \\ \Delta_B(1) &= 1 \otimes 1, & \Delta_B(ab) &= \Delta_B(a)\Delta_B(b). \end{aligned}$$

Here $B \otimes H$ is equipped with the product algebra structure.

² Any product of injectives over a ring is injective; an infinite direct sum of injectives is injective if and only if the ring is Noetherian [Lam99, Theorem 3.46].

Let V be an H -module, and M be a B -module. The tensor product $M \otimes V$ is naturally a B -module, via Δ_B . The tensor product gives rise to a bifunctor

$$B\text{-mod} \times H\text{-mod} \longrightarrow B\text{-mod}$$

compatible with the monoidal structure of $H\text{-mod}$, and in turn this makes $B\text{-mod}$ into a (right) module-category over $H\text{-mod}$.

DEFINITION 2.5. Let $B_{H\text{-mod}}$ be the quotient category of $B\text{-mod}$ by the ideal of morphisms that factor through a B -module of the form $N \otimes H$, where N is some B -module.

More precisely, we call a morphism of B -modules $f : M_1 \longrightarrow M_2$ *null-homotopic* if there exists a B -module N such that f factors as

$$M_1 \longrightarrow N \otimes H \longrightarrow M_2.$$

The space of null-homotopic morphisms forms an ideal in $B\text{-mod}$. The quotient category $B_{H\text{-mod}}$ by this ideal, by definition, has the same objects as $B\text{-mod}$, while the space of morphisms in $B_{H\text{-mod}}$ between any two objects M_1, M_2 is the quotient of $\text{Hom}_{B\text{-mod}}(M_1, M_2)$ by the subspace of null-homotopic morphisms.

We also recall the following useful lemma, which gives an alternative characterization of the ideal of null-homotopic homomorphism.

LEMMA 2.6. A map $f : M \longrightarrow N$ of B -modules is null-homotopic if and only if it factors through the map $M \xrightarrow{\text{Id}_M \otimes \Lambda} M \otimes H$.

Proof. This is [Kho06, § 1, Lemma 1]. □

As a matter of notation, we will denote the canonical B -module map in the lemma by $\lambda_M : M \xrightarrow{\text{Id}_M \otimes \Lambda} M \otimes H$ for any B -module M , as such maps will appear repeatedly in what follows.

PROPOSITION 2.7. The category $B_{H\text{-mod}}$ is a (right) module-category over $H\text{-mod}$.

Proof. The tensor product $B\text{-mod} \times H\text{-mod} \longrightarrow B\text{-mod}$ descends to a bifunctor

$$B_{H\text{-mod}} \times H\text{-mod} \longrightarrow B_{H\text{-mod}},$$

compatible with the monoidal structure of $H\text{-mod}$. □

We will be mainly interested in the following class of examples. See [Kho06, § 1, Example (g)], or [Mon93, ch. 4].

Example 2.8 (The main example). Let A be a left H -module algebra. This means that A is a left H -module, and the multiplication and unit maps of A are left H -module maps. An excellent treatise for such algebras is [Mon93].

DEFINITION. The *smash product algebra* $B = A \# H$ is the \mathbb{k} -vector space $A \otimes H$ with the multiplication,

$$(a \otimes h)(b \otimes l) = \sum_{(h)} a(h_{(1)} \cdot b) \otimes h_{(2)}l.$$

Here ‘ \cdot ’ denotes the left H action of $h_{(1)}$ on b .

Note that B has the structure of a right H -module algebra by setting $\Delta_B : B \longrightarrow B \otimes H$, $\Delta_B(a \otimes h) := a \otimes \Delta(h)$ for any $a \otimes h \in B$. We will loosely refer to the class of modules over this kind of smash product ring B as *hopfological modules*.

As special cases of this class of examples, if $A = \mathbb{k}$ with the trivial module structure over H , then $A = \mathbb{k}\#H = H$. We recover the usual stable category of H , that is, $B_{H\text{-mod}} = H\text{-mod}$.

Slightly more generally, let A be any \mathbb{k} -algebra with the trivial H -module structure. Then $B = A \otimes H$. We will see later that the usual notion of chain complexes of modules over the algebra B , or their ‘ n -complex’ analogs [Dub96, Kap96], are examples of this particular case. We will deal with a more specific class of examples of this kind in the last section.

3. Triangular structure

Now let us recall the shift functor, the cone construction, and the triangles in $B_{H\text{-mod}}$. See [Kho06, §1]. We refer the reader to [GM03, ch. IV] and [Hap88, ch. I] for more information about triangulated categories.

3.1 The shift functor

The shift (or translation) functor T on $B_{H\text{-mod}}$ is the functor that $B_{H\text{-mod}}$ inherits from T of $H\text{-mod}$, where we regard $B_{H\text{-mod}}$ as a module category over $H\text{-mod}$ (see Proposition 2.7 above). More precisely, we define the following result.

DEFINITION 3.1. For any left B -module M , let $T(M)$ be

$$T(M) := M \otimes (H/(\mathbb{k}\Lambda)).$$

This defines a functor on $B\text{-mod}$ and it descends to be the shift endo-functor on $B_{H\text{-mod}}$.

The above definition is justified thanks to the following.

PROPOSITION 3.2. T is an invertible functor on $B_{H\text{-mod}}$, whose inverse T^{-1} is given by

$$T^{-1}(M) := M \otimes \ker(\epsilon).$$

Proof. The proof is omitted. This proposition is [Kho06, §1, Proposition 3]. □

3.2 Distinguished triangles

For any B -module morphism $u : X \rightarrow Y$, denote by \underline{u} its residue class in the stable category $B_{H\text{-mod}}$ (this and the following \bar{u} notation etc. are taken from [Hap88]).

DEFINITION 3.3. The cone C_u is defined as the pushout of u and λ_X in $B\text{-mod}$, so that it fits into the following Cartesian diagram.

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ \lambda_x \downarrow & & \downarrow v \\ X \otimes H & \xrightarrow{\bar{u}} & C_u \end{array}$$

Now, let $u : X \rightarrow Y$ be a morphism of B -modules. We denote by $\overline{\lambda_X}$ the quotient map from $X \otimes H$ to TX , so that there is the following diagram of short exact sequences in $B\text{-mod}$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{\lambda_x} & X \otimes H & \xrightarrow{\overline{\lambda_x}} & TX & \longrightarrow & 0 \\ \downarrow & & \downarrow u & & \downarrow \bar{u} & & \parallel & & \downarrow \\ 0 & \longrightarrow & Y & \xrightarrow{v} & C_u & \xrightarrow{w} & TX & \longrightarrow & 0 \end{array}$$

DEFINITION 3.4. A *standard distinguished triangle* in $B_H\text{-mod}$ is defined to be the sextuple,

$$X \xrightarrow{u} Y \xrightarrow{v} C_u \xrightarrow{w} TX,$$

associated with some morphism u of B -modules. A sextuple $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$ in $B_H\text{-mod}$ of objects and morphisms in $B_H\text{-mod}$ is called a *distinguished triangle* if it is isomorphic in $B_H\text{-mod}$ to a standard distinguished triangle.

THEOREM 3.5. *The category $B_H\text{-mod}$ is triangulated, with the shift functor T and the class of distinguished triangles defined as above.*

Proof. The proof is omitted. This theorem is [Kho06, § 1, Theorem 1]. □

3.3 Triangulated module category

Recall that an additive functor $F : \mathcal{C} \rightarrow \mathcal{D}$ between triangulated categories is called *exact* if it commutes with the respective shift functors and takes distinguished triangles to distinguished triangles. The lemma below implies that, if V is an H -module, then tensoring a distinguished triangle $X \rightarrow Y \rightarrow Z \rightarrow TX$ with V gives a distinguished triangle in $B_H\text{-mod}$

$$X \otimes V \rightarrow Y \otimes V \rightarrow Z \otimes V \rightarrow T(X \otimes V),$$

so that tensoring with any H -module V is an exact functor on $B_H\text{-mod}$. We say informally that $B_H\text{-mod}$ is a ‘triangulated (right) module-category’ over $H\text{-mod}$.

LEMMA 3.6. *There exists a functorial-in- V isomorphism of H -modules*

$$r : H \otimes V \rightarrow V \otimes H$$

intertwining the H -module inclusions $\Lambda \otimes \text{Id}_V : V \rightarrow H \otimes V$, and $\text{Id}_V \otimes \Lambda : V \rightarrow V \otimes H$.

Proof. The proof is omitted. See [Kho06, § 1, Lemma 2]. We take r to be the inverse of the functorial intertwiner in the lemma there. □

Remark 3.7 (Graded versions). Before proceeding to other hopfological constructions, we remark here that all of our constructions above apply without much change to finite-dimensional graded Hopf algebras, finite-dimensional graded Hopf super algebras, or more generally, any finite-dimensional Hopf-algebra object in a symmetric monoidal category which admits integrals (see [Kup96, § 3] where a diagrammatic construction of integrals in these cases are exhibited). A good example to keep in mind is when $H = \mathbb{k}[d]/(d^2)$ is the \mathbb{Z} -graded Hopf super algebra where $\text{deg}(d) = 1$. As we will see, a \mathbb{Z} -graded algebra A being an H -module algebra means that A is a differential graded (DG) algebra over the field \mathbb{k} , as defined in [BL94, § 10]. The categories $A\#H\text{-mod}$, $\mathcal{C}(A, H)$, and $\mathcal{D}(A, H)$ correspond respectively to the abelian category of complexes of DG modules over A , the homotopy category of complexes of DG modules over A , and the derived category of DG modules over A , with the latter two being triangulated. The morphism spaces in these cases are slightly different: as we will see later, the morphism spaces are given by the usual $\mathbf{R}\text{Hom}$ of complexes in $\mathcal{C}(A, H)$ and $\mathcal{D}(A, H)$, at least between ‘nice’ complexes. See the first example of [Kho06, § 2] for more details.

3.4 Examples

We now describe the objects of $B_H\text{-mod}$ more explicitly for some particular smash product algebras $B = A\#H$ (see the main example, Example 2.8). We will see that modules over these algebras are natural generalizations of DG modules in hopfological algebra.

- Let $H = \mathbb{k}[d]/(d^2)$ be the graded Hopf super algebra over \mathbb{k} , where $\deg(d) = 1$. One can easily check that the notion of H -module algebra coincides with that of the usual DG algebra. Modules over the smash product algebra are none other than the usual DG modules.
- Let $H = H_n$ be the Taft algebra over $\mathbb{Q}[\zeta]$ (see Example 2.2), and let A be an H_n -module algebra. Since K generate a subalgebra of H_n isomorphic to the group algebra of $\mathbb{Z}/n\mathbb{Z}$, A must be $\mathbb{Z}/n\mathbb{Z}$ -graded and the multiplication on A must respect this grading. For any homogeneous element $a \in A$ of degree $|a|$, K acts on a by $K \cdot a = \zeta^{|a|}a$. Furthermore, the relation $Kd = \zeta dK$ applied to a gives us $Kd(a) = \zeta^{|a|+1}d(a)$, i.e., $d(a)$ is homogeneous of degree $|a| + 1$. Equivalently, d has to increase the degree by one. The comultiplication equation on d , $\Delta(d) = d \otimes 1 + K \otimes d$, when applied to any product of homogeneous elements $a_1, a_2 \in A$, imposes the differential condition that $d(a_1a_2) = d(a_1)a_2 + \zeta^{|a_1|}a_1d(a_2)$. Finally, $d^n = 0$ just says that $d^n(a) = 0$ for all $a \in A$. Thus we conclude that an H_n -module algebra is just a $\mathbb{Z}/n\mathbb{Z}$ -graded algebra equipped with a degree-one differential such that

$$d(a_1a_2) = d(a_1)a_2 + \zeta^{|a_1|}a_1d(a_2), \quad d^n(a) = 0.$$

Following [Bic03, Dub96, Kap96, KW98], we say that A is an n -differential graded (n -DG) algebra over $\mathbb{Q}[\zeta]$. Notice that A could have a \mathbb{Z} -grading since any such grading collapses into a $\mathbb{Z}/n\mathbb{Z}$ -grading. Similarly to the the DG case, an $A\#H_n$ -module is equivalent to a $\mathbb{Z}/n\mathbb{Z}$ -graded A -module, equipped with a degree-one differential d , such that, for any homogeneous $a \in A$, $m \in M$,

$$d(am) = d(a)m + \zeta^{|a|}ad(m), \quad d^n(m) = 0.$$

Likewise, we will call such a module an n -DG module.

- Let \mathbb{k} be a field of positive characteristic p , and $H = \mathbb{k}[\partial]/(\partial^p)$. This case is entirely analogous to the above n -DG algebra case, and we just state the results. An H -module algebra A comes with differential ∂ such that for all $a, a_1, a_2 \in A$,

$$\partial(a_1a_2) = \partial(a_1)a_2 + a_1\partial(a_2), \quad \partial^p(a) = 0.$$

Notice the lack of minus signs before a_1 on the right-hand side of the first equation. Similarly, an $A\#H$ -module M is an A -module equipped with a differential ∂ on it compatible with the A -module differential, i.e., for all $a \in A$, $m \in M$,

$$\partial(am) = \partial(a)m + a\partial(m) \quad \partial^p(m) = 0.$$

Algebras and modules of this kind will be refereed to as p -DG algebras and p -DG modules. We can also require some compatible grading on ∂ , A and M , but the formulas remain unchanged. We leave the details to the reader.

4. Derived categories

From now on, we will focus on the case of the main example above, Example 2.8, where derived categories can be defined.

4.1 Quasi-isomorphisms

Suppose $B = A\#H$ is the smash product of H and a left H -module algebra A . Since $H \cong \mathbb{k} \otimes H$ is a subalgebra of B , we have the restriction functor from B -mod to H -mod,

$$\text{Res} : B\text{-mod} \longrightarrow H\text{-mod}.$$

This descends to an exact functor on the quotient categories

$$\underline{\text{Res}} : B_H\text{-mod} \longrightarrow H\text{-mod}.$$

In what follows, we will introduce a new notation for the triangulated category $B_H\text{-mod}$ for the special case of the main example, Example 2.8:

$$\mathcal{C}(A, H) := B_H\text{-mod}.$$

The notation stands informally for ‘the category of chain complexes of A -modules up to homotopy’. The reason for using this term will be clear once we understand the Hom spaces better in the next section.

DEFINITION 4.1. (i) We define the *total cohomology functor* to be the restriction functor,

$$\underline{\text{Res}} : \mathcal{C}(A, H)\text{-mod} \longrightarrow H\text{-mod}.$$

(ii) A morphism $f : M \longrightarrow N$ in $\mathcal{C}(A, H)$ is called a *quasi-isomorphism* if its restriction $\underline{\text{Res}}(f)$ is an isomorphism in $H\text{-mod}$.

(iii) A B -module M is called *acyclic* if $0 \longrightarrow M$ is a quasi-isomorphism.

THEOREM 4.2. (1) *Quasi-isomorphisms in $\mathcal{C}(A, H)$ constitute a localizing class.*

(2) *The localization of $\mathcal{C}(A, H)$ with respect to the quasi-isomorphisms, denoted $\mathcal{D}(A, H)$, is triangulated. Tensoring with any H -module (on the right) is an exact functor in $\mathcal{D}(A, H)$.*

We will call $\mathcal{D}(A, H)$ the derived category of $B\text{-mod}$.

Proof. The proof is omitted. See [Kho06, § 1, Proposition 4 and Corollary 2]. □

4.2 Constructing distinguished triangles

Now we describe how short exact sequences in the abelian category $B\text{-mod}$ lead to distinguished triangles in $\mathcal{C}(A, H)$ and $\mathcal{D}(A, H)$. We start with the construction in $\mathcal{C}(A, H)$.

LEMMA 4.3. *Let*

$$0 \longrightarrow X \xrightarrow{u} Y \xrightarrow{v} Z \longrightarrow 0$$

be a short exact sequence in $B\text{-mod}$, which is split exact as a sequence of A -modules. Then associated to it there is a distinguished triangle in $\mathcal{C}(A, H)$:

$$X \xrightarrow{u} Y \xrightarrow{v} Z \longrightarrow TX$$

(the connecting homomorphism on the third arrow is described in the proof below). Conversely, any distinguished triangle in $\mathcal{C}(A, H)$ is isomorphic to one that arises in this way.

Proof. The last part of the lemma holds by construction, since $\lambda_X : X \longrightarrow X \otimes H$ is always a split injection of A -modules. We prove the first half.

According to Definition 3.4, the map $u : X \rightarrow Y$ gives rise to a commutative diagram in $B\text{-mod}$.

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & X & \xrightarrow{\lambda_X} & X \otimes H & \longrightarrow & TX \longrightarrow 0 \\
 & & \downarrow u & & \downarrow \bar{u} & & \parallel \\
 0 & \longrightarrow & Y & \longrightarrow & C_u & \longrightarrow & TX \longrightarrow 0 \\
 & & \downarrow v & & \downarrow \bar{v} & & \\
 & & Z & \xlongequal{\quad} & Z & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array} \tag{*}$$

Therefore the cone C_u fits into a short exact sequence of B -modules,

$$0 \longrightarrow X \otimes H \xrightarrow{\bar{u}} C_u \xrightarrow{\bar{v}} Z \longrightarrow 0,$$

which is split exact as a sequence of A -modules. Thus, we will be done with the first half of the lemma once we establish it in the following special case: in the short exact sequence as above, \bar{v} becomes an isomorphism in $\mathcal{C}(A, H)$. The connecting homomorphism is then taken to be the composition of the inverse of \bar{v} and $C_u \rightarrow TX$.

To prove the last claim, consider the cone of \bar{v} , which fits into the following commutative diagram.

$$\begin{array}{ccccccc}
 0 & \longrightarrow & X \otimes H & \xrightarrow{\bar{u}} & C_u & \xrightarrow{\bar{v}} & Z \longrightarrow 0 \\
 \downarrow & & \parallel & & \downarrow \lambda_{C_u} & & \downarrow \\
 0 & \longrightarrow & X \otimes H & \xrightarrow{u'} & C_u \otimes H & \xrightarrow{v'} & C_{\bar{v}} \longrightarrow 0
 \end{array} \tag{**}$$

By assumption, the top short exact sequence splits in $A\text{-mod}$, and so does the bottom one since the third square in $(**)$ is a pushout. We will show that $C_{\bar{v}} \cong 0$ in $\mathcal{C}(A, H)$, and the special case will follow, since, by construction,

$$C_u \xrightarrow{\bar{v}} Z \longrightarrow C_{\bar{v}} \longrightarrow T(C_u),$$

is a distinguished triangle in $\mathcal{C}(A, H)$.

Now we examine the B -module structure of $C_u \otimes H$. By tensoring the top short exact sequence with H in the above diagram, we obtain the exact sequence

$$0 \longrightarrow X \otimes H \otimes H \longrightarrow C_u \otimes H \longrightarrow Z \otimes H \longrightarrow 0,$$

which is A -split. By commutativity of the second square in $(**)$, $u' : X \otimes H \rightarrow C_u \otimes H$ factors through

$$u' : X \otimes H \xrightarrow{\lambda_{X \otimes H}} X \otimes H \otimes H \longrightarrow C_u \otimes H.$$

Now notice that the map $H \rightarrow H \otimes H$ which sends $h \mapsto h \otimes \Lambda$ is an H -module injection, whose quotient $H \otimes (H/\mathbb{k}\Lambda) \cong H^{(\dim(H)-1)}$ is an injective and projective summand in $H \otimes H$ which we write as H' . This is true since H is self-injective (see [Kho06, Lemma 1] for an explicit splitting). Modding out the submodule $X \otimes H \otimes \Lambda$ in $C_u \otimes H$, resulting in no other than $C_{\bar{v}}$, we get a short

exact sequence of B -modules,

$$0 \longrightarrow X \otimes H' \xrightarrow{\alpha} C_{\bar{v}} \xrightarrow{\beta} Z \otimes H \longrightarrow 0,$$

which is also A -split. The next lemma then shows that

$$C_{\bar{v}} \cong X \otimes H' \oplus Z \otimes H,$$

and the result follows. □

LEMMA 4.4. *Let $\beta : C \longrightarrow Z \otimes H$ be a surjective map of B -modules which admits a section in A -mod. Then $Z \otimes H$ is a direct summand of C in B -mod.*

Proof. Let $\gamma' : Z \otimes H \longrightarrow C$ be a section of β as a map of A -modules, so that $\beta \circ \gamma' = \text{Id}_{Z \otimes H}$. Define

$$\gamma : Z \otimes H \longrightarrow C, \quad z \otimes h \mapsto h_{(2)}\gamma'(S^{-1}(h_{(1)})z \otimes 1).$$

Then we claim that γ is a section of β in B -mod.

To prove the claim, we first show that γ is A -linear. For any $a \in A$ and $z \otimes h \in Z \otimes H$, we have

$$\begin{aligned} \gamma(az \otimes h) &= h_{(2)}\gamma'(S^{-1}(h_{(1)})(az) \otimes 1) = h_{(3)}\gamma'((S^{-1}(h_{(2)})a)(S^{-1}(h_{(1)})z) \otimes 1) \\ &= h_{(3)}((S^{-1}(h_{(2)})a)\gamma'((S^{-1}(h_{(1)})c) \otimes 1)) \\ &= h_{(3)}(S^{-1}(h_{(2)})a)h_{(4)}(\gamma'((S^{-1}(h_{(1)})z) \otimes 1)) \\ &= (\epsilon(h_{(2)})a)h_{(3)}(\gamma'((S^{-1}(h_{(1)})z) \otimes 1)) = ah_{(2)}\gamma'(S^{-1}(h_{(1)})z \otimes 1) = a\gamma(z \otimes h), \end{aligned}$$

with third equality holding because γ' is A -linear.

Then we show that it is H -linear as well. If $l \in H$, $z \otimes h \in Z \otimes H$, then

$$\begin{aligned} \gamma(l(z \otimes h)) &= \gamma(l_{(1)}z \otimes l_{(2)}h) = l_{(3)}h_{(2)}\gamma'(S^{-1}(l_{(2)}h_{(1)})(l_{(1)}z) \otimes 1) \\ &= l_{(3)}h_{(2)}\gamma'(S^{-1}(h_{(1)})S^{-1}(l_{(2)})l_{(1)}z \otimes 1) = l_{(2)}h_{(2)}\gamma'(S^{-1}(h_{(1)})\epsilon(l_{(1)})z \otimes 1) \\ &= lh_{(2)}\gamma'(S^{-1}(h_{(1)})z \otimes 1) = l\gamma(z \otimes h). \end{aligned}$$

Finally, we show that γ is a B -module section of β . Taking $z \otimes h \in Z \otimes H$, we have

$$\begin{aligned} \beta(\gamma(z \otimes h)) &= \beta(h_{(2)}\gamma'(S^{-1}(h_{(1)})z \otimes 1)) = h_{(2)}\beta\gamma'(S^{-1}(h_{(1)})z \otimes 1) = h_{(2)}(S^{-1}(h_{(1)})z \otimes 1) \\ &= h_{(2)}S^{-1}h_{(1)}z \otimes h_{(3)} = \epsilon(h_{(1)})z \otimes h_{(2)} = z \otimes h, \end{aligned}$$

where, in the third equality, we have used the fact that β is H -linear. The claim follows. □

Following Happel [Hap88, § 2.7], we describe the class of distinguished triangles in the derived category $\mathcal{D}(A, H)$.

After localization, any short exact sequence of B -modules, not necessarily A -split, will lead to a distinguished triangle in $\mathcal{D}(A, H)$, as below. Let

$$0 \longrightarrow X \xrightarrow{u} Y \xrightarrow{v} Z \longrightarrow 0$$

be a short exact sequence of B -modules. Then, similarly to the the proof of Lemma 4.3, there is a distinguished triangle in $\mathcal{C}(A, H)$,

$$X \longrightarrow Y \longrightarrow C_u \longrightarrow T(X),$$

coming from the diagram (\star) , and C_u fits into a short exact sequence of B -modules

$$0 \longrightarrow X \otimes H \longrightarrow C_u \longrightarrow Z \longrightarrow 0.$$

Proposition 2.1 shows that $X \otimes H$, as an H -module, is projective and injective. It follows that $\bar{v} : C_u \rightarrow Z$ is a quasi-isomorphism so that it is invertible in the derived category. Therefore we obtain a distinguished triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X),$$

where w is taken to be the composition of $(\bar{v})^{-1}$ by $C_u \rightarrow TX$.

LEMMA 4.5. *In the same notation as in the above discussion, given any short exact sequence of B -modules $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$,*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$$

is a distinguished triangle in $\mathcal{D}(A, H)$. Conversely, any distinguished triangle in $\mathcal{D}(A, H)$ is isomorphic to one that arises in this way.

Remark 4.6. An alternative proof of Lemma 4.5 can be obtained by combining Lemma 4.3 with the ‘bar-resolution’ Theorem 6.6 of § 6. In particular it will tell us that all distinguished triangles in $\mathcal{D}(A, H)$ are isomorphic to the ones arising as the image of a split short exact sequence of A -modules.

4.3 Examples

As an immediate application of the above construction, we calculate the Grothendieck rings of the stable categories $H\text{-mod}$ ($H\text{-gmod}$) where H is among the examples we gave in Example 2.2. Note that in our notation, $H\text{-mod} \cong \mathcal{C}(\mathbb{k}, H) \cong \mathcal{D}(\mathbb{k}, H)$. Recall that $K_0(H\text{-mod})$ ($K_0(H\text{-gmod})$) is the abelian group generated by the symbols $[X]$, where the $[X]$ are isomorphism classes of finite-dimensional objects in $H\text{-mod}$ ($H\text{-gmod}$), modulo the relations $[Y] = [X] + [Z]$ whenever $X \rightarrow Y \rightarrow Z \rightarrow T(X)$ is a distinguished triangle in $H\text{-mod}$ ($H\text{-gmod}$). More general discussion about the Grothendieck groups of $\mathcal{D}(A, H)$ will be given in § 7.2.

As a matter of notation, for any graded module X over some graded ring, we will denote by $X\{r\}$ the same underlying module but with its grading shifted up by r .

- When H is the graded Hopf super algebra of dual numbers, the Grothendieck ring of its stable category, or rather, its homotopy category, is isomorphic to the ring of quantum integers $\mathbb{Z}[q, q^{-1}]$.

- Consider the graded Hopf algebra $H = \mathbb{k}[\partial]/(\partial^p)$, where \mathbb{k} is of positive characteristic p . As shown in [Kho06, § 3], $K_0(H\text{-gmod})$ is again generated by the graded simple one-dimensional module $\mathbb{k}_0 := H/(\partial)$, subject to the only relation

$$0 = [H] = [\mathbb{k}_0] + q[\mathbb{k}_0] + \cdots + q^{p-1}[\mathbb{k}_0].$$

Therefore we have the Grothendieck group $K_0(H\text{-gmod}) \cong \mathbb{Z}[q, q^{-1}]/(1 + q + \cdots + q^{p-1}) \cong \mathbb{Z}[\zeta]$, the ring of p th cyclotomic integers (ζ , being the image of q , is a primitive p th root of unity). If we forget about the grading, the same reasoning as above gives us $K_0(H\text{-mod}) \cong \mathbb{Z}/p\mathbb{Z}$, the field of p elements. It was this observation that lead Khovanov to initiate the program of categorification at certain roots of unity. See [Kho06] for more details about the motivation.

- Let $H = H_n$ be the Taft algebra as in Example 2.2. Inverting Majid’s bosonization process [Maj97], one can identify the category of H_n -modules with the category whose objects are $\mathbb{Z}/n\mathbb{Z}$ -graded $\mathbb{Q}[\zeta]$ -vector spaces $\bigoplus_{i=0}^{n-1} V_i$, together with a map $d : V_i \rightarrow V_{i+1}$ of degree one such that $d^n = 0$, and morphisms are homogenous degree-zero maps of graded vector spaces commuting with d . Under this identification, it is readily seen that the indecomposable projective

modules are precisely the shifts of the module

$$P_0 := (\mathbb{Q}[\zeta] \xrightarrow{\cdot 1} \mathbb{Q}[\zeta] \xrightarrow{\cdot 1} \dots \xrightarrow{\cdot 1} \mathbb{Q}[\zeta]),$$

where there are n terms of \mathbb{Q} and the starting term sits in degree zero. The simple modules are the grading shifts of the one-dimensional module $\mathbb{Q}[\zeta]_0 := \mathbb{Q}[\zeta]$, with d acting as zero. Using the same argument as above, we see that $K_0(H_n\text{-mod})$ is generated as an $\mathbb{Z}[q, q^{-1}]/(q^n - 1)$ -module by $[\mathbb{Q}[\zeta]_0]$ subject to the only relation

$$0 = [P_0] = [\mathbb{Q}[\zeta]_0] + q[\mathbb{Q}[\zeta]_0] + \dots + q^{n-1}[\mathbb{Q}[\zeta]_0],$$

and thus $K_0(H_n\text{-mod}) \cong \mathbb{Z}[q]/(1 + q + \dots + q^{n-1})$. In particular, when $n = p$, this gives rise to a characteristic zero categorification of the rings of the p th cyclotomic integers.

5. Morphism spaces

In this section we further analyze the Hom spaces introduced previously for the categories $B\text{-mod}$ and $\mathcal{C}(A, H)$. We will see that they are in fact the spaces of H -invariants of some naturally enriched Hom spaces that we will introduce in this section.

5.1 The Hopf module Hom

As before, we assume that H is a finite-dimensional (graded) Hopf algebra over \mathbb{k} , or, more generally, a finite-dimensional Hopf-algebra object in some \mathbb{k} -linear symmetric monoidal category (for an example of such an object, take a graded super Hopf algebra in the category of graded super vector spaces). Throughout we will continue with the assumption that A is a left H -module algebra and the notation $B = A\#H$ (see the main example, Example 2.8).

DEFINITION 5.1. Let M, N be B -modules. The vector space $\text{Hom}_A(M, N)$ becomes an H -module by defining, for any $f \in \text{Hom}_A(M, N)$, $m \in M$ and $h \in H$,

$$(h \cdot f)(m) := \sum h_{(2)} f(S^{-1}(h_{(1)})m).$$

When A and H are \mathbb{Z} -graded and M, N are graded modules, we define the enriched HOM_A space to be

$$\text{HOM}_A(M, N) = \bigoplus_{r \in \mathbb{Z}} \text{Hom}_A(M, N\{r\}),$$

where $N\{r\}$ denotes the same underlying A -module N with grading shifted up by r , and the Hom space on the right-hand side stands for the space of degree preserving maps of graded A -modules. The graded H -module structure on $\text{HOM}_A(M, N)$ is given by the same formula as in the ungraded case above.

It is readily seen that when $M = A$, we have $\text{Hom}_A(A, N) \cong N$, and, in the graded case, $\text{HOM}_A(A, N) \cong N$, both as (graded) H -modules.

5.2 The space of chain maps

The newly defined H -module $\text{Hom}_A(M, N)$ (respectively graded H -module $\text{HOM}_A(M, N)$ in the graded case) for any hopfological modules M, N is closely related to the Hom spaces in the abelian category $B\text{-mod}$ and the homotopy category $\mathcal{C}(A, H)$. We clarify this relation in this section. We will mostly consider the ungraded case, as the graded case follows by similar arguments.

To avoid potential confusion, we will denote the abstract one-dimensional trivial H -module by \mathbb{k}_0 , i.e., $\mathbb{k}_0 \cong \mathbb{k} \cdot v_0$, where for any $h \in H$

$$h \cdot v_0 = \epsilon(h)v_0.$$

When H is graded, we let v_0 be homogeneous of degree zero.

LEMMA 5.2. *Let M, N be hopfological modules over B . Any $f \in \text{Hom}_B(M, N)$, regarded as an element in $\text{Hom}_A(M, N)$, spans a trivial submodule of H , i.e., for all $h \in H$,*

$$h \cdot f = \epsilon(h)f.$$

Conversely, any $f \in \text{Hom}_A(M, N)$ on which H acts trivially extends to a B -module homomorphism. In other words, we have a canonical isomorphism of \mathbb{k} -vector spaces:

$$\text{Hom}_B(M, N) = \text{Hom}_H(\mathbb{k}_0, \text{Hom}_A(M, N)).$$

Proof. Since B contains H as a subalgebra, f is H -linear. Therefore, for any $h \in H, m \in M$, we have

$$(h \cdot f)(m) = h_{(2)}f(S^{-1}(h_{(1)}) \cdot m) = h_{(2)}S^{-1}(h_{(1)})f(m) = \epsilon(h)f(m).$$

For the converse, it suffices to see that f is H -linear:

$$\begin{aligned} f(h \cdot m) &= \epsilon(h_{(2)})f(h_{(1)} \cdot m) = (h_{(2)} \cdot f)(h_{(1)} \cdot m) = h_{(3)} \cdot f(S^{-1}(h_{(2)}) \cdot h_{(1)} \cdot m) \\ &= h_{(2)} \cdot f(\epsilon(h_{(1)})m) = h \cdot f(m). \end{aligned}$$

This finishes the proof of the first part of the lemma. The last claim is clear. □

The right-hand side of the canonical identification in the lemma involves taking H -invariants, of which we now recall the definition.

DEFINITION 5.3. For any H -module V , its *space of H -invariants*, denoted $\mathcal{Z}(V)$, is defined to be the \mathbb{k} -vector space (in fact an H -submodule),

$$\mathcal{Z}(V) := \text{Hom}_H(\mathbb{k}_0, V) \cong \{v \in V \mid h \cdot v = \epsilon(h)v, \forall h \in H\} \cong V^H.$$

Likewise, when H and V are graded, we define the *total space of homogeneous H -invariants* $\mathcal{Z}^*(V)$ to be the graded \mathbb{k} -vector space

$$\mathcal{Z}^*(V) := \text{HOM}_{H\text{-gmod}}(\mathbb{k}_0, V) \cong V^H.$$

Moreover, in the graded case, the *subspace of homogeneous degree n invariants* is defined to be the homogeneous degree n part of $\mathcal{Z}^*(V)$,

$$\mathcal{Z}^n(V) := \{v \in V \mid \text{deg}(v) = n, h \cdot v = \epsilon(h)v, \forall h \in H\},$$

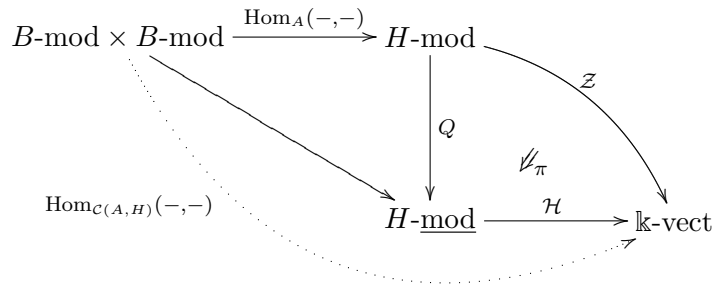
so that $\mathcal{Z}^*(V) = \bigoplus_{n \in \mathbb{Z}} \mathcal{Z}^n(V)$.

The previous lemma allows us to interpret the subspace of H -invariants in $\text{Hom}_A(M, N)$ as an analogue of the space of chain maps between DG-modules in the hopfological context. From now on, we will refer to $\mathcal{Z}(\text{Hom}_A(M, N)) = \text{Hom}_B(M, N)$ as the *space of chain maps* between the two hopfological modules M and N .

This immediately raises the related question: what is the analogue of the *space of chain maps up to homotopy*?

5.3 The space of chain maps up to homotopy

Our main goal in this section is to exhibit and explain the following commutative diagram.



Here Q is the natural localization (Verdier quotient) functor, the slanted arrow on the left is the composition of Q with $\text{Hom}_A(-, -)$, and \mathcal{H} is the functor of taking ‘stable invariants’ (see Definition 5.6). We put π on a double arrow to indicate that it is a natural transformation of the two functors $\pi : \mathcal{Z} \Rightarrow \mathcal{H} \circ Q$. In the usual DG case, π plays the role of passing from the space of cocycles to cohomology, and its definition will be given after the next lemma. The composition of \mathcal{Z} with $\text{Hom}_A(-, -)$ gives the bifunctor $\text{Hom}_B(-, -)$, while the functor $\mathcal{H} \circ Q \circ \text{Hom}_A(-, -)$ (we will omit Q when no confusion can arise) is just the previously defined $\text{Hom}_{\mathcal{C}(A, H)}(-, -)$ of the homotopy category, which is labeled as the dotted arrow. Therefore, we can roughly summarize the diagram as saying that, the functor \mathcal{Z} of taking H -invariants descends to a functor \mathcal{H} on the stable category $H\text{-mod}$ (this explains the terminology we use for \mathcal{H}), and the space of stable invariants $\mathcal{H}(\text{Hom}_A(M, N))$ computes the ‘chain maps up to homotopy’, which turns out to be the same as the hom space from M to N in the homotopy category $\mathcal{C}(A, H)$.

To do this, we first need to take a closer look at the ideal of null-homotopic morphisms in $B\text{-mod}$.

LEMMA 5.4. Under the canonical isomorphism of Lemma 5.2, for any two hopfological modules M and N , the space $I(M, N)$ of null-homotopic morphisms in $\text{Hom}_B(M, N)$ is naturally identified with

$$I(M, N) \cong \Lambda \cdot \text{Hom}_A(M, N),$$

where the right-hand side is regarded as a \mathbb{k} -subspace of $\mathcal{Z}(\text{Hom}_A(M, N))$. A similar result holds in the graded case as well.

Proof. That $\Lambda \cdot \text{Hom}_A(M, N)$ is contained in $\mathcal{Z}(\text{Hom}_A(M, N))$ follows easily from the left integral property $h \cdot \Lambda = \epsilon(h)\Lambda$. We need to show that, if $f \in \mathcal{Z}(\text{Hom}_A(M, N)) \cong \text{Hom}_B(M, N)$ satisfies $f = \Lambda \cdot g$ for some $g \in \text{Hom}_A(M, N)$, then f is null-homotopic as a B -module map, i.e., it factors through as $f : M \xrightarrow{\lambda_M} M \otimes H \xrightarrow{\tilde{g}} N$ for some B -module map \tilde{g} , and vice versa. To do this, we extend g to be a B -module map $\tilde{g} : M \otimes H \rightarrow N$, by setting $\tilde{g}(m \otimes h) := (h \cdot g)(m)$. This map \tilde{g} is H -linear since for any $h, l \in H$ and $m \in M$

$$\begin{aligned} \tilde{g}(h \cdot (m \otimes l)) &= \tilde{g}(h_{(1)} \cdot m \otimes h_{(2)}l) = (h_{(2)}l \cdot g)(h_{(1)} \cdot m) \\ &= h_{(3)}l_{(2)}g(S^{-1}(h_{(2)}l_{(1)}) \cdot h_{(1)} \cdot m) = h_{(3)}l_{(2)}g(S^{-1}(l_{(1)})S^{-1}(h_{(2)})h_{(1)} \cdot m) \\ &= h_{(2)}l_{(2)}g(\epsilon(h_{(1)})S(l_{(1)}) \cdot m) = h(l_{(2)}g(S^{-1}(l_{(1)}) \cdot m)) \\ &= h((l \cdot g)(m)) = h\tilde{g}(m \otimes l). \end{aligned}$$

Conversely, given an $f \in \text{Hom}_B(M, N) = \text{Hom}_A(M, N)^H$ which is null-homotopic, we need to exhibit a $g \in \text{Hom}_A(M, N)$ so that $f = \Lambda \cdot g$. The hint is to reverse the above equalities and define g to be the composition

$$g : M \cong M \otimes 1 \hookrightarrow M \otimes H \xrightarrow{\tilde{g}} N.$$

This is only an A -module map, since the first identification is only an A -linear. Then, for any $h \in H, m \in M$, we have

$$\begin{aligned} \tilde{g}(m \otimes h) &= \tilde{g}(\epsilon(h_{(1)})m \otimes h_{(2)}) = \tilde{g}(h_{(2)}S^{-1}(h_{(1)}) \cdot m \otimes h_{(3)}) \\ &= \tilde{g}(h_{(2)} \cdot (S^{-1}(h_{(1)}) \cdot m \otimes 1)) = h_{(2)}\tilde{g}(S^{-1}(h_{(1)}) \cdot m \otimes 1) \\ &= h_{(2)}g(S^{-1}(h_{(1)}) \cdot m) = (h \cdot g)(m), \end{aligned}$$

where the fourth equality uses that \tilde{g} is H -linear by assumption, and the fifth equality holds by definition of g . Now the lemma follows since $f(m) = \tilde{g}(\lambda_M(m)) = \tilde{g}(m \otimes \Lambda) = (\Lambda \cdot g)(m)$. \square

In particular, when $A = \mathbb{k}$, we obtain an explicit way of computing morphism spaces in the category $\mathcal{C}(\mathbb{k}, H) = H\text{-mod}$.

COROLLARY 5.5. *Let H be a finite-dimensional Hopf algebra over \mathbb{k} . The morphism space of two H -modules M, N in the stable category $H\text{-mod}$ is canonically isomorphic to the quotient space $(\text{Hom}_{\mathbb{k}}(M, N))^H / (\Lambda \cdot \text{Hom}_{\mathbb{k}}(M, N))$. In other words, we have a bifunctorial isomorphism:*

$$\text{Hom}_{H\text{-mod}}(M, N) \cong \mathcal{Z}(\text{Hom}_{\mathbb{k}}(M, N)) / (\Lambda \cdot \text{Hom}_{\mathbb{k}}(M, N)).$$

Likewise, in the graded case,

$$\text{HOM}_{H\text{-gmod}}(M, N) \cong \mathcal{Z}^*(\text{HOM}_{\mathbb{k}}(M, N)) / (\Lambda \cdot \text{HOM}_{\mathbb{k}}(M, N)).$$

The right-hand side of the above isomorphism is defined for any H -module V in place of $\text{Hom}_A(M, N)$, which we formalize in the following definition.

DEFINITION 5.6. For any H -module V we define its *space of stable invariants* to be the \mathbb{k} -vector space

$$\mathcal{H}(V) := \mathcal{Z}(V) / (\Lambda \cdot V) \cong V^H / (\Lambda \cdot V).$$

Note that $\mathcal{H} : H\text{-mod} \rightarrow \mathbb{k}\text{-vect}$ is a functor. Likewise, in the graded case, we define the *total space of graded stable invariants* to be

$$\mathcal{H}^*(V) := \mathcal{Z}^*(V) / (\Lambda \cdot V) \cong V^H / (\Lambda \cdot V),$$

while the *space of degree n stable invariants*, denoted $\mathcal{H}^n(V)$, is defined to be the homogeneous degree n part of $\mathcal{H}^*(V)$, for any $n \in \mathbb{Z}$.

COROLLARY 5.7. *The functor $\mathcal{H} : H\text{-mod} \rightarrow \mathbb{k}\text{-vect}$ descends to a cohomological functor*

$$\mathcal{H} : H\text{-mod} \rightarrow \mathbb{k}\text{-vect}.$$

Here, by cohomological we mean that \mathcal{H} takes distinguished triangles in $H\text{-mod}$ into long exact sequences of \mathbb{k} -vector spaces. Likewise, in the graded case,

$$\begin{aligned} \mathcal{H}^* : H\text{-gmod} &\rightarrow \mathbb{k}\text{-gvect}, \\ \mathcal{H}^n : H\text{-gmod} &\rightarrow \mathbb{k}\text{-vect} \end{aligned}$$

are cohomological functors as well.

Proof. Taking M to be the trivial module \mathbb{k}_0 in Corollary 5.5, we obtain

$$\mathcal{H}(N) \cong \text{Hom}_{H\text{-mod}}(\mathbb{k}_0, N).$$

Thus \mathcal{H} descends to the stable category, and it takes distinguished triangles into long exact sequences. The graded case follows similarly. \square

Remark 5.8. One possible confusion about the definition of $\mathcal{H}(\text{Hom}_A(M, N))$ is that, although this space plays the role analogous as the total space of chain maps up to homotopy of all different degrees in the DG case, the latter in turn being the total cohomology group of the usual $\mathbf{R}\text{Hom}$ complex, it is in general different from the total cohomology we defined earlier using the (‘stabilized’) restriction functor $\underline{\text{Res}} : \mathcal{C}(A, H) \rightarrow H\text{-gmod}$ for an arbitrary H . In fact by Corollary 5.7, \mathcal{H} is cohomological, and we lose information if we forget about its derived terms. We will return to this point later when discussing derived functors.

We summarize the previous results of this subsection in the next proposition, which is just a reformulation of the commutative diagram we exhibited at the beginning of this subsection.

PROPOSITION 5.9. *Let H be a finite-dimensional Hopf algebra over \mathbb{k} and A be a left H -module algebra. There are identifications of bifunctors,*

$$\begin{aligned} \mathcal{Z}(\text{Hom}_A(-, -)) &\cong \text{Hom}_B(-, -) : B\text{-mod} \times B\text{-mod} \rightarrow \mathbb{k}\text{-vect}, \\ \mathcal{H}(\text{Hom}_A(-, -)) &\cong \text{Hom}_{\mathcal{C}(A, H)}(-, -) : B\text{-mod} \times B\text{-mod} \rightarrow \mathbb{k}\text{-vect}, \end{aligned}$$

i.e., for any hopfological modules M, N over $B = A\#H$, there are isomorphism of \mathbb{k} -vector spaces

$$\begin{aligned} \mathcal{Z}(\text{Hom}_A(M, N)) &\cong \text{Hom}_B(M, N), \\ \mathcal{H}(\text{Hom}_A(M, N)) &\cong \text{Hom}_{\mathcal{C}(A, H)}(M, N), \end{aligned}$$

bifunctorial in M and N .

The identifications in the proposition above also show that taking \mathcal{Z} or \mathcal{H} commutes with direct sums of Hom spaces. The following corollary will be needed later when dealing with compact objects.

COROLLARY 5.10. *Let I be any index set and $M_i, N_i, i \in I$ be hopfological modules. Then*

$$\begin{aligned} \mathcal{Z}(\oplus_{i \in I} \text{Hom}_A(M_i, N_i)) &\cong \oplus_{i \in I} \mathcal{Z}(\text{Hom}_A(M_i, N_i)), \\ \mathcal{H}(\oplus_{i \in I} \text{Hom}_A(M_i, N_i)) &\cong \oplus_{i \in I} \mathcal{H}(\text{Hom}_A(M_i, N_i)). \end{aligned}$$

Proof. This follows readily from the proposition and the fact that $\text{Hom}_H(\mathbb{k}_0, -)$ commutes with arbitrary direct sums. \square

5.4 Examples

We will give three examples on what homotopic morphisms look like for some of the Hopf algebras we discussed in § 3. By Lemma 5.4 these are precisely the morphisms of the form $f = \Lambda \cdot h$ for some $h \in \text{Hom}_A(M, N)$. Recall that

$$(\Lambda \cdot h)(-) = \sum \Lambda_{(2)} h(S^{-1}(\Lambda_{(1)})(-)).$$

- When H is the Hopf super algebra $\mathbb{k}[d]/(d^2)$ (i.e., we are in the usual DG algebra case), $\Lambda = d$ and, for any homogeneous $h \in \text{Hom}_A(M, N)$ of degree $|h|$,

$$d \cdot h = dh + (-1)^{|h|+1}hd.$$

The minus signs come from switching d and f in the category of super-vector spaces and $S^{-1}(d) = -d$.

- Let $H = H_n$ be the Taft algebra. In the examples of §2, we have seen that a left integral of H is given by $\Lambda = (\sum_{i=0}^{n-1} K^i) d^{n-1}$. Notice that if $g = \sum_{i=0}^{n-1} g_i \in \text{Hom}_A(M, N)$ is a decomposition of g into its homogeneous components, $(\sum_{i=0}^{n-1} K^i) g_j = (\sum_{i=0}^{n-1} \zeta^{ij}) g_j$, which can be non-zero only when $j = 0$, in which case it equals g_0 . That is, $\sum_{i=0}^{n-1} K^i$ projects any vector onto its degree-zero component. Thus the effect of applying Λ to any $h \in \text{Hom}_A(M, N)$ will only be seen in its homogeneous of degree $(1 - n)$ part. Without loss of generality we will assume $\deg(h) = 1 - n$. Then using the commutator relations, we obtain that, on such an h ,

$$\begin{aligned} d^{n-1} \cdot h &= \sum_{j=0}^{n-1} (-1)^{n-1-j} \zeta^{(1-n)(n-1-j)} \binom{n-1}{j}_\zeta d^j \circ h \circ d^{n-1-j} \\ &= \sum_{j=0}^{n-1} (-1)^n \zeta^{-(j+1)(j+2)/2} d^j \circ h \circ d^{n-1-j}, \end{aligned}$$

where $\binom{n}{k}_\zeta = (n-1)_\zeta! / ((k)_\zeta!(n-1-k)_\zeta!)$ and, for any $j \in \mathbb{N}$, $(j)_\zeta := 1 + \zeta + \dots + \zeta^{j-1}$ is the un-symmetrized quantum integer j . In the last step, we may rescale h componentwise by a non-zero scalar to obtain the formula for a null-homotopic f (c.f. [Kap96, Sar95]):

$$f = \sum_{j=0}^{n-1} d^j \circ h \circ d^{n-1-j}.$$

- Consider the (graded) Hopf algebra $H = \mathbb{k}[\partial]/(\partial^p)$, where \mathbb{k} is of positive characteristic p . As above, if $h \in \text{Hom}_A(M, N)$, we have

$$\partial^{p-1}(h) = \sum_{i=0}^{p-1} (-1)^{p-1-i} \binom{p-1}{i} \partial^i \circ h \circ \partial^{p-1-i} = \sum_{i=0}^{p-1} \partial^i \circ h \circ \partial^{p-1-i}.$$

The last equality holds because $(-1)^i \binom{p-1}{i} = 1$ in \mathbb{k} . We depict such a morphism in the following diagram, in comparison with the usual notion.

$$\begin{array}{cccccccccccc} \dots & \xrightarrow{\partial_M} & M^{i-p+1} & \xrightarrow{\partial_M} & M^{i-p+2} & \xrightarrow{\partial_M} & \dots & \xrightarrow{\partial_M} & M^i & \xrightarrow{\partial_M} & M^{i+1} & \xrightarrow{\partial_M} & \dots & \xrightarrow{\partial_M} & M^{i+p-1} & \xrightarrow{\partial_M} & \dots \\ & & \downarrow f & & \downarrow f & \nearrow h & & \downarrow f & \nearrow h & & \downarrow f & \nearrow h & & \downarrow f & \nearrow h & & \downarrow f \\ \dots & \xrightarrow{\partial_N} & N^{i-p+1} & \xrightarrow{\partial_N} & N^{i-p+2} & \xrightarrow{\partial_N} & \dots & \xrightarrow{\partial_N} & N^i & \xrightarrow{\partial_N} & N^{i+1} & \xrightarrow{\partial_N} & \dots & \xrightarrow{\partial_N} & N^{i+p-1} & \xrightarrow{\partial_N} & \dots \end{array}$$

6. Cofibrant modules

Adapting the corresponding definition from Keller [Kel94, Kel07] on the DG case, we define the notion of cofibrant hopfological modules and give a functorial cofibrant resolution (i.e., quasi-isomorphism) $\mathbf{p}M \rightarrow M$ for any hopfological module M . This will be utilized later when discussing compact hopfological modules, derived functors and derived equivalences between hopfological module categories.

In this section, H will be assumed, as before, to be a finite-dimensional Hopf algebra over a base field \mathbb{k} , A will be assumed to be an H -module algebra, and we set $B = A\#H$.

6.1 Cofibrant modules

First we introduce the notion of ‘cofibrant hopfological modules’ in analogy with the DG case.

DEFINITION 6.1. A B -module P is called *cofibrant* if, for any surjective quasi-isomorphism $M \twoheadrightarrow N$ of B -modules, the induced map of \mathbb{k} -vector spaces

$$\mathcal{Z}(\mathrm{Hom}_A(P, M)) \longrightarrow \mathcal{Z}(\mathrm{Hom}_A(P, N))$$

is surjective. In the graded case, we require instead that the graded H -module map

$$\mathcal{Z}^*(\mathrm{HOM}_A(P, M)) \longrightarrow \mathcal{Z}^*(\mathrm{HOM}_A(P, N))$$

is surjective in the category of graded \mathbb{k} -vector spaces. Notice that this is equivalent to imposing the same condition on \mathcal{Z}^0 , as $M\{r\} \twoheadrightarrow N\{r\}$ is a surjective quasi-isomorphism, for any $r \in \mathbb{Z}$, whenever $M \twoheadrightarrow N$ is.

It is rather straightforward to see that being a ‘cofibrant module’ in the case of DG modules implies the usual sense of being ‘K-projective module’, as described, for instance, by Bernstein and Lunts [BL94]. It says that, for any acyclic DG-module M , the complex $\mathrm{HOM}_A(P, M)$ is acyclic as a $\mathbb{k}[d]/(d^2)$ -module, i.e., the homology of this complex is 0. Indeed, it can be verified by applying the defining property to the surjective quasi-isomorphism

$$\mathrm{Cone}(\mathrm{Id}_M) \longrightarrow M,$$

and observing that $\mathrm{HOM}_A(P, \mathrm{Cone}(\mathrm{Id}_M)) = \mathrm{Cone}(\mathrm{Id}_{\mathrm{HOM}_A(P, M)})$ is contractible. The following lemma is motivated by this discussion.

LEMMA 6.2. *Let P be a cofibrant hopfological module. Then, for any acyclic module $M \in B\text{-mod}$ (respectively $B\text{-gmod}$), the H -module $\mathrm{Hom}_A(P, M)$ (respectively $\mathrm{HOM}_A(P, M)$) has trivial stable invariants,*

$$\mathcal{H}(\mathrm{Hom}_A(P, M)) = 0 \quad (\text{respectively } \mathcal{H}^*(\mathrm{HOM}_A(P, M)) = 0),$$

and thus, in the homotopy category, we have

$$\mathrm{Hom}_{\mathcal{C}(A, H)}(P, M) = 0.$$

Proof. Let P be a cofibrant hopfological module. Applying $\mathrm{Hom}_A(P, -)$ to the B -module map $M \otimes H \xrightarrow{\mathrm{Id}_M \otimes \epsilon} M$, we obtain the induced map,

$$\mathcal{Z}(\mathrm{Hom}_A(P, M \otimes H)) \twoheadrightarrow \mathcal{Z}(\mathrm{Hom}_A(P, M)),$$

which is a surjection by the cofibrance assumption. Therefore, given $\phi \in \mathcal{Z}(\mathrm{Hom}_A(P, M))$, we can find $\Phi \in \mathcal{Z}(\mathrm{Hom}_A(P, M \otimes H))$ which when composed with $\mathrm{Id} \otimes \epsilon$ gives us ϕ . Since $\mathrm{Hom}_A(P, M \otimes H) \cong \mathrm{Hom}_A(P, M) \otimes H$ is contractible, $\Phi = \Lambda \cdot \Psi$ for some $\Psi \in \mathrm{Hom}_A(P, M \otimes H)$ (Lemma 5.4). Then, for any $x \in P$, we have

$$\begin{aligned} (\Lambda \cdot ((\mathrm{Id} \otimes \epsilon) \circ \Psi))(x) &= \Lambda_{(2)} \cdot ((\mathrm{Id} \otimes \epsilon) \circ \Psi(S^{-1}(\Lambda_{(1)}) \cdot x)) = (\mathrm{Id} \otimes \epsilon)(\Lambda_{(2)} \cdot \Psi(S^{-1}(\Lambda_{(1)}) \cdot x)) \\ &= (\mathrm{Id} \otimes \epsilon)((\Lambda \cdot \Psi)(x)) = (\mathrm{Id} \otimes \epsilon)(\psi(x)) = \phi(x), \end{aligned}$$

where the second equality holds since $\mathrm{Id} \otimes \epsilon$ is H -linear. Therefore by Corollary 5.5, $\phi = 0$ after passing to the stable category. The last claim follows from Proposition 5.9. \square

Notice that, when H is a finite-dimensional local Hopf algebra, $\mathcal{H}(\mathrm{Hom}_A(P, M)) = 0$ actually implies that the total cohomology $\mathrm{Hom}_A(P, M)$ is 0 in the stable category $H\text{-mod}$. This follows from the observation that any indecomposable module in the case contains a trivial submodule. Therefore, for such Hopf algebras H , we know that the H -module $\mathrm{Hom}_A(P, M)$ is projective and injective as an H -module (we will just call such H -modules acyclic when no confusion

could arise). In fact, this will turn out to be true for any H and any cofibrant module P . We will show this after introducing some necessary tools.

Our main goal in this section is to construct, for each A -module M , a functorial cofibrant replacement. We make the following definition.

DEFINITION 6.3. We say that a B -module satisfies *property (P)* if it is isomorphic to a module P in the category $\mathcal{C}(A, H)$ for which the following three conditions hold (c.f. [Kel94, § 3]).

(P1) There is a filtration on P

$$0 \subset F_0 \subset F_1 \subset \cdots \subset F_r \subset F_{r+1} \subset \cdots \subset P,$$

which is exhaustive in the sense that $P = \bigcup_{r \in \mathbb{N}} F_r$.

(P2) The inclusion $F_r \subset F_{r+1}$ splits as a map of left A -modules (respectively graded left A -modules when they are graded) for all $r \in \mathbb{N}$.

(P3) The submodule F_0 , as well as the quotients F_{r+1}/F_r for all $r \in \mathbb{N}$, is isomorphic to direct sums

of B -modules of the form $A \otimes V$, where V is an indecomposable H -module (respectively $A \otimes V \in B\text{-gmod}$ and $V \in H\text{-gmod}$ in the graded case).

Equivalently, in the last condition (P3), we may drop the direct sum requirement for indecomposable V but instead allow V to be any H -module.

We need to clarify the relation between modules with property (P) and cofibrant modules. First of all we will show that modules with property (P) are cofibrant.

LEMMA 6.4. *Let $P \in B\text{-mod}$ (respectively $B\text{-gmod}$) be a module satisfying property (P), and K be an acyclic B -module. Then the H -module $\text{Hom}_A(P, K)$ is projective and injective as an H -module.*

Proof. The proof is divided into three steps. First off, we check that free modules of the form $A \otimes V$ have the claimed property of the lemma. As H -modules, we have a canonical isomorphism:

$$\text{Hom}_A(A \otimes V, K) \cong \text{Hom}_{\mathbb{k}}(V, K).$$

Thus the result for $A \otimes V$ follows from Lemma 2.3.

Secondly, we use induction to prove that $\text{Hom}_A(F_r, K)$ is projective and injective (acyclic for short) for any $r \geq 0$. In fact, assuming so for F_r , applying $\text{Hom}_A(-, K)$ to the short exact sequence of free A -modules,

$$0 \longrightarrow F_r \longrightarrow F_{r+1} \longrightarrow \bigoplus_{j \in J} A \otimes V_j \longrightarrow 0,$$

we obtain a short exact sequence of H -modules:

$$0 \longrightarrow \prod_{j \in J} \text{Hom}_A(A \otimes V_j, K) \longrightarrow \text{Hom}_A(F_{r+1}, K) \longrightarrow \text{Hom}_A(F_r, K) \longrightarrow 0.$$

By the inductive hypothesis and the previous step, $\text{Hom}_A(F_r, K)$ and $\prod_{j \in J} \text{Hom}_A(A \otimes V_j, K)$ are acyclic. Thus $\text{Hom}_A(F_{r+1}, K)$ is acyclic, since in $H\text{-mod}$ it is isomorphic to the direct sum of these acyclic modules.

Finally, by definition, we have the following short exact sequence of free A -modules:

$$0 \longrightarrow \bigoplus_{r \in \mathbb{N}} F_r \xrightarrow{\Psi} \bigoplus_{s \in \mathbb{N}} F_s \longrightarrow P \longrightarrow 0,$$

where the map Ψ is defined similarly as in Lemma 2.3. Applying $\text{Hom}_A(-, K)$ to the short exact sequence of free A -modules, we obtain a short exact sequence of H -modules:

$$0 \longrightarrow \text{Hom}_A(P, K) \longrightarrow \prod \text{Hom}_A(F_s, K) \longrightarrow \prod \text{Hom}_A(F_r, K) \longrightarrow 0.$$

By the second step, the two terms on the right are acyclic. Hence $\text{Hom}_A(P, K)$ is acyclic and the lemma follows. \square

COROLLARY 6.5. *If P is a B -module with property (P), then it is cofibrant.*

Proof. Let $M \longrightarrow N$ be a surjective quasi-isomorphism in $B\text{-mod}$. We have a short exact sequence of B -modules:

$$0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0,$$

where K is acyclic by our assumption. Applying $\text{Hom}_A(P, -)$ to this short exact sequence, we obtain a short exact sequence of H -modules,

$$0 \longrightarrow \text{Hom}_A(P, K) \longrightarrow \text{Hom}_A(P, M) \longrightarrow \text{Hom}_A(P, N) \longrightarrow 0,$$

since P is projective as an A -module. The above Lemma 6.4 says that $\text{Hom}_A(P, K)$ considered as an H -module is projective and injective, and thus the sequence splits and we have a direct sum decomposition:

$$\text{Hom}_A(P, M) \cong \text{Hom}_A(P, K) \oplus \text{Hom}_A(P, N).$$

Taking H -invariants on both sides (Proposition 5.9) gives us

$$\mathcal{Z}(\text{Hom}_A(A \otimes V, M)) \cong \mathcal{Z}(\text{Hom}_A(A \otimes V, K)) \oplus \mathcal{Z}(\text{Hom}_A(A \otimes V, N)),$$

whence the surjectivity $\mathcal{Z}(\text{Hom}_A(P, M)) \rightarrow \mathcal{Z}(\text{Hom}_A(P, N))$ follows. \square

6.2 The bar resolution

Now we formulate the main result of this section and its immediate consequences.

THEOREM 6.6. *Let H be a finite-dimensional (graded) Hopf (super-)algebra, A be a left H -module algebra, and set $B = A\#H$. For each module $M \in B\text{-mod}$ (respectively $B\text{-gmod}$), there is a short exact sequence in $B\text{-mod}$ (respectively $B\text{-gmod}$) which is split exact as a sequence of A -modules,*

$$0 \longrightarrow M \longrightarrow \mathbf{a}M \longrightarrow \tilde{\mathbf{p}}M \longrightarrow 0,$$

where $\tilde{\mathbf{p}}M$ satisfies property (P) and $\mathbf{a}M$ is an acyclic B -module. Moreover the construction of the short exact sequence is functorial in M .

We will refer to the construction of the theorem, as well as the cofibrant replacement in the next corollary, as the ‘bar resolution’ of any hopfological module M , which is the functorial cofibrant replacement we claimed at the beginning of this section.

COROLLARY 6.7. *Under the same conditions as in Theorem 6.6, let M be a hopfological module.*

(i) *There is an associated distinguished triangle, functorial in M inside $\mathcal{C}(A, H)$:*

$$M \longrightarrow \mathbf{a}M \longrightarrow \tilde{\mathbf{p}}M \longrightarrow TM.$$

(ii) *In the derived category $\mathcal{D}(A, H)$, there is a functorial isomorphism*

$$\mathbf{p}M \xrightarrow{\cong} M,$$

where $\mathbf{p}M := T^{-1}(\tilde{\mathbf{p}}M)$ is a module with property (P).

(iii) The isomorphism in part (ii) arises as the image of a surjective quasi-isomorphism $\mathbf{p}M \rightarrow M$ in $B\text{-mod}$.

Proof. By applying Lemma 4.3 to the short exact sequence of the theorem, we obtain a distinguished triangle in $\mathcal{C}(A, H)$,

$$M \longrightarrow \mathbf{a}M \longrightarrow \tilde{\mathbf{p}}M \longrightarrow T(M),$$

which is functorial in M by Theorem 6.6. Since $\mathbf{a}M$ is acyclic, it is isomorphic to 0 in the derived category. By passing to the derived category $\mathcal{D}(A, H)$ we obtain a functorial isomorphism

$$\tilde{\mathbf{p}}M \xrightarrow{\cong} T(M).$$

Then apply T^{-1} to this isomorphism $\tilde{\mathbf{p}}M \rightarrow T(M)$, and define

$$\mathbf{p}M := T^{-1}(\tilde{\mathbf{p}}M) = \tilde{\mathbf{p}}M \otimes \ker(\epsilon),$$

which satisfies property (P) since $\tilde{\mathbf{p}}M$ does. This proves parts (i) and (ii). We will postpone the proof of part (iii) until the end of this section, where the explicit surjective quasi-isomorphism is constructed. \square

We reap some other direct consequences of the bar construction, the first of which is the promised relationship between cofibrant modules and modules with property (P).

COROLLARY 6.8. *Let M be a cofibrant hopfological module. Then M is a direct summand of a B -module with property (P). Conversely, any $B\text{-mod}$ direct summand of a module with property (P) is cofibrant. In other words, the class of cofibrant modules is the idempotent completion of the class of modules with property (P) in the abelian category $B\text{-mod}$.*

Proof. By Corollary 6.7(iii), we have a surjective quasi-isomorphism $\mathbf{p}M \rightarrow M$. Applying the $\text{Hom}_B(M, -)$ to this surjection and using the cofibrance condition, we see immediately that M is a direct summand of $\mathbf{p}M$, the latter being a module with property (P) by the same corollary. The converse is easy. \square

The next result gives the promised characterization of cofibrant modules as an analogue of ‘ K -projective modules’ due to Bernstein and Lunts [BL94].

COROLLARY 6.9. *A hopfological module M is cofibrant if and only if M is projective as an A -module, and, for any acyclic module K , the H -module $\text{Hom}_A(M, K)$ is projective and injective.*

Proof. The ‘if’ direction follows from the same argument we used in Corollary 6.5. The ‘only if’ part follows from the above Corollary 6.8, the corresponding result for property (P) modules Lemma 6.4, and the fact that an injective submodule of any H -module is an H -direct summand. \square

The last immediate consequence of the theorem we record here is the equivalence between $\mathcal{D}(A, H)$ and the homotopy category of property (P) (respectively cofibrant) objects. We leave the proof to the reader as exercise.

COROLLARY 6.10. *Let $\mathcal{P}(A, H)$ (respectively $\mathcal{CF}(A, H)$) be the full triangulated subcategory of $\mathcal{C}(A, H)$ whose objects consist property (P) modules (respectively cofibrant modules). Then we have the following.*

(1) The morphism space between any two objects P_1, P_2 in $\mathcal{P}(A, H)$ (respectively $\mathcal{CF}(A, H)$) coincides with the morphism space of these objects in the derived category:

$$\text{Hom}_{\mathcal{P}(A,H)}(P_1, P_2) \cong \text{Hom}_{\mathcal{D}(A,H)}(P_1, P_2).$$

In fact, for any module P with property (P) (respectively which is cofibrant), localization induces a functorial isomorphism:

$$\text{Hom}_{\mathcal{C}(A,H)}(P, -) \cong \text{Hom}_{\mathcal{D}(A,H)}(P, -).$$

(2) The composition of functors,

$$\mathcal{P}(A, H) \subset \mathcal{C}(A, H) \xrightarrow{Q} \mathcal{D}(A, H) \quad (\text{respectively } \mathcal{CF}(A, H) \subset \mathcal{C}(A, H) \xrightarrow{Q} \mathcal{D}(A, H)),$$

where Q is the localization functor, is an equivalence of triangulated categories.

(3) The bar resolution is a functor $\mathbf{p} : \mathcal{D}(A, H) \rightarrow \mathcal{P}(A, H)$ which is the left adjoint to the composition functor $\mathcal{P}(A, H) \subset \mathcal{C}(A, H) \xrightarrow{Q} \mathcal{D}(A, H)$.

Remark 6.11. To summarize the notions we introduced in this section, we have an inclusion of diagrams inside the abelian category $B\text{-mod}$:

$$(\text{Modules with property (P)}) \subset (\text{Cofibrant modules}) \subset (\text{Hopfological modules}).$$

The previous corollary can be summarized as saying that these inclusions in turn give equivalences of the homotopy categories $\mathcal{P}(A, H)$ and $\mathcal{CF}(A, H)$ with the derived category $\mathcal{D}(A, H)$.

6.3 Proof of Theorem 6.6

The simplicial bar resolution of an algebra. Recall that, for an algebra A over \mathbb{k} (the construction works more generally over \mathbb{Z}), the simplicial bar resolution of A is a projective resolution of A as a module over the envelope algebra $A \otimes A^{\text{op}}$. More details about bar resolutions can be found in Loday’s monograph [Lod98, ch. I].

Let (C_\bullet, d_i, s_i) be a simplicial module over the base field \mathbb{k} , where d_i is the face map, and s_i is the degeneration map, satisfying the commutator relation,

$$d_i d_j = d_{j-1} d_i \quad \text{if } i < j, \quad d_i s_j = \begin{cases} s_{j-1} d_i & \text{if } i < j, \\ \text{id} & \text{if } i = j, j + 1, \\ s_j d_{i-1} & \text{if } i > j + 1. \end{cases}$$

One can naturally associate with such a simplicial module a complex by defining the differential $\delta : C_n \rightarrow C_{n+1}$ as the alternating sum of the face maps $\delta = \sum_{i=0}^{n-1} (-1)^i d_i$. One then checks readily using the commutator relations in the definition that (C_\bullet, δ) becomes a complex. Now we apply this construction to the Hochschild complex.

DEFINITION 6.12. The Hochschild simplicial module of a \mathbb{k} -algebra A is the simplicial module $(C(A), d_i, s_i)$, where, for each $n \geq 0$, $C_{-n} = A^{\otimes(n+1)}$, and $C_{n+1} = 0$. The face and degeneration maps are defined by

$$d_i(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \begin{cases} a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n & \text{if } 0 \leq i \leq n - 1, \\ a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1} & \text{if } i = n, \end{cases}$$

and

$$s_i(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = a_0 \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_n.$$

We have the following well-known lemma.

LEMMA 6.13. *The associated simplicial bar complex $(C_{-n} = A^{\otimes(n+1)}, \delta_n)$ is a contractible complex, giving a resolution of A as an (A, A) -bimodule by free bimodules.*

Proof. A homotopy is given by the ‘extra-degeneracy’ $s : A^{\otimes n} \rightarrow A^{\otimes(n+1)}$, $a_0 \otimes \cdots \otimes a_{n-1} \mapsto 1 \otimes a_0 \otimes \cdots \otimes a_{n-1}$, for $n \in \mathbb{N}$. □

Proof of Theorem 6.6: construction. Now we begin with the construction of the bar resolution. The first observation to make is that, in the recap above, when A is a left H -module algebra, all the face and degeneration maps are H -module maps. For instance, the map $\delta_0 : A \otimes A \rightarrow A$, $\delta_0(a_0 \otimes a_1) = a_0 a_1$ is the multiplication map, which is an H -module map by definition. Now we apply the cone construction (Definition 3.3) to this map and obtain

$$\text{Cone}(\delta_0) \cong A \otimes A \otimes (H/\mathbb{k}\Lambda) \oplus A,$$

the isomorphism viewed as a (A, A) -bimodule map (A acts trivially on the $(H/\mathbb{k}\Lambda)$ factor). However, this isomorphism is not an H -module isomorphism. The H -module structure on the cone is defined in a slightly abstract way using the pushout property, which is not preserved under this identification.

Next, we will lift the map $\delta_1 : A \otimes A \otimes A \rightarrow A \otimes A$ to a map $\tilde{\delta}_1 : A \otimes A \otimes A \otimes (H/\mathbb{k}\Lambda) \rightarrow \text{Cone}(\delta_0)$, as follows. First off we define a map:

$$\begin{aligned} A \otimes A \otimes A \otimes H &\rightarrow A \otimes A \otimes H \oplus A \\ a \otimes a' \otimes a'' \otimes h &\mapsto (\delta_1(a \otimes a' \otimes a'') \otimes h, 0). \end{aligned}$$

Since $\delta_0 \delta_1 = 0$, this map descends to the quotient and gives rise to $\tilde{\delta}_1$:

$$\tilde{\delta}_1 : A \otimes A \otimes A \otimes (H/\mathbb{k}\Lambda) \rightarrow \text{Cone}(\delta_0).$$

Also observe that $\tilde{\delta}_1$ kills elements in the submodule $\text{Im}(\delta_2) \otimes (H/\mathbb{k}\Lambda)$.

Then we can construct the cone of $\tilde{\delta}_1$. Recall from the definition of the cone construction that, in $\text{Cone}(\delta_0)$, A is naturally an H -submodule, while the quotient $\text{Cone}(\delta_0)/A$ is isomorphic to the H -module $A \otimes A \otimes (H/\mathbb{k}\Lambda)$. Thus the cone of $\tilde{\delta}_1$ has a filtration by (A, A) -bimodules,

$$0 \subset A \subset \text{Cone}(\delta_0) \subset \text{Cone}(\tilde{\delta}_1),$$

whose subquotients are respectively A , $A^{\otimes 2} \otimes (H/\mathbb{k}\Lambda)$, and $A^{\otimes 3} \otimes (H/\mathbb{k}\Lambda)^{\otimes 2}$. These observations will allow us to construct the bar resolution inductively.

Then an easy induction argument shows that:

- (1) $C_n = \text{Cone}(\tilde{\delta}_n : A^{\otimes(n+2)} \otimes (H/\mathbb{k}\Lambda)^{\otimes n} \rightarrow C_{n-1}) \in B\text{-mod}$;
- (2) for any $x \in (H/\mathbb{k}\Lambda)^n$, $a \in A^{\otimes(n+3)}$ we have $\tilde{\delta}_n(\delta_{n+1}(a) \otimes x) = 0$.

This assumption implies that C_{n-1} is a submodule of C_n . It follows that C_n has an exhaustive filtration

$$F^\bullet : 0 = F^{-1} \subset F^0 \subset \cdots \subset F^{p-1} \subset F^p \subset \cdots \subset F^{n+1} = C_n,$$

whose subquotients F^n/F^{n-1} are isomorphic to $A^{\otimes(n+1)} \otimes (H/\mathbb{k}\Lambda)^{\otimes n}$. In particular, this says that C_n satisfies ‘property (P)’, and therefore is a cofibrant B -module as defined earlier.

Then we define

$$\mathbf{a}A := \bigcup_{n=-1}^{\infty} C_n,$$

which fits into a short exact sequence,

$$0 \longrightarrow A \longrightarrow \mathbf{a}A \longrightarrow \tilde{\mathbf{p}}A \longrightarrow 0.$$

We may regard any left B -module M as an A -module by restriction. Tensoring the above sequence by M gives rise to the short exact sequence

$$0 \longrightarrow M \longrightarrow \mathbf{a}M \longrightarrow \tilde{\mathbf{p}}M \longrightarrow 0$$

claimed in the theorem. Our next goal would then be to show that $\mathbf{a}M$ in the above short exact sequence is contractible as an H -module, for any hopfological module M .

Proof of Theorem 6.6: contractibility. Now we show that $\mathbf{a}M$ is acyclic, for any A -module M . To do this we may safely forget about the B -module structures involved and regard the modules as H -modules. We will show this for $\mathbf{a}A$; the general case follows by the same argument.

Observe that in the Lemma 6.13, the homotopy $s : A^{\otimes n} \longrightarrow A^{\otimes(n+1)}$ is an H -module map since A is an H -module algebra. Thus the homotopy allows us to split the terms in the original bar complex of A into H -modules summands

$$A^{\otimes n} \cong A^{(n)} \oplus A^{(n-1)}$$

so that the boundary map $\delta : A^{\otimes n} \longrightarrow A^{\otimes(n-1)}$ (an H -module map again) kills the $A^{(n)}$ factor and identifies the $A^{(n-1)}$ factor with that in $A^{\otimes(n-1)}$. Now if we go back to the definition of the cone C_0 as in the previous part, we see that it was constructed as a pushout, and therefore, as H -modules, we can identify it with

$$\begin{aligned} C_0 &\cong (A^{\otimes 2} \otimes H \oplus A) / (\{a \otimes a' \otimes \Lambda, aa' \mid a, a' \in A\}) \\ &\cong ((A^{(2)} \oplus A) \otimes H \oplus A) / (\{(a^{(2)}, a) \otimes \Lambda, a \mid a^{(2)} \in A^{(2)}, a \in A\}) \\ &\cong (A^{(2)} \otimes H \oplus A \otimes H \oplus A) / (\{(a^{(2)} \otimes \Lambda) \mid a^{(2)} \in A^{(2)}\} \oplus \{(a \otimes \Lambda, a) \mid a \in A\}) \\ &\cong A \otimes (H/\mathbb{k}\Lambda) \oplus A \otimes H. \end{aligned}$$

Then, an easy induction argument shows that there is an H -module isomorphism:

$$C_n = \text{Cone}(\tilde{\delta}_n) \cong A^{(n+2)} \otimes (H/\mathbb{k}\Lambda)^{\otimes(n+1)} \oplus \bigoplus_{i=1}^{n+1} A^{(i)} \otimes (H/\mathbb{k}\Lambda)^{\otimes(i-1)} \otimes H.$$

Taking the union of all n gives us

$$\mathbf{a}(A) \cong \bigoplus_{i=1}^{\infty} A^{(i)} \otimes (H/\mathbb{k}\Lambda)^{\otimes(i-1)} \otimes H \cong \left(\bigoplus_{i=1}^{\infty} A^{(i)} \otimes (H/\mathbb{k}\Lambda)^{\otimes(i-1)} \right) \otimes H,$$

which is of the form $N \otimes H$ for some H -module N , and the acyclicity follows. This finishes the proof of Theorem 6.6. □

Proof of part (iii) of Corollary 6.7. Now we finish the proof of the corollary. Notice that as (A, A) bimodules,

$$\begin{aligned} \mathbf{p}(A) &= \bigcup_{n=0}^{\infty} C_n \otimes \ker(\epsilon) \\ &\cong A \otimes A \otimes H/\mathbb{k}\Lambda \otimes \ker(\epsilon) \oplus \dots \oplus A^{\otimes(n+2)} \otimes (H/\mathbb{k}\Lambda)^{\otimes(n+1)} \otimes \ker(\epsilon) \oplus \dots \\ &\cong A \otimes A \otimes (\mathbb{k} \oplus Q) \oplus \dots \oplus A^{\otimes(n+2)} \otimes (H/\mathbb{k}\Lambda)^{\otimes(n)} \otimes (\mathbb{k} \oplus Q) \oplus \dots, \end{aligned}$$

where Q is a projective H -module (see [Kho06, Proposition 3]). It is then easily seen that the map $A \otimes A \otimes \mathbb{k} \cong A \otimes A \xrightarrow{\delta_0=m} A$ extends to $\mathbf{p}A \twoheadrightarrow A$. The cone of this map, when ignoring

the contributions from factors containing tensor products with Q , is just $\mathbf{a}A$, which is contractible. The corollary follows by inducing $(\mathbf{p}A \twoheadrightarrow A)$ up to the resolution $\mathbf{p}M \twoheadrightarrow M$. \square

Remark 6.14. The more general notion of H -module algebra would be ‘ H -module category’, which is a graded category (including the cyclic $\mathbb{Z}/(n)$ -graded case as well) with a finite-dimensional (graded) Hopf (super-)algebra action on the Hom spaces between objects. A first example of such a category which is not an H -module algebra (i.e., there are infinitely many objects) is the category $H\text{-mod}$. More generally, the graded module category over $B = A\#H$ is another example of such a category. The algebra A itself is an H -module category with a single object whose endomorphism space is given by A , together with the defining H action. In the next two sections, our logical treatment follows Keller’s on DG categories [Kel94] closely and the above story generalizes without much difficulty to the categorical case.

7. Compact modules

In this section, we follow Neeman’s original treatment in [Nee96] to discuss compact hopfological modules. Thankfully, Neeman’s original setup was general enough that it can be applied here without essential modification. See also Keller [Kel94, § 5] for another account of Neeman’s treatment, where the notion of generators of a triangulated category appears to be slightly different. However, it turns out that the two notions are equivalent.

Throughout this section, we make the same assumption as in the previous section that H is a finite-dimensional Hopf algebra over the base field \mathbb{k} , and A is an H -module algebra. We let \mathcal{D} denote a \mathbb{k} -linear triangulated category that admits infinite direct sums.

7.1 Generators

We begin with a discussion of the notion of compact generators for $\mathcal{D}(A, H)$.

DEFINITION 7.1. An object $X \in \mathcal{D}$ is said to be *compact* if $\text{Hom}_{\mathcal{D}}(X, -) : \mathcal{D} \rightarrow \mathbb{k}\text{-vect}$ commutes with arbitrary direct sums.

The following lemma is obvious from the definition and the axioms of triangulated categories.

LEMMA 7.2. *In any distinguished triangle in \mathcal{D} , if two out of the three objects in the distinguished triangle are compact, then so is the third.* \square

The next lemma gives us the easiest examples of compact objects in $\mathcal{D}(A, H)$.

LEMMA 7.3. *For any finite-dimensional H -module V , the hopfological module $A \otimes V$ is compact in $\mathcal{D}(A, H)$.*

Proof. Since $A \otimes V$ is cofibrant, using Lemma 6.10, we have

$$\begin{aligned} \text{Hom}_{\mathcal{D}(A,H)}\left(A \otimes V, \bigoplus_{i \in I} M_i\right) &\cong \text{Hom}_{\mathcal{C}(A,H)}\left(A \otimes V, \bigoplus_{i \in I} M_i\right) \cong \mathcal{H}\left(\text{Hom}_A\left(A \otimes V, \bigoplus_{i \in I} M_i\right)\right) \\ &\cong \mathcal{H}\left(\text{Hom}_{\mathbb{k}}\left(V, \bigoplus_{i \in I} M_i\right)\right) \cong \bigoplus_{i \in I} \mathcal{H}(\text{Hom}_{\mathbb{k}}(V, M_i)) \cong \bigoplus_{i \in I} \mathcal{H}(\text{Hom}_A(A \otimes V, M_i)) \\ &\cong \bigoplus_{i \in I} \text{Hom}_{\mathcal{D}(A,H)}(A \otimes V, M_i), \end{aligned}$$

where, in the fourth isomorphism, we used the fact that V is finite dimensional (thus compact) and taking \mathcal{H} commutes with direct sums (Corollary 5.10). The lemma follows. \square

COROLLARY 7.4. *Let $A \otimes V$ be as in the previous lemma. Then $T^n(A \otimes V)$ is compact for any $n \in \mathbb{Z}$.*

Proof. Of course, this can be seen without the previous lemma since the shift functors are automorphisms of $\mathcal{D}(A, H)$. Alternatively, recall that the shifts T, T^{-1} are given by right tensoring $A \otimes V$ with the finite-dimensional H -modules $H/\mathbb{k}\Lambda, \text{Ker}(\epsilon)$ respectively (Proposition 3.2). The compactness of $T(A \otimes V) = A \otimes V \otimes (H/\mathbb{k}\Lambda)$ etc. then follows directly from the previous lemma. \square

DEFINITION 7.5 (Neeman). Let \mathcal{D} be as above. We say that \mathcal{D} is *generated by a set of objects* if there exists a set $\mathcal{G} = \{G_i \in \mathcal{D} \mid i \in I\}$ so that, for any $X \in \mathcal{D}, X \cong 0$ if and only if

$$\text{Hom}_{\mathcal{D}}(T^n(G_i), X) = 0$$

for all $n \in \mathbb{Z}$ and $G_i \in \mathcal{G}$; \mathcal{D} is said to be *compactly generated* if \mathcal{D} is generated by a set \mathcal{G} consisting of compact objects.

As an example of this definition, we show that $\mathcal{D}(A, H)$ admits a set of compact generators.

PROPOSITION 7.6. *The derived category $\mathcal{D}(A, H)$ is compactly generated by the finite set of objects $\mathcal{G} := \{A \otimes V\}$, where V ranges over a finite set of representatives of isomorphism classes of simple H -modules.*

Proof. It suffices to show that, if an object $X \in \mathcal{D}(A, H)$ satisfies $\text{Hom}_{\mathcal{D}(A,H)}(A \otimes V, X) = 0$ for all $A \otimes V \in \mathcal{G}$, then $X \cong 0$ in $\mathcal{D}(A, H)$.

Firstly, we show that the hypothesis implies that $\text{Hom}_{\mathcal{D}(A,H)}(A \otimes W, X) = 0$ for any finite-dimensional H -module W . We prove this by induction on the length of W , the length 1 case following by the assumption. Inductively, take any finite-dimensional irreducible submodule W' of W and form the quotient $W'' = W/W'$. W'' has shorter length by construction, and we have a short exact sequence of cofibrant modules

$$0 \longrightarrow A \otimes W' \longrightarrow A \otimes W \longrightarrow A \otimes W'' \longrightarrow 0.$$

This short exact sequence becomes a distinguished triangle of cofibrant modules in $\mathcal{D}(A, H)$ and applying $\text{Hom}_{\mathcal{D}(A,H)}(-, X)$ to the triangle leads to a long exact sequence

$$\begin{aligned} \cdots \longrightarrow \text{Hom}_{\mathcal{D}(A,H)}(T^n(A \otimes W''), X) &\longrightarrow \text{Hom}_{\mathcal{D}(A,H)}(T^n(A \otimes W), X) \\ &\longrightarrow \text{Hom}_{\mathcal{D}(A,H)}(T^n(A \otimes W'), X) \longrightarrow \cdots \end{aligned}$$

The two end terms vanish by assumption and inductive hypothesis, and therefore so does the middle term.

Next, we show that $\text{Hom}_{\mathcal{D}(A,H)}(T^n(A \otimes W), X)$ vanishes for any indecomposable H -module W (W could be infinite-dimensional). The strategy is to filter W by finite-dimensional submodules, which we used in the proof of Lemma 2.3. Tensoring the short exact sequence there with A , we obtain a short exact sequence of B -modules

$$0 \longrightarrow \bigoplus_{i \in I} A \otimes W_i \xrightarrow{\text{Id}_A \otimes \Psi} \bigoplus_{i \in I} A \otimes W_i \longrightarrow A \otimes W \longrightarrow 0,$$

where each W_i is finite dimensional. Applying $\text{Hom}_{\mathcal{D}(A,H)}(-, X)$ to the corresponding distinguished triangle and using the previous step finishes this step.

Thirdly, we prove the vanishing of $\text{Hom}_{\mathcal{D}(A,H)}(T^n(P), X)$ for all P with property (P). Consider the following short exact sequence of $B = A\#H$ modules used in Lemma 6.4:

$$0 \longrightarrow \bigoplus_{r \in \mathbb{N}} F_r \xrightarrow{\Psi} \bigoplus_{s \in \mathbb{N}} F_s \longrightarrow P \longrightarrow 0.$$

An induction argument on q using the previous step shows that $\text{Hom}_{\mathcal{D}(A,H)}(T^n(F_r), X) = 0$ for all $r \in \mathbb{N}, n \in \mathbb{Z}$. Then applying $\text{Hom}_{\mathcal{D}(A,H)}(-, X)$ to the distinguished triangle associated with the above short exact sequence gives us a long exact sequence

$$\begin{aligned} \cdots \longrightarrow \prod_{s \in \mathbb{N}} \text{Hom}_{\mathcal{D}(A,H)}(T^n(F_s), X) &\longrightarrow \text{Hom}_{\mathcal{D}(A,H)}(T^n(P), X) \\ &\longrightarrow \prod_{r \in \mathbb{N}} \text{Hom}_{\mathcal{D}(A,H)}(T^{n+1}(F_r), X) \longrightarrow \cdots \end{aligned}$$

Both ends vanish and the claim follows

Finally, for any object $X \in \mathcal{D}(A, H)$, take its bar resolution $\mathbf{p}X \cong X$ (Corollary 6.7), where $\mathbf{p}X$ satisfies property (P). Then

$$\text{Hom}_{\mathcal{D}(A,H)}(X, X) \cong \text{Hom}_{\mathcal{D}(A,H)}(\mathbf{p}X, X),$$

and the right-hand side vanishes by the previous step. It follows that $\text{Id}_X \cong 0$ and $X \cong 0$, finishing the proof of the lemma. \square

Remark 7.7 (On the notion of generators). In the above proposition, we can equivalently take one compact generator $A \otimes W$ where W is a direct sum of simple H -modules, one from each of the isomorphism classes. Notice that, when H is a local Hopf algebra of *finite type*, we can replace condition (P3) of property (P) (Definition 6.3) with the equivalent requirement that $F_r/F_{r+1} \cong A$ instead. Here by finite type we mean that the set of isomorphism classes of indecomposable modules over H is finite. Indeed, in this case, the dimensions of indecomposable modules are bounded, and thus any direct sum of indecomposable H -modules V admits a finite step filtration whose subquotients are isomorphic to the trivial H -module. Therefore by refining the original filtration of condition (P3) by inducing this filtration on V , we obtain a new filtration whose subquotients are just isomorphic to the free module A (with appropriate grading shifts in the graded case). In particular, this allows us to see immediately that A generates $\mathcal{D}(A, H)$ in the stronger sense of Keller [Kel94, § 4.2]:

- The category ‘ $\mathcal{D}(A, H)$ is the smallest strictly³ full triangulated subcategory in itself which contains A and is closed under taking arbitrary direct sums and forming distinguished triangles’.

It is readily seen that this seemingly stronger version of generators implies the notion we used in Definition 7.5.

By contrast, for almost all finite-dimensional Hopf algebras H , the set of isomorphism classes of indecomposable H -modules may well be infinite, and there is in general no good parametrization of these isomorphism classes. Over such an H , it seems that the definition of property (P) using all indecomposable modules is more natural and fits the construction of the bar resolution we gave previously. Moreover, using the bar resolution, Proposition 7.6 shows that a natural set of compact generators is given by $\{A \otimes V\}$, where V ranges over the representatives of isomorphism classes of simple H -modules. Thus one might wonder whether in the generic case

³ A subcategory \mathcal{D}' of \mathcal{D} is called strictly full if any object of \mathcal{D} that is isomorphic to some object in \mathcal{D}' must itself be in \mathcal{D}' .

of H there would still be a similar relation between the two notions of generators. By a localization theorem of Thomason and Neeman, they are always equivalent.

COROLLARY 7.8. *The category $\mathcal{D}(A, H)$ is the smallest strictly full triangulated subcategory in itself that contains $\mathcal{G} = \{A \otimes V\}$ and is closed under taking arbitrary direct sums and forming distinguished triangles.*

Proof. The proof is just a corollary of the following theorem, where we take $R = T^{\mathbb{Z}}(\mathcal{G}) := \{T^n(G) \mid G \in \mathcal{G}, n \in \mathbb{Z}\}$, and \mathcal{G} is the set of compact generators we exhibited in Proposition 7.6. \square

THEOREM 7.9 (Thomason–Neeman). *Let \mathcal{D} be a compactly generated triangulated category. Let R be a set of compact objects of \mathcal{D} closed under the shift functor T of \mathcal{D} . Let \mathcal{R} be the smallest full subcategory of \mathcal{D} containing R and closed with respect to taking coproducts and forming triangles. Then we have the following.*

- (1) *The category \mathcal{R} is compactly generated by the set of generators R .*
- (2) *If R is also a set of generators for \mathcal{D} , then $\mathcal{R} = \mathcal{D}$.*
- (3) *The compact objects in \mathcal{R} equals $\mathcal{R}^c = \mathcal{D}^c \cap \mathcal{R}$. In particular, if R is closed under forming triangles and taking direct summands, it coincides with \mathcal{R}^c .*

Proof. This is part of [Nee96, Theorem 2.1]. \square

7.2 Compact modules

Brown representability theorem. We recall the notion of homotopy colimits in a triangulated category that admits infinite direct sums. Homotopy colimits are used in the construction of representable functors on the triangulated category (Brown’s representability theorem).

DEFINITION 7.10. Let \mathcal{D} be as before. Let $\{f_n : X_n \rightarrow X_{n+1} \mid n \in \mathbb{N}\}$ be a sequence of morphisms in \mathcal{D} . A homotopy colimit of this sequence is an object $X \in \mathcal{D}$ that fits into a distinguished triangle as follows:

$$\bigoplus_{n \in \mathbb{N}} X_n \xrightarrow{\Psi} \bigoplus_{n \in \mathbb{N}} X_n \longrightarrow X \longrightarrow T\left(\bigoplus_{n \in \mathbb{N}} X_n\right),$$

where Ψ is given by an infinite matrix similar as in Lemma 2.3. Notice that such X is unique up to isomorphisms in \mathcal{D} .

THEOREM 7.11 (Brown representability). *Let \mathcal{D} be a triangulated category that admits infinite direct sums. Suppose \mathcal{D} is compactly generated by a set of generators \mathcal{G} . A cohomological functor $F : \mathcal{D} \rightarrow (\mathbb{k}\text{-vect})^{\text{op}}$ is representable if and only if it commutes with direct sums. When representable, such an F is represented by the homotopy colimit of a sequence $\{f_r : X_r \rightarrow X_{r+1} \mid r \in \mathbb{N}\}$ where X_1 as well as the cone of any f_n is represented by a possibly infinite direct sum of objects of the form $T^n(G)$, with $G \in \mathcal{G}$ and $n \in \mathbb{Z}$.*

Proof. See [Nee96, Theorem 3.1]. \square

Characterizing compact modules. The fact that $\mathcal{D}(A, H)$ is compactly generated allows us to give an alternative characterization of compact hopfological modules as summands of iterated extensions of a finite number of free modules of the form $T^n(A \otimes V)$ where V belongs to the set of simple H -modules. The original idea of the proof is due to Ravenel [Rav84] and Neeman [Nee92], and a very readable account of the proof is given by Keller [Kel94, § 5.3], which we follow.

DEFINITION 7.12. Let \mathcal{D} be a triangulated category as above and \mathcal{U}, \mathcal{V} be two classes of objects of \mathcal{D} . Let $\mathcal{U} * \mathcal{V}$ be the class of objects X in \mathcal{D} that fit into a distinguished triangle of the form

$$G_1 \longrightarrow X \longrightarrow G_2 \longrightarrow T(G_1),$$

where $G_1 \in \mathcal{U}$ and $G_2 \in \mathcal{V}$. The lemma below says that the operation $*$ is associative, and therefore we can define unambiguously the class of length n objects generated by \mathcal{W} to be the class of objects in

$$\mathcal{W} * \mathcal{W} * \cdots * \mathcal{W},$$

where there are n copies of \mathcal{W} . We will refer to each object belonging to $\mathcal{W} * \mathcal{W} * \cdots * \mathcal{W}$ for some $n \in \mathbb{N}$ as a *finite extension of objects in \mathcal{W}* .

LEMMA 7.13. *The above operation $*$ is associative in the sense that the two classes of objects $(\mathcal{U} * \mathcal{V}) * \mathcal{W}, \mathcal{U} * (\mathcal{V} * \mathcal{W})$ coincide.*

Proof. We leave the proof as an exercise. □

THEOREM 7.14 (Ravenel–Neeman). *Let \mathcal{D} be a triangulated category compactly generated by a set of generators \mathcal{G} . Any compact object of \mathcal{D} is then a direct summand of a finite extension of objects of the form $T^n(G)$, where $G \in \mathcal{G}$ and $n \in \mathbb{Z}$.*

Proof. See [Kel94, Nee92, Rav84]. □

COROLLARY 7.15. *Let $\mathcal{D}^c(A, H)$ denote the strictly full subcategory of compact hopfological modules in $\mathcal{D}(A, H)$. It is triangulated and idempotent complete. Any $X \in \mathcal{D}^c(A, H)$ is a direct summand of an object which is a finite extension of modules in $T^{\mathbb{Z}}(\mathcal{G}) = \{T^n(A \otimes V)\}$, where $n \in \mathbb{Z}$ and V ranges over the set of representatives of isomorphism classes of simple H -modules. Furthermore, $\mathcal{D}^c(A, H)$ is the smallest strictly full triangulated subcategory of $\mathcal{D}(A, H)$ that contains \mathcal{G} which is closed under taking direct summands.*

Proof. Combine the previous theorem with Proposition 7.6. The last statement follows from Theorem 7.9. □

DEFINITION 7.16. Let A be an H -module algebra over a finite-dimensional Hopf algebra H over the base field \mathbb{k} . We define the Grothendieck group $K_0(\mathcal{D}^c(A, H))$ (or $K_0(A, H)$ for short) to be the abelian group generated by the symbols of isomorphism classes of objects in $\mathcal{D}^c(A, H)$, modulo the relations

$$[Y] = [X] + [Z],$$

whenever there is a distinguished triangle inside $\mathcal{D}^c(A, H)$ of the form

$$X \longrightarrow Y \longrightarrow Z \longrightarrow T(X).$$

Remark 7.17. Since $\mathcal{D}^c(A, H)$ is a (right) triangulated module-category over $H\text{-mod}$, on the Grothendieck group level, $K_0(\mathcal{D}^c(A, H))$ is a (right) module over $K_0(H\text{-mod})$. When H is cocommutative, $K_0(H\text{-mod})$ is a commutative ring and there is no need to distinguish right or left modules over it.

More generally, we can define higher K -groups of A by applying Waldhausen, Thomason and Trobaugh’s construction to $\mathcal{D}^c(A, H)$. We expect a large chunk of the K-theoretic results of Thomason and Trobaugh [TT07] and Schlichting [Sch06] to generalize to our case.

7.3 A useful criterion

As another application of the Thomason–Neeman theorem (Theorem 7.9) and the notion of compactly generated categories (Definition 7.5), we give a useful criterion concerning the fully-faithfulness of exact functors on a compactly generated triangulated category and natural transformations between these functors. Of course the main example of such categories we have in mind are the derived categories of H -module algebras. The criterion will be needed in the next section.

LEMMA 7.18. *Let $\mathcal{D}_1, \mathcal{D}_2$ be triangulated categories, let $F, F' : \mathcal{D}_1 \rightarrow \mathcal{D}_2$ be exact functors between them, and let $\mu : F \Rightarrow F'$ be a natural transformation of these functors. Suppose furthermore that \mathcal{D}_1 admits arbitrary direct sums and is compactly generated by a set of generators \mathcal{G} , and that F, F' commute with direct sums.⁴ Then the following statements hold.*

- (1) *The functor F is fully-faithful if F restricted to the full subcategory consisting of objects in $T^{\mathbb{Z}}(\mathcal{G}) := \bigcup_{n \in \mathbb{Z}} T^n(\mathcal{G})$ is fully faithful and $F(G)$ is compact for all $G \in \mathcal{G}$. The converse holds if F is essentially surjective on objects.⁵*
- (2) *The natural transformation μ is invertible if and only if $\mu(G) : F(G) \rightarrow F'(G)$ is invertible for all $G \in \mathcal{G}$.*

Proof. We prove the first part and leave the second as an exercise. Observe that the full subcategory consisting of objects X on which the functor F induces an isomorphism of vector spaces,

$$\text{Hom}_{\mathcal{D}_1}(T^n(G), X) \cong \text{Hom}_{\mathcal{D}_2}(F(T^n(G)), F(X)),$$

forms a strictly full triangulated subcategory of \mathcal{D}_1 . By the compactness assumption on $F(G)$, this subcategory contains arbitrary direct sums. Now Theorem 7.9 applies since \mathcal{D}_1 is compactly generated. The converse is true since if F is essentially surjective on objects, $F(G)$ is then automatically compact whenever G is. □

The following result is immediate.

COROLLARY 7.19. *Let $F : \mathcal{D}_1 \rightarrow \mathcal{D}_2$ be an exact functor between \mathbb{k} -linear triangulated categories which are compactly generated and admit arbitrary direct sums. Suppose F also commutes with direct sums. Let $\mathcal{G} = \{G\}$ be a set of compact generators for \mathcal{D}_1 . Then F induces an equivalence of triangulated categories if and only if when restricted to the full subcategory consisting of objects $T^{\mathbb{Z}}(\mathcal{G}) := \bigcup_{n \in \mathbb{Z}} T^n(\mathcal{G})$ it is fully-faithful, and $F(\mathcal{G}) := \{F(G) \mid G \in \mathcal{G}\}$ is a set of compact generators for \mathcal{D}_2 .*

8. Derived functors

In this section, we define the derived functors associated with hopfological bimodules. Then we proceed to prove a sufficient condition for two H -module algebras to be derived Morita equivalent. As a corollary, we discuss when a morphism of H -module algebras induces an equivalence of derived categories. The arguments we use are modeled on the DG case, as in the work of Keller [Kel94, § 6].

Throughout this section, we need to further assume that H is also a (co)commutative Hopf algebra. This condition is needed when we define a left module algebra structure on the opposite

⁴ This amounts to saying that $F(\bigoplus_{i \in I} X_i)$ is a direct sum object for $F(X_i), i \in I$ inside \mathcal{D}_2 although \mathcal{D}_2 may not admit arbitrary direct sums.

⁵ By ‘essentially surjective’ we mean that any object of \mathcal{D}_2 is isomorphic to an object in the image of F .

algebra A^{op} of a left module algebra A , and when dealing with derived functors and derived equivalences. We will make some further remarks on this assumption later.

8.1 The opposite algebra and tensor product

By the construction of $B = A\#H$, it is readily seen that the opposite algebra of B is isomorphic to the smash product ring $B^{\text{op}} = H^{\text{op, cop}}\#A^{\text{op}}$, where $H^{\text{op, cop}}$ denotes the Hopf algebra H with the opposite multiplication and opposite comultiplication. Therefore, A^{op} is naturally a right $H^{\text{op, cop}}$ -module algebra, or equivalently, a left H^{cop} -module algebra (H^{cop} becomes a Hopf algebra if we equip with it the antipode map S^{-1}). By our assumption H is (co)commutative, and we can naturally identify $H^{\text{cop}} \cong H$ ($S^{-1} = S$ in this case). Therefore, we have a left H -module algebra structure on A^{op} .

DEFINITION 8.1. Let H be a cocommutative Hopf algebra, and A be an H -module algebra as in the main example, Example 2.8. We define the *opposite H -module algebra* A^{op} to be the same H -module as A but with the opposite multiplication. An analogous definition applies when A, H are compatibly \mathbb{Z} -graded.

Example 8.2. We give an example showing the necessity of assuming H to be cocommutative. Consider an n -DG algebra A equipped with a differential d of degree one (see the second example of § 4). For any $a, b \in A$, we have

$$d(ab) = (da)b + \nu^{|a|}a(db),$$

where ν is an n th root of unity and $|a| \in \mathbb{Z}$ denotes the degree of a . As such, an algebra can be regarded as a graded module algebra over the Taft algebra H_n at the n th root of unity ν (see [Bic03] and the second example of § 3.4), which is non-commutative and non-cocommutative. Now in A^{op} , whose multiplication will be denoted by \circ , we have $a \circ b = \xi^{|b||a|}ba$, where we allow ξ to be some other n th root of unity. Then

$$\begin{aligned} d(a \circ b) &= d(\xi^{|b||a|}ba) = \xi^{|b||a|}((db)a + \nu^{|b|}b(da)) \\ &= \xi^{|b||a|}(\xi^{(|b|+1)|a|}a \circ (db) + \nu^{|b|}\xi^{(|a|+1)|b|}(da) \circ b) \\ &= \xi^{(2|b|+1)|a|}a \circ (db) + \nu^{|b|}\xi^{|b|(2|a|+1)}(da) \circ b. \end{aligned}$$

Compare this with the relation we need to make A^{op} differential graded: $d(a \circ b) = (da) \circ b + \eta^{|a|}a \circ (db)$. Now assuming A has non-zero terms in each degree, it is easy to see that in order to make these expressions equal, we need $\xi = \pm 1$ and $\nu = \xi^{-1}$. Thus it appears that the opposite algebra does not carry a natural n -DG structure if $\nu \neq \pm 1$.

DEFINITION 8.3. Let H be a cocommutative Hopf algebra. Let M be a left $A^{\text{op}}\#H$ -module and N be a left $A\#H$ -module. The tensor product space $M \otimes_A N$ is naturally an H -module by setting, for any $m \in M, n \in N$ and $h \in H$,

$$h(m \otimes n) := \sum (h_{(1)}m) \otimes (h_{(2)}n).$$

The H -module $M \otimes_A N$ is graded if H, A, M, N are compatibly graded.

One checks easily that, when H is cocommutative, we have an equivalence between the categories of right $A\#H$ -modules and the category of left $A^{\text{op}}\#H$ -modules. Indeed, for any right $A\#H$ -module M , we define the corresponding left $A^{\text{op}}\#H$ -module to be the same underlying H -module with the left A^{op} action given by $a \circ m := ma$, for any element $a \in A$ and $m \in M$. The compatibility of this left A^{op} -structure with the H -module structure is guaranteed by the cocommutativity of H .

Now, if M is a B -module which is finitely presented as an A -module (finitely generated if A is Noetherian), we have a canonical isomorphism of H -modules

$$\text{Hom}_A(M, N) \cong M^\vee \otimes_A N,$$

where M^\vee denotes the H -module $\text{Hom}_A(M, A)$, equipped with the right A -module structure from that of the target A . A similar identification holds in the graded case.

8.2 Derived tensor

Our first task is to define the derived tensor functor associated with a hopfological bimodule and determine when it induces an equivalence of derived categories. We will denote by A_1, A_2 two H -module algebras over a finite-dimensional (graded, super) cocommutative Hopf algebra H , and set $B_1 = A_1 \# H, B_2 = A_2 \# H$.

DEFINITION 8.4. Let A_1, A_2 be as above, and define their *tensor product H -module algebra* $A_1 \otimes A_2$ to be the usual tensor product of A_1, A_2 as a \mathbb{k} -vector space and the algebra structure given by

$$(a_1 \otimes a_2) \cdot (b_1 \otimes b_2) := (a_1 b_1) \otimes (a_2 b_2),$$

for any $a_1, b_1 \in A_1, a_2, b_2 \in A_2$. We equip it with the H -action that, for any $h \in H, a_1 \otimes a_2 \in A_1 \otimes A_2$,

$$h \cdot (a_1 \otimes a_2) := \sum h_{(1)} a_1 \otimes h_{(2)} a_2.$$

It is readily checked that $A_1 \otimes A_2$ indeed satisfies the axioms of an H -module algebra under the assumption that H is cocommutative.

Now let A_1, A_2 be as above and ${}_A X_{A_2}$ be an (A_1, A_2) hopfological bimodule, i.e., a module over the ring $(A_1 \otimes A_2^{\text{op}}) \# H$. We define the associated tensor and hom functors to be

$$\begin{aligned} {}_{A_1} X_{A_2} \otimes_{A_2} (-) : A_2\text{-mod} &\longrightarrow A_1\text{-mod}, & {}_{A_2} N &\mapsto {}_{A_1} X \otimes_{A_2} N, \\ \text{Hom}_{A_1}({}_{A_1} X_{A_2}, -) : A_1\text{-mod} &\longrightarrow A_2\text{-mod}, & {}_{A_1} M &\mapsto \text{Hom}_{A_1}({}_{A_1} X_{A_2}, M). \end{aligned}$$

In the above definition and what follows, we omit some of the subscripts whenever no confusion can arise. For instance, $\text{Hom}_{A_1}({}_{A_1} X_{A_2}, M) := \text{Hom}_{A_1}({}_{A_1} X_{A_2}, {}_{A_1} M)$. The natural left A_2 -module structure on the right-hand side is compatible with the H -action under the assumption that H is cocommutative. Therefore $\text{Hom}_{A_1}({}_{A_1} X_{A_2}, M) \in B_2\text{-mod}$, and more generally one easily checks that both maps above are compatible with the H -actions on the algebras and modules, thus inducing functors on the corresponding B -module categories. We leave the analogous statements and their verification in the graded case to the reader; their proofs are similar to the argument we use in the next lemma.

LEMMA 8.5. *The canonical adjunction between the tensor and hom functors in the above definition associated with the bimodule ${}_A X_{A_2}$,*

$$\text{Hom}_{A_1}(X \otimes_{A_2} N, M) \cong \text{Hom}_{A_2}(N, \text{Hom}_{A_1}(X_{A_2}, M)),$$

is an isomorphism of H -modules, functorial in M and N for any $M \in B_1\text{-mod}, N \in B_2\text{-mod}$. A similar statement holds in the graded case.

Proof. Recall that under the tensor-hom adjunction, we associate with any element $f \in \text{Hom}_{A_2}(N, \text{Hom}_{A_1}(X_{A_2}, M))$ the element of $\text{Hom}_{A_1}(X \otimes_{A_2} N, M)$, still denoted f , which sends $x \otimes n$ to $f(n)(x)$. On one hand, for any $h \in H, h \cdot f \in \text{Hom}_{A_2}(N, \text{Hom}_{A_1}(X_{A_2}, M))$

is given by

$$(h \cdot f)(-) = h_{(2)}f(S^{-1}(h_{(1)}) \cdot -) : N \longrightarrow \text{Hom}_{A_1}(X_{A_2}, M).$$

Thus, for any $n \in N$, $x \in X$, we have from the above assignment

$$(h \cdot f)(x \otimes n) = (h_{(2)} \cdot f(S^{-1}(h_{(1)}) \cdot n))(x) = h_{(3)}f(S^{-1}(h_{(1)}) \cdot n)(S^{-1}(h_{(2)}) \cdot x).$$

On the other hand, when regarding f as an element of $\text{Hom}_{A_1}(X \otimes_{A_2} N, M)$ using the adjunction, the H -action has the effect

$$\begin{aligned} (h \cdot f)(x \otimes n) &= h_{(2)}f(S^{-1}(h_{(1)}) \cdot (x \otimes n)) = h_{(3)}(f(S^{-1}(h_{(2)}) \cdot x \otimes S^{-1}(h_{(1)}) \cdot n)) \\ &= h_{(3)}(f(S^{-1}(h_{(1)}) \cdot n)(S^{-1}(h_{(2)}) \cdot x)). \end{aligned}$$

This shows that the two expressions are equal and the lemma follows. □

Taking stable invariants \mathcal{H} (Proposition 5.9) of the above canonical isomorphism gives us the corresponding adjunction in the homotopy categories.

COROLLARY 8.6. *The functors ${}_{A_1}X \otimes_{A_2} (-)$, $\text{Hom}_{A_1}(X_{A_2}, -)$ descend to adjoint functors in the homotopy category:*

$$\text{Hom}_{\mathcal{C}(A_1, H)}(X \otimes_{A_2} N, M) \cong \text{Hom}_{\mathcal{C}(A_2, H)}(N, \text{Hom}_{A_1}(X_{A_2}, M))$$

functorially in $M \in B_1\text{-mod}$, $N \in B_2\text{-mod}$. □

DEFINITION 8.7. Let ${}_{A_1}X_{A_2}$ be as above. We define the (left) derived tensor functor ${}_{A_1}X \otimes_{A_2}^{\mathbf{L}} (-)$ to be the composition,

$$\begin{array}{c} {}_{A_1}X \otimes_{A_2}^{\mathbf{L}} (-) : \mathcal{D}(A_2, H) \xrightarrow{\mathbf{p}} \mathcal{P}(A_2, H) \xrightarrow{X \otimes_{A_2} (-)} \mathcal{C}(A_1, H) \xrightarrow{Q} \mathcal{D}(A_1, H) \\ A_2 M \mapsto {}_{A_1}X \otimes_{A_2} \mathbf{p}M \end{array}$$

where \mathbf{p} is the functorial bar resolution of Corollary 6.7 and Q is the canonical localization functor.

PROPOSITION 8.8. *Let ${}_{A_1}X_{A_2}$, ${}_{A_1}Y_{A_2}$ be (A_1, A_2) hopfological bimodules, and let*

$$\mu : {}_{A_1}X_{A_2} \longrightarrow {}_{A_1}Y_{A_2}$$

be a map of hopfological bimodules. Then we have the following.

(1) *Suppose ${}_{A_1}X_{A_2}$ is cofibrant when regarded as a B_1 -module. The functor ${}_{A_1}X \otimes_{A_2}^{\mathbf{L}} (-)$ is an equivalence of categories if and only if $A_2 \longrightarrow \text{Hom}_{A_1}(X_{A_2}, X_{A_2})$ is a quasi-isomorphism, and $\{{}_{A_1}X \otimes V\}$, when regarded as left B_1 -modules, is a set of compact cofibrant generators $\mathcal{D}(A_1, H)$. Here V ranges over a finite set of representatives of isomorphism classes of simple H -modules.*

(2) *The map of bimodules μ induces an invertible natural transformation of functors*

$$\mu^{\mathbf{L}} : {}_{A_1}X \otimes_{A_2}^{\mathbf{L}} (-) \Rightarrow {}_{A_1}Y \otimes_{A_2}^{\mathbf{L}} (-)$$

if and only if μ is a quasi-isomorphism in $(A_1 \otimes A_2^{\text{op}})\#H\text{-mod}$.

Proof. The first statement of the proposition is a consequence of Corollary 7.19, provided we know that $\mathcal{D}(A_i, H)$ is compactly generated by the set of generators $\mathcal{G} = \{A_i \otimes V\}$, $i = 1, 2$ (Proposition 7.6). We check that under our assumption, the conditions of the corollary are satisfied. Since $T^n(M) \cong M \otimes W$ for some finite-dimensional H -module W (see Definition 3.1

and Proposition 3.2), we have, for any $A_2 \otimes V, A_2 \otimes V' \in \mathcal{G}$, which are property (P) modules,

$$\begin{aligned} \text{Hom}_{A_2}(T^n(A_2 \otimes V), T^m(A_2 \otimes V')) &\cong \text{Hom}_{A_2}(A_2 \otimes V \otimes W, A_2 \otimes V' \otimes W') \\ &\cong A_2 \otimes \text{Hom}_{\mathbb{k}}(V \otimes W, V' \otimes W') \xrightarrow{\alpha} \text{Hom}_{A_1}(X, X) \otimes \text{Hom}_{\mathbb{k}}(V \otimes W, V' \otimes W') \\ &\cong \text{Hom}_{A_1}(X \otimes V \otimes W, X \otimes V' \otimes W') \\ &\cong \text{Hom}_{A_1}(X \otimes_{A_2} (A_2 \otimes V \otimes W), X \otimes_{A_2} (A_2 \otimes V' \otimes W')), \end{aligned}$$

where α is a quasi-isomorphism by assumption. Since V, V', W and W' are finite dimensional, we can pull $\text{Hom}(V \otimes W, V' \otimes W')$ in and out of the A_1 -hom spaces. Taking stable invariants of the first and last hom-spaces shows that the morphism spaces in the derived categories are isomorphic as well (here we use the fact that ${}_{A_1}X$ is cofibrant), thereby establishing the fully-faithfulness of the tensor functor when restricted to $T^{\mathbb{Z}}(\mathcal{G})$. Furthermore, the hypothesis says that the modules ${}_{A_1}X \otimes_{A_2}^{\mathbf{L}} (A_2 \otimes V) \cong {}_{A_1}X \otimes_{A_2} (A_2 \otimes V) \cong {}_{A_1}X \otimes V$ for the V as in the assumption constitute a set of compact cofibrant generators of $\mathcal{D}(A_1, H)$. Finally, the functor commutes with direct sums since the tensor product does so.

For the second part, note that $X \otimes_{A_2} (A_2 \otimes V) \cong X \otimes V$ is quasi-isomorphic to $Y \otimes_{A_2} (A_2 \otimes V) \cong Y \otimes V$ for all simple H -modules V if and only if X is quasi-isomorphic to Y . Now use part (2) of Lemma 7.18. \square

COROLLARY 8.9. *Let ${}_{A_1}X_{A_2}$ be a hopfological bimodule and ${}_{A_1}(\mathbf{p}X)_{A_2}$ be its bar resolution in $(A_1 \otimes A_2^{\text{op}})\#H\text{-mod}$. Then $\mathbf{p}X \rightarrow X$ induces a canonical isomorphism of functors*

$${}_{A_1}X \otimes_{A_2}^{\mathbf{L}} (-) \cong {}_{A_1}(\mathbf{p}X) \otimes_{A_2} (-) : \mathcal{D}(A_2, H) \rightarrow \mathcal{D}(A_1, H).$$

Proof. By the previous result, we have an isomorphism of functors

$${}_{A_1}X \otimes_{A_2}^{\mathbf{L}} (-) \cong {}_{A_1}(\mathbf{p}X) \otimes_{A_2}^{\mathbf{L}} (-) : \mathcal{D}(A_2, H) \rightarrow \mathcal{D}(A_1, H).$$

To this end, it suffices to show that, if a bimodule ${}_{A_1}P_{A_2}$ has property (P), then $P_{A_2} \otimes_{A_2} M$ is quasi-isomorphic to $P_{A_2} \otimes_{A_2} \mathbf{p}M$ for any $M \in B_2\text{-mod}$.

Tensoring the short exact sequence of free $A_1 \otimes A_2^{\text{op}}$ -modules

$$0 \rightarrow \bigoplus_{r \in \mathbb{N}} F_r \rightarrow \bigoplus_{s \in \mathbb{N}} F_s \rightarrow P \rightarrow 0$$

we used in Lemma 6.4 with the bar resolution $\mathbf{p}M \rightarrow M$ and passing to the homotopy category $\mathcal{C}(A_1, H)$ (Lemma 4.3), we have a morphism of distinguished triangles.

$$\begin{array}{ccccccc} \bigoplus_{r \in \mathbb{N}} (F_r \otimes_{A_2} \mathbf{p}M) & \longrightarrow & \bigoplus_{s \in \mathbb{N}} (F_s \otimes_{A_2} \mathbf{p}M) & \longrightarrow & P \otimes_{A_2} \mathbf{p}M & \longrightarrow & T\left(\bigoplus_{r \in \mathbb{N}} (F_r \otimes_{A_2} \mathbf{p}M)\right) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \bigoplus_{r \in \mathbb{N}} (F_r \otimes_{A_2} M) & \longrightarrow & \bigoplus_{s \in \mathbb{N}} (F_s \otimes_{A_2} M) & \longrightarrow & P \otimes_{A_2} M & \longrightarrow & T\left(\bigoplus_{r \in \mathbb{N}} (F_r \otimes_{A_2} M)\right) \end{array}$$

Taking cohomology (passing to $H\text{-mod}$ via Res) and using the ‘two-out-of-three’ property of triangulated categories (see, for instance, [GM03, Corollary 4, §IV.1]), we are reduced to exhibiting the claimed property for each $F_r, r \in \mathbb{N}$. An induction argument on r further reduces us to the special case when $P = A_1 \otimes A_2 \otimes V$, which is easily seen to be true:

$$(A_1 \otimes A_2 \otimes V) \otimes_{A_2} \mathbf{p}M \cong A_1 \otimes V \otimes \mathbf{p}M \cong A_1 \otimes V \otimes M \cong (A_1 \otimes A_2 \otimes V) \otimes_{A_2} M,$$

where the first and last isomorphisms are that of modules, while the middle one is only a quasi-isomorphism. \square

COROLLARY 8.10. *Let A_1, A_2, A_3 be H -module algebras, and ${}_{A_1}X_{A_2}, {}_{A_2}Y_{A_3}$ be hopfological bimodules. Then there is an isomorphism of functors*

$${}_{A_1}X_{A_2} \otimes_{A_2}^{\mathbf{L}} ({}_{A_2}Y_{A_3} \otimes_{A_3}^{\mathbf{L}} (-)) \cong ({}_{A_1}Z_{A_3} \otimes_{A_3}^{\mathbf{L}} (-)) : \mathcal{D}(A_3, H) \longrightarrow \mathcal{D}(A_1, H),$$

where ${}_{A_1}Z_{A_3} = {}_{A_1}(\mathbf{p}X) \otimes_{A_2} Y_{A_3}$ and ${}_{A_1}(\mathbf{p}X)_{A_2}$ stands for the bar resolution of X as an (A_1, A_2) -bimodule.

Proof. The corollary follows easily from Corollary 8.9. \square

8.3 Derived hom

We next focus on the derived hom functor and exhibit a derived version of the adjunctions Lemma 8.5 and Corollary 8.6.

DEFINITION 8.11. Let ${}_{A_1}X_{A_2}$ be a hopfological bimodule as before. Let $\mathbf{p}X$ be the bar resolution of X as a left B_1 -module. By our construction, $\mathbf{p}X = \mathbf{p}A_1 \otimes_{A_1} X$ is also a right B_2 -module. We define the derived hom functor $\mathbf{RHom}_{A_1}(X_{A_2}, -)$ to be the composition,

$$\begin{aligned} \mathbf{RHom}_{A_1}(X_{A_2}, -) : \mathcal{D}(A_1, H) &\xrightarrow{\text{Hom}_{A_1}(\mathbf{p}X, -)} \mathcal{C}(A_2, H) \xrightarrow{Q} \mathcal{D}(A_2, H) \\ {}_{A_1}M &\mapsto \text{Hom}_{A_1}((\mathbf{p}X)_{A_2}, M). \end{aligned}$$

The next lemma guarantees that $\text{Hom}_{A_1}(\mathbf{p}X, -)$ is well defined on the derived category $\mathcal{D}(A_1, H)$.

LEMMA 8.12. *If ${}_{A_1}\tilde{X}_{A_2}$ has property (P) as a left B_1 -module, then $\text{Hom}_{A_1}(\tilde{X}_{A_2}, K)$ is an acyclic B_2 -module whenever $K \in B_1\text{-mod}$ is acyclic. Consequently, $\mathbf{RHom}_{A_1}(\tilde{X}_{A_2}, -)$ descends to a functor:*

$$\mathbf{RHom}_{A_1}(\tilde{X}_{A_2}, -) : \mathcal{D}(A_1, H) \longrightarrow \mathcal{D}(A_2, H).$$

Likewise, the result holds when the property (P) assumption is replaced by cofibrance in the statement.

Proof. The proof is similar to that of Corollary 8.9. Consider the short exact sequence of B_1 -modules $0 \longrightarrow \bigoplus_{r \in \mathbb{N}} F_r \longrightarrow \bigoplus_{s \in \mathbb{N}} F_s \longrightarrow \tilde{X} \longrightarrow 0$ associated with \tilde{X} . Since each $F_r, r \in \mathbb{N}$, and \tilde{X} are free as A_1 -modules, applying $\text{Hom}_{A_1}(-, K)$ yields a short exact sequence of B_2 -modules:

$$0 \longrightarrow \text{Hom}_{A_1}(\tilde{X}, K) \longrightarrow \prod_{s \in \mathbb{N}} \text{Hom}_{A_1}(F_s, K) \longrightarrow \prod_{r \in \mathbb{N}} \text{Hom}_{A_1}(F_r, K) \longrightarrow 0.$$

Thus it suffices to show that $\text{Hom}_{A_1}(F_r, K)$ is acyclic for each $r \in \mathbb{N}$. An induction on r further reduces us to the case of free modules of the form $A_1 \otimes N$ where N is some indecomposable H -module. This case now follows from Lemma 2.3 since $\text{Hom}_{A_1}(A_1 \otimes N, K) \cong \text{Hom}_{\mathbb{k}}(N, K)$.

The last claim follows readily from the first part of the lemma and Corollary 6.8. \square

Remark 8.13. More generally, it is easy to see that $\mathbf{RHom}_{A_1}(-, -)$ is a bifunctor

$$\mathbf{RHom}_{A_1}(-, -) : \mathcal{D}(A_1 \otimes A_2^{\text{op}}, H)^{\text{op}} \times \mathcal{D}(A_1, H) \longrightarrow \mathcal{D}(A_2, H).$$

In particular, when $A_2 \cong \mathbb{k}$, we have a bifunctor,

$$\mathbf{RHom}_{A_1}(-, -) : \mathcal{D}(A_1, H)^{\text{op}} \times \mathcal{D}(A_1, H) \longrightarrow H\text{-mod}.$$

There is another derived Hom-space one can associate with any two hopfological modules M and N , namely the space of chain maps up to homotopy $\mathcal{H}(\text{Hom}_A(M, N))$. By Proposition 5.9 and the remark that follows it, this is the space of (stable) invariants in $\text{Hom}_A(\mathbf{p}M, N)$, and thus it usually contains less information than the $\mathbf{R}\text{Hom}$ above. Another reason that we use the definition above is that it satisfies the right adjunction property with the derived tensor product functor, as shown in the next lemma. Notice that in the DG case, i.e., $H = \mathbb{k}[d]/(d^2)$, the natural map of HOM-spaces $\mathbf{R}\text{HOM}_A(M, N) \xrightarrow{\mathcal{H}} \text{HOM}_{\mathcal{C}(A, H)}(M, N)$ is an isomorphism since the only stably non-zero modules are the graded shifts of the trivial module \mathbb{k}_0 .

LEMMA 8.14. *The functor $\mathbf{R}\text{Hom}(X, -)$ is right adjoint to $X \otimes_{A_1}^{\mathbf{L}} (-)$ as functors between $\mathcal{D}(A_i, H)$, $i = 1, 2$.*

Proof. Notice that $\mathbf{p}X \otimes_{A_2} N$ has property (P) as a B_1 -module whenever $N \in B_2\text{-mod}$ does (check for $N = A_2 \otimes V$). Therefore if $M \in B_1\text{-mod}$ and $N \in B_2\text{-mod}$, we have

$$\begin{aligned} \text{Hom}_{\mathcal{D}(A_1, H)}(X \otimes_{A_2}^{\mathbf{L}} N, M) &\cong \text{Hom}_{\mathcal{D}(A_1, H)}(\mathbf{p}X \otimes_{A_2} \mathbf{p}N, M) \cong \text{Hom}_{\mathcal{C}(A_1, H)}(\mathbf{p}X \otimes_{A_2} \mathbf{p}N, M) \\ &\cong \text{Hom}_{\mathcal{C}(A_2, H)}(\mathbf{p}N, \text{Hom}_{A_1}(\mathbf{p}X, M)) \\ &\cong \text{Hom}_{\mathcal{D}(A_2, H)}(N, \mathbf{R}\text{Hom}_{A_1}(X, M)). \end{aligned}$$

Here the first isomorphism holds by Corollary 8.9; the second holds since $\mathbf{p}X \otimes_{A_1} \mathbf{p}N$ has property (P) so that we can use Corollary 6.10; the third holds by adjunction Corollary 8.6 in the homotopy category, while the fourth holds by Corollary 6.10 and the definition of $\mathbf{R}\text{Hom}$. \square

DEFINITION 8.15. Let ${}_{A_1}X_{A_2}$ be a hopfological bimodule as before. We define its A_1 -dual to be

$${}_{A_2}\check{X}_{A_1} := \text{Hom}_{A_1}(\mathbf{p}X_{A_2}, A_1),$$

where its left A_2 structure is inherited from the right A_2 -module structure of $\mathbf{p}X$, while the right A_1 structure comes from that of A_1 .

Notice that there is a canonical map

$${}_{A_2}\check{X} \otimes_{A_1}^{\mathbf{L}} M \cong \text{Hom}_{A_1}(\mathbf{p}X, A_1) \otimes_{A_1} M \longrightarrow \text{Hom}_{A_1}(\mathbf{p}X, M) \cong \mathbf{R}\text{Hom}_{A_1}(X, M),$$

which is an isomorphism whenever M is of the form $A_1 \otimes V$ for any finite-dimensional H -module V .

PROPOSITION 8.16. *If $X \otimes_{A_1}^{\mathbf{L}} (-) : \mathcal{D}(A_1, H) \longrightarrow \mathcal{D}(A_2, H)$ is an equivalence, its quasi-inverse is given by ${}_{A_2}\check{X}_{A_1} \otimes^{\mathbf{L}} (-) : \mathcal{D}(A_2, H) \longrightarrow \mathcal{D}(A_1, H)$.*

Proof. By the adjunction lemma, Lemma 8.14, if $X \otimes_{A_1}^{\mathbf{L}} (-)$ is an equivalence, its quasi-inverse is given by $\mathbf{R}\text{Hom}_{A_1}(X_{A_2}, -)$. Therefore $\mathbf{R}\text{Hom}_{A_1}(X_{A_2}, -)$ commutes with direct sums, and the corollary now follows from part (2) of Lemma 7.18 and the observation we made before this proposition. \square

8.4 A special case

We specialize the previous results to the case of H -module algebras $\phi : A_2 \longrightarrow A_1$, and the bimodule ${}_{A_1}X_{A_2} := {}_{A_1}A_1A_2$. Here the right A_2 -module structure on A_1 is realized via the morphism ϕ , i.e., $a_1 \cdot a_2 := a_1\phi(a_2)$ where $a_i \in A_i$, $i = 1, 2$.

DEFINITION 8.17. We define the *induction functor*

$$\phi^* : \mathcal{D}(A_2, H) \longrightarrow \mathcal{D}(A_1, H), \phi^*(M) := A_1 \otimes_{A_2}^{\mathbf{L}} M$$

and the restriction functor

$$\phi_* : \mathcal{D}(A_1, H) \longrightarrow \mathcal{D}(A_2, H), \phi_*(N) := \mathbf{R}\mathrm{Hom}_{A_1}(A_2, N) \cong_{A_2} N.$$

Note that $\mathbf{R}\mathrm{Hom}_{A_1}(A_1 A_2, N) \cong \mathrm{Hom}_{A_1}(A_1 A_2, N) \cong_{A_2} N$ where A_2 acts on N via the morphism ϕ . The first isomorphism holds since A_1 has property (P) as a left B_1 -module.

The derived adjunction (Lemma 8.14) gives us

$$\mathrm{Hom}_{\mathcal{D}(A_2, H)}(\phi^*(N), M) \cong \mathrm{Hom}_{\mathcal{D}(A_1, H)}(N, \phi_*(M)).$$

We have the following immediate corollary, concerning when a morphism of H -module algebras induces a derived equivalence of their module categories. The result in the DG case is already proven in [BL94, Theorem 10.12.5.1].

COROLLARY 8.18. *Let $\phi : A_2 \longrightarrow A_1$ be a quasi-isomorphism of H -module algebras. Then the induction and restriction functors,*

$$\begin{aligned} \phi^* : \mathcal{D}(A_2, H) &\longrightarrow \mathcal{D}(A_1, H), \\ \phi_* : \mathcal{D}(A_1, H) &\longrightarrow \mathcal{D}(A_2, H), \end{aligned}$$

are mutually-inverse equivalences of triangulated categories.

COROLLARY 8.19. *Let A be a left H -module algebra. Then $\mathcal{D}(A, H) \cong 0$ if and only if there exists an element $x \in A$ such that*

$$\Lambda \cdot x = 1.$$

Furthermore, if x is central in A , $\mathcal{C}(A, H) \cong 0$.

Proof. One shows that, as in the proof of Lemma 5.4, under the assumption, the H -module map $A \xrightarrow{\lambda_A} A \otimes H$ now admits an H -module retract, defined as

$$A \otimes H \longrightarrow A, \quad a \otimes h \mapsto (h \cdot r_x)(a),$$

where the $r_x : A \longrightarrow A$ is the right multiplication on A by x . The rest of the proof is left as an exercise to the reader. □

9. Special examples

In this section, we apply the previous results to a very special class of H -module algebras on which H acts trivially. As a consequence we deduce that the Grothendieck group for the ground field $K_0(\mathcal{D}^c(\mathbb{k}, H))$ coincides with $K_0(H\text{-mod})$.

9.1 Variants of derived categories

First off, we introduce the analogue of the usual notion of the bounded derived category in the hopfological case. For simplicity, we will only do this when the H -module algebra A is Noetherian. Since H is finite dimensional, $B = A \otimes H$ is a finite A -module, and therefore the Noetherian condition on A is equivalent to that on B .

DEFINITION 9.1. Let A be a Noetherian H -module algebra. The bounded derived category $\mathcal{D}^b(A, H)$ is the strictly full subcategory of $\mathcal{D}(A, H)$ consisting of objects which are isomorphic to some finitely generated A -module.

Likewise, define the finite derived category $\mathcal{D}^f(A, H)$ to be the strictly full subcategory of $\mathcal{D}(A, H)$ consisting of objects which are isomorphic to some finite length A -module.

Notice that if A is finite dimensional, the two notions $\mathcal{D}^b(A, H)$ and $\mathcal{D}^f(A, H)$ coincide with each other. In any case, it is readily seen that there is an embedding $\mathcal{D}^f(A, H) \subset \mathcal{D}^b(A, H)$, and there is always a bifunctorial pairing

$$\mathcal{D}^c(A, H) \times \mathcal{D}^f(A, H) \longrightarrow \mathcal{D}^f(\mathbb{k}, H), \quad (P, M) \mapsto \mathbf{R}\mathrm{Hom}_A(P, M),$$

where the category $\mathcal{D}^f(\mathbb{k}, H) \subset H\text{-mod}$ is just the bounded (also finite) derived category of \mathbb{k} .

DEFINITION 9.2. Let A be a Noetherian H -module algebra. We define the bounded Grothendieck group of A , denoted $G_0(\mathcal{D}^b(A, H))$ (or $G_0(A, H)$ for short) to be the abelian group generated by the symbols of isomorphism classes of objects in $\mathcal{D}^b(A, H)$, modulo the relations

$$[Y] = [X] + [Z]$$

whenever there is a distinguished triangle inside $\mathcal{D}^b(A, H)$ of the form

$$X \longrightarrow Y \longrightarrow Z \longrightarrow T(X).$$

Likewise, we define the finite Grothendieck group $G_0^f(A, H) := G_0(\mathcal{D}^f(A, H))$ in an analogous fashion.

9.2 Smooth basic algebras

Now we exhibit a class of examples where the Grothendieck groups $K_0(A, H)$ can be recovered from the usual Grothendieck group $K_0(A)$.

DEFINITION 9.3. Let A be an (graded) artinian algebra over a ground field \mathbb{k} . We say that A is *basic in its Morita equivalence class* if all simple modules over A are one-dimensional over \mathbb{k} .

Equivalently, A is basic in its Morita equivalence class if and only if $A/J(A) \cong \mathbb{k} \times \cdots \times \mathbb{k}$, where $J(A)$ is the (graded) Jacobson radical. Here the number of copies of \mathbb{k} equals the number of isomorphism classes of simple A -modules, or, equivalently, that of indecomposable projective A modules.

DEFINITION 9.4. A \mathbb{k} -algebra A is called *smooth* if it has a finite projective resolution as an (A, A) -bimodule.

In this section we mainly focus on the class of (graded finite-dimensional) smooth, basic artinian algebras. Some examples of such algebras are provided by the path algebras over oriented quivers without oriented cycles. In fact, such path algebras are hereditary and have length one (i.e., two-term) projective resolutions as bimodules over themselves. In what follows, we will abbreviate the above hypothesis on our algebra A by simply saying that A is a *smooth basic algebra*, meaning that it is artinian (or graded finite-dimensional), smooth, and basic in its Morita equivalence class. We will regard such an A as an H -module algebra by letting H act trivially on it. Notice that a B -module may carry some non-trivial H -action.

LEMMA 9.5. *Let A be an H -module algebra with H acting trivially on it, and let P be a finitely generated projective A -module with trivial H action. Then, given any finite-dimensional H -module V , $P \otimes V$ is cofibrant in $B\text{-mod}$.*

Proof. It suffices to show that $A \otimes V$ is cofibrant since in this situation P is a direct summand of A^n (with trivial H -module structure) for some $n \in \mathbb{N}$. The cofibrance of $A \otimes V$ is clear. \square

By the characterization of compact modules in $\mathcal{D}(A, H)$ (Corollary 7.15), compact cofibrant modules are direct summands of free modules in the derived category. When A is artinian, the

direct summand can be taken in the abelian category $B\text{-mod}$, as shown in the next result. Note that here we do not assume the H -action on A is trivial.

LEMMA 9.6. *Let A be an artinian H -module algebra and $M \in \mathcal{D}^c(A, H)$ be a compact object. Then M is isomorphic to a finite projective A -module in the derived category.*

Proof. A direct summand of a finitely generated free A -module P in the derived category is given by an endomorphism $e : P \rightarrow P$ such that $e^2 = e$ in $\text{Hom}_{\mathcal{D}(A, H)}(P, P) = \text{Hom}_{\mathcal{C}(A, H)}(P, P)$. Therefore, by Lemma 5.4, $e^2 - e = \Lambda \cdot f$ for some $f \in \text{Hom}_A(P, P)$. By the artinian assumption, the endomorphism algebra of a free module is finite dimensional. Using the classical Fitting’s lemma,⁶ we can decompose P into a direct sum of B -modules (since $\Lambda \cdot f$ is a map of B -modules),

$$P \cong \text{Im}(\Lambda \cdot f)^N \oplus \text{Ker}(\Lambda \cdot f)^N,$$

for N sufficiently large. Here $\Lambda \cdot f$ acts as an automorphism on $\text{Im}(\Lambda \cdot f)^N$, and it acts on $\text{Ker}(\Lambda \cdot f)^N$ nilpotently. We may remove the summand $\text{Im}(\Lambda \cdot f)^N$ since it is contractible by Corollary 8.19. Note that $\text{Ker}(\Lambda \cdot f)^N$ is still a projective A -module. Now $\Lambda \cdot f$ is nilpotent on $\text{Ker}(\Lambda \cdot f)^N$ and we may lift the idempotent e easily using Newton’s method, which we leave to the reader as an exercise (see [Ben91, Theorem 1.7.3]). \square

Therefore, $\mathcal{D}^c(A, H)$ consists of modules which are images of finitely generated, projective A -modules under the localization map. We now look at these modules more closely.

LEMMA 9.7. *Let A be a smooth basic algebra, and M be a finitely dimensional B -module. Then M is quasi-isomorphic to some finite-dimensional projective A -module.*

Proof. Since A is smooth, it has a finite projective (A, A) -bimodule resolution $P_\bullet \rightarrow A \rightarrow 0$. Now, as in the bar construction of Theorem 6.6, we can lift this resolution to a hopfological resolution $\tilde{P}_\bullet \rightarrow A$, since the differentials in the chain complex are (trivially) H -module maps. Now, for each finite-dimensional B -module M , we tensor this complex with M to obtain $\tilde{P}_\bullet \otimes_A M \rightarrow M \rightarrow 0$. The total space of $\tilde{P}_\bullet \otimes_A M$ is finite-dimensional since P_\bullet , A and M are. It is also cofibrant by Lemma 9.5. The claim follows. \square

PROPOSITION 9.8. *If A is smooth basic, then there is an equivalence of triangulated categories*

$$\mathcal{D}^c(A, H) \cong \mathcal{D}^f(A, H).$$

Proof. Lemma 9.6 shows that any compact module is isomorphic to a finite-dimensional projective A -module. Since $\mathcal{D}^f(A, H)$ is, by definition, strictly full, there is an inclusion functor $\mathcal{D}^c(A, H) \subset \mathcal{D}^f(A, H)$. On the other hand, any object in $\mathcal{D}^f(A, H)$, being isomorphic to some finite-dimensional module, has a finite cofibrant replacement by the previous Lemma 9.7. Hence the inclusion functor is essentially surjective. The proposition follows. \square

The following corollary is immediate by taking $A = \mathbb{k}$ in the above proposition.

COROLLARY 9.9. *Under the canonical isomorphism $\mathcal{D}(\mathbb{k}) \cong H\text{-mod}$, $\mathcal{D}^c(\mathbb{k})$ is isomorphic to the strictly full subcategory of $H\text{-mod}$ which consists of objects that are quasi-isomorphic to finite-dimensional H -modules.*

When A is artinian, the \mathbf{RHom} -pairing

$$\mathbf{RHom}(-, -) : \mathcal{D}^c(A, H) \times \mathcal{D}^f(A, H) \rightarrow H\text{-mod}$$

⁶ See, for instance, Benson [Ben91, Lemma 1.4.4] for the form of the lemma that is used here.

descends to the Grothendieck groups

$$[\mathbf{R}\mathrm{Hom}_A(-, -)] : K_0(A, H) \times G_0(A, H) \longrightarrow K_0(H\text{-mod}).$$

Write $R := K_0(H\text{-mod})$ for the moment. Notice that if V is a finite-dimensional H -module algebra, and P, M are B -modules, there is a canonical isomorphism of H -modules

$$\mathrm{Hom}_A(P \otimes V, M) \cong \mathrm{Hom}_A(P, M \otimes V^*) \cong \mathrm{Hom}_A(P, M) \otimes V^*.$$

On the Grothendieck group level, this says that the pairing above is sesquilinear in the sense that it is linear in the second argument, and $*$ -linear in the first argument, where $* : R \longrightarrow R, [V] \mapsto [V^*]$ is an involution of the ring R .

PROPOSITION 9.10. *Let A be a smooth basic algebra. Then there is an isomorphism of Grothendieck groups:*

$$K_0(\mathcal{D}^c(A, H)) \cong K_0(A) \otimes_{\mathbb{Z}} K_0(H\text{-mod}),$$

where $K_0(A)$ denotes the usual Grothendieck group of the algebra A . Likewise, when A is graded finite dimensional,

$$K_0(\mathcal{D}^c(A, H)) \cong K_0(A) \otimes_{\mathbb{Z}[q, q^{-1}]} K_0(H\text{-mod}).$$

Proof. Let $\{P_i, i = 1, \dots, n\}$ and $\{S_j, j = 1, \dots, n\}$ be a complete list of isomorphism classes of indecomposable projective and simple A -modules respectively, and $R = K_0(H\text{-mod})$. Lemma 9.6 says that $K_0(A, H)$ as an R -module is generated by the symbols $[P_i], i = 1, \dots, n$. In the usual $K_0(A)$, $\{[P_i] \mid i = 0, \dots, n\}$ forms a basis. Thus it suffices to show that the symbols $[P_i]$ are linearly independent over R in $K_0(A, H)$. To do this, we use the above sesquilinear pairing

$$[\mathbf{R}\mathrm{Hom}_A(-, -)] : K_0(A, H) \times K_0(A, H) \longrightarrow R.$$

Here we identify K_0 with G_0 using the previous Proposition 9.8. Since A is basic, we have

$$\mathrm{Hom}_A(P_i, S_j) = \begin{cases} \mathbb{k} & i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Since P_i is cofibrant (Lemma 9.5), $\mathrm{Hom}_A(P_i, S_j) \cong \mathbf{R}\mathrm{Hom}_A(P_i, S_j)$ (Lemma 8.12). Hence the sesquilinear pairing is perfect and $\{[P_i] \mid i = 1, \dots, n\}$ forms an R -basis of $K_0(A, H)$. The graded analogue is proved in a similar way using the pairing $\mathbf{R}\mathrm{HOM}_A$, and the proposition follows. \square

In the special case when $A = \mathbb{k}$, the proposition says that $K_0(H\text{-mod})$ is just the Grothendieck group of the ground field.

COROLLARY 9.11. *We have an isomorphism of abelian groups:*

$$K_0(\mathbb{k}, H) \cong K_0(H\text{-mod}).$$

Remark 9.12. When the ring A is a commutative algebra, the usual tensor product of A -modules descends to an internal tensor product on $\mathcal{D}^c(A, H)$. On the Grothendieck group level, it turns $K_0(A, H)$ into a ring (not necessarily commutative). The above corollary can then be strengthened into an isomorphism of rings. We leave the details to the reader.

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