Canad. Math. Bull. Vol. 36 (1) 1993, pp. 96–102

ON A MAXIMALITY PROPERTY OF PARTITION REGULAR SYSTEMS OF EQUATIONS

HANNO LEFMANN AND HAMZA SI KADDOUR

ABSTRACT. In this note we will study the following problem. For a given partition regular system of equations, which equations can be added to this system without introducing new variables, such that the new augmented system is again partition regular. It turns that the Hindman system on finite sums as well as the Deuber-Hindman system on finite sums of (m, p, c)-sets are maximal in this sense.

1. **Introduction.** An important topic in Ramsey theory are partition regular systems of equations, *cf.* [GRS 80]. These systems show certain regularities with respect to colorings of the set N of positive integers with a finite number of colors, namely, independent of the particular coloring one can always find a monochromatic solution:

DEFINITION. Let $A \in Z^{u \times v}$, $1 \leq u, v \leq \omega$, be a $u \times v$-matrix with entries from the set Z of integers. A system of equations $A\mathbf{x} = \mathbf{0}$, $\mathbf{x} = (x_1, x_2, \ldots, x_v)^T$, is *partition regular* if and only if for every positive integer r and for every coloring $\Delta: N \longrightarrow r$ of the positive integers with r colors, $r = \{0, 1, \ldots, r-1\}$, there exists a monochromatic solution of $A\mathbf{x} = \mathbf{0}$, *i.e.* there exist positive integers $x_1, x_2, \ldots, x_v$ such that $A(x_1, x_2, \ldots, x_v)^T = \mathbf{0}$ and $\Delta(x_1) = \Delta(x_2) = \cdots = \Delta(x_v)$.

For finite matrices $A \in Z^{u \times v}$, where u, v are positive integers, a complete characterization of all partition regular systems of equations was given by Rado [Ra 33]. Concerning infinite matrices, only partial results are known. The first such infinite partition regular system was given by Hindman:

THEOREM 1 ([Hin 74]). *Let r be a positive integer. Then for every coloring* $\Delta: N \longrightarrow r$ *of the positive integers with r colors there exist infinitely many positive integers* $x_1, x_2, \ldots$ *which are with all their nonempty finite sums colored the same, i.e. the system* $\langle \sum_{i \in I} x_i = x_I \mid I \subset \{1, 2, \ldots\}, I$ *finite and nonempty* $\rangle$ *is partition regular.*

Later, Deuber and Hindman proved in [DH 87] that the system given by all finite sums of (m, p, c)-sets is partition regular. It is not known how far this system is away from a maximal partition regular system of equations with denumerably many finite equations, *cf.* [De 89]. In order to get more insight to what extent these two systems could be maximal, we will study the question, whether new equations can be added

This work has been done during the second author's visit at the SFB 343, Universität Bielefeld.
Received by the editors November 5, 1991.
AMS subject classification: 05A99.
Key words and phrases: Ramsey Theory, infinite sums, (m, p, c)-sets.

https://doi.org/10.4153/CMB-1993-014-2 Published online by Cambridge University Press

to the Hindman-system, resp. the Deuber-Hindman system—without introducing new variables—such that the augmented system is again partition regular. It will turn out, that both systems are maximal in this sense:

MAIN THEOREM. *If no new variables are introduced, then the Hindman-system and the Deuber-Hindman system are maximal.*

2. **Finite sums.** Extending earlier results of Schur [Sch 16] on the partition regularity of the equation $x + y = z$ and of van der Waerden [vdW 27] on arithmetic progressions, Rado gave in [Ra 33] a complete characterization of all finite partition regular systems of equations. His characterization is of an algebraic nature, given by a certain property of the underlying matrix:

DEFINITION. Let u, v be positive integers. Let $A = (\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_v}) \in Z^{u\times v}$ be a $u \times v$-matrix with integral entries and with columns $\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_v}$.

The matrix A has the *columns property* if and only if there exists a partition $\{1, 2, \ldots, v\} = I_0 \dot\cup I_1 \dot\cup \cdots \dot\cup I_m$ of the set of column indices of A such that the following is valid:

(i) $\sum_{i\in I_0} \mathbf{a_i} = \mathbf{0}$, *i.e.* the sum of all columns in class I_0 add up to the zero-vector $\mathbf{0}$, and

(ii) for every positive integer j, $1 \leq j \leq m$, the sum $\sum_{i\in I_j} \mathbf{a_i}$ of all columns in I_j is a rational linear combination of all columns $\mathbf{a_k}$, $k \in I_0 \cup I_1 \cup \cdots \cup I_{j-1}$, in former classes.

Rado showed that the columns property is central for the characterization of partition regular systems of equations:

THEOREM 2 ([Ra 33]). *Let u, v be positive integers and let $A \in Z^{u\times v}$ be a matrix with integral entries. Then the following statements are equivalent:*

(i) The system $A\mathbf{x} = \mathbf{0}$ is partition regular.

(ii) The matrix A has the columns property.

We will show next that the Hindman-system is maximal with respect to adding new equations. Before doing so, we have to make the notion of maximality precise:

DEFINITION. Let $A \in Z^{u\times v}$, $1 \leq u, v \leq \omega$, be a matrix, such that the system $A\mathbf{x} = \mathbf{0}$, $\mathbf{x} = (x_1, x_2, \ldots, x_v)^T$, is partition regular. Then the system $A\mathbf{x} = \mathbf{0}$ is *maximal* if and only if for every choice of positive integers $\alpha_1, \alpha_2, \ldots, \alpha_v$ the following is valid:

If the combined system $\langle A\mathbf{x} = \mathbf{0}, \sum_{i=1}^{v} \alpha_i x_i = 0\rangle$ is partition regular, then the vector $(\alpha_1, \alpha_2, \ldots, \alpha_v)$ is an element of the rowspace of A over the rationals, *i.e.* there is a rational linear combination of some equations of the system $A\mathbf{x} = \mathbf{0}$, which gives exactly $\sum_{i=1}^{v} \alpha_i x_i = 0$.

THEOREM 3. *The Hindman-system*

$$\left\langle \sum_{i\in I} x_i = x_I \mid I \subset \{1, 2, \ldots\}, I \textit{ finite and nonempty} \right\rangle$$

on finite sums is maximal.

PROOF. Assume to the contrary that there exists an equation $\sum_{j\in J} \alpha_j x_j = 0$, $\alpha_j \in Z$ for all $j \in J$ (where each $j \in J$ is a nonempty finite subset of $\{1, 2, \ldots\}$), such that the system

$$(1) \qquad \Big\langle \sum_{i\in I} x_i = x_I, \sum_{j\in J} \alpha_j x_j = 0 \mid I \subset \{1, 2, \ldots\}, I \text{ finite and nonempty} \Big\rangle$$

is partition regular. Notice, that each $j \in J$ is either an element i of the set $\{1, 2, \ldots\}$ or a finite subset I of $\{1, 2, \ldots\}$. Suppose that for some $j \in J$ it is $j = I$ for some finite set I with $|I| \geq 2$. Then by (1) we know that $x_j = x_I = \sum_{i\in I} x_i$. Hence by evaluating all such x_j's as the corresponding sum, we can assume that $j \in \{1, 2, \ldots\}$ for all $j \in J$. We will show next that $\alpha_j = 0$ for all $j \in J$ which implies that the Hindman-system is maximal. Consider the following subsystem of the system (1)

$$(2) \qquad \begin{aligned} \sum_{j\in J} \alpha_j x_j &= 0 \\ -x_J + \sum_{j\in J} x_j &= 0. \end{aligned}$$

This is a finite system, hence Theorem 2 applies. By assumption the system (1) is partition regular, hence the same applies to (2). As (2) is a finite system, Theorem 2 gives that the corresponding matrix must have the columns property. In particular, some columns have to add up to the zero-vector. This implies that $\alpha_{j_0} = 0$ for some element $j_0 \in J$. Now consider the subsystem of (1) given by

$$\begin{aligned} \sum_{j\in J\setminus\{j_0\}} \alpha_j x_j &= 0 \\ -x_{J\setminus\{j_0\}} + \sum_{j\in J\setminus\{j_0\}} x_j &= 0. \end{aligned}$$

apply again Theorem 2 and iterate. This gives finally $\alpha_j = 0$ for all $j \in J$. Hence the original equation $\sum_{j\in J} \alpha_j x_j = 0$ was a rational linear combination of equations from (1) and therefore the matrix A is maximal. ■

The proof of Theorem 3 shows also that it is not relevant that the Hindman-system is infinite, hence we have the following

COROLLARY 1. *Let k be a positive integer. Then the Hindman-system $\langle \sum_{i\in I} x_i = x_I \mid I \subseteq \{1, 2, \ldots, k\}, I \neq \emptyset \rangle$ is maximal.*

3. **Finite sums of (m, p, c)-systems.** Rado's theorem gives a complete characterization of all finite partition regular systems of equations in terms of a certain property of the underlying matrices, namely the columns property. In [De 73] Deuber made a new approach by characterizing these systems through their solutions. For this he introduced the concept of (m, p, c)-sets:

DEFINITION. Let m, p, c be positive integers. A finite subset $X \subset N$ of the set of positive integers is an *(m, p, c)-set* if and only if there exist positive integers $x_0, x_1, \ldots, x_m$ serving as generators such that X consists of all elements occurring in the following list:

$$\begin{array}{r} cx_0 + \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_m x_m \\ cx_1 + \lambda_2 x_2 + \cdots + \lambda_m x_m \\ \vdots \\ cx_m \end{array}$$

with integers λ_i, $-p \leq \lambda_i \leq p$ for each $i = 1, 2, \ldots, m$.

For example, a $(1, p, c)$-set consists of an arithmetic progression with $(2p + 1)$ terms together with c-times the difference of the arithmetic progression.

The importance of (m, p, c)-sets is due to the fact, that they approximate the solutions of finite partition regular systems of equations:

THEOREM 4 ([De 73]). *Let u, v be positive integers and let $A \in Z^{u \times v}$ be a matrix with integral coefficients. Then the following statements are equivalent:*

(i) The system $A\mathbf{x} = \mathbf{0}$ is partition regular.

(ii) The matrix A has the columns property.

(iii) There exists a triple (m, p, c) of positive integers such that every (m, p, c)-set contains a solution of the system $A\mathbf{x} = \mathbf{0}$.

Hence, (m, p, c)-sets give universal approximations from above of the solutions of finite partition regular systems of equations. Moreover, for a given triple (m, p, c) of positive integers there exists a partition regular system $A\mathbf{x} = \mathbf{0}$ such that every solution of this system contains an (m, p, c)-set, namely the system given by

$$(3) \qquad \left\langle cx_i + \sum_{j=i+1}^{m} \lambda_j x_j = cx_{(c,i,\lambda_{i+1},\lambda_{i+2},\ldots,\lambda_m)} \mid i = 0, 1, \ldots, m, -p \leq \lambda_j \leq p \right\rangle.$$

It is well known that if in a triple (m, p, c) at least one of the values of m, p is ω, then the corresponding system (3) is no longer partition regular. This can be easily seen as follows. In case $p = \omega$ consider a coloring of N with two colors, which colors the intervals $[2^i, 2^i - 1]$such that elements in the same interval are colored the same, but the elements of both adjacent intervals are colored differently from the elements in the middle interval. In case $m = \omega$, take again a coloring of N with two colors, defined in the following way: write every positive integer in base 3-representation, and color according to the last non-zero digit.

On the other hand, these (m, p, c)-sets also turned out to be handy for infinite partition regular systems of equations. In [DH 87] Deuber and Hindman proved that, roughly said, that replacing each variable x_i in the Hindman-system by an (m_i, p_i, c_i)-set results in a partition regular system of equations:

THEOREM 5 ([DH 87]). *Let r be a positive integer and let (m_i, p_i, c_i), $i < \omega$, be an enumeration of all triples of positive integers. Then for every coloring $\Delta: N \longrightarrow r$ there exists a family $\{X_{(m_i,p_i,c_i)} \mid i < \omega\}$ of pairwise disjoint (m_i, p_i, c_i)-sets $X_{(m_i,p_i,c_i)}$ such that the following is valid:*

https://doi.org/10.4153/CMB-1993-014-2 Published online by Cambridge University Press

All finite sums $\sum x_j$ are colored the same, where $x_j \in \bigcup_{i<\omega} X_{(m_i,p_i,c_i)}$ for each j and no two different x_j's belong to the same set $X_{(m_i,p_i,c_i)}$.

The corresponding system of equations can be described as follows. Take all pairwise independent systems corresponding to (m_i, p_i, c_i)-sets as in (3) together with all finite sum systems $\sum_{j\in J} x_j = x_J$, where the variables x_j are variables from the (m_i, p_i, c_i)-systems, but no two x_j's in one such equation belong to the same (m_i, p_i, c_i)-set equation. The Hindman-system is a subsystem of this Deuber-Hindman system.

THEOREM 6. *The Deuber-Hindman system on finite sums is maximal.*

PROOF. Assume to the contrary that the Deuber-Hindman system is not maximal. Then there exists an equation

$$\sum_{j\in J} \alpha_j x_j = 0, \tag{4}$$

$\alpha_j \in J$ for all $j \in J$, which can be added to the Deuber-Hindman system, without introducing new variables, such that the augmented system is again partition regular.

Assume first that the variables x_j occur in pairwise different (m, p, c)-sets. Then we know that for every subset $I \subseteq J$, $|I| \geq 2$, the system

$$-x_I + \sum_{i\in I} x_i = 0 \tag{5}$$

is included in the Deuber-Hindman system. Now as in the proof of Theorem 3 it follows for the system determined by (4) and (5) that $\alpha_j = 0$ for all $j \in J$.

Suppose now that there are at least two x_j's which belong to the same (m, p, c)-set. Let the variables of the Deuber-Hindman system have indices describing their positions in the corresponding (m, p, c)-set, *cf.* (3). Hence, for $x_j = x^{(m_k,p_k,c_k)}_{(c_k,i,\lambda_{i+1},\ldots,\lambda_{m_k})}$ we have by assumption

$$x^{(m_k,p_k,c_k)}_{(c_k,i,\lambda_{i+1},\ldots,\lambda_{m_k})} = x_i^{(m_k,p_k,c_k)} + \sum_{l=i+1}^{p_k} \frac{\lambda_l}{c_k} x_l^{(m_k,p_k,c_k)}. \tag{6}$$

Evaluating (4) according to this representation we can assume w.l.o.g. that for every $j \in J$ there are i, m_k, p_k, c_k such that $x_j = x_i^{(m_k,p_k,c_k)}$.

Partition the set $X_J = \{x_j \mid j \in J\} = J_0 \dot\cup J_1 \dot\cup \cdots \dot\cup J_t$ such that $x_j, x_{j'}$ are in the same set J_i if and only if $x_j, x_{j'}$ belong to the same (m_i, p_i, c_i)-set for some $i < \omega$. Define MIN J_i = min $\{l \mid x_l^{(.)} \in J_i\}$, where *min* denotes the usual minimum.

For every $0 \leq i \leq t$ let $J_i = \{x_0^{J_i}, x_1^{J_i}, \ldots, x_{r_i}^{J_i}\}$, where $x_{MINJ_i} = x_0^{J_i}$. By renumbering the α_j's appropriately, (4) becomes

$$\sum_{i=0}^{t} \sum_{l=0}^{r_i} \alpha_l^{J_i} x_l^{J_i} = 0.$$

Consider the following matrix A

$$\begin{matrix} 0 & 0 & \alpha_0^{J_0} & \alpha_1^{J_0} & \alpha_2^{J_0} & \cdots & \alpha_{r_0}^{J_0} & \alpha_0^{J_1} & \alpha_1^{J_1} & \cdots & \alpha_{r_1}^{J_1} & \cdots & \alpha_0^{J_t} & \alpha_1^{J_t} & \cdots & \alpha_{r_t}^{J_t} \\ 0 & -1 & 1 & \frac{1}{c_0} & \frac{1}{c_0} & \cdots & \frac{1}{c_0} & 1 & \frac{1}{c_1} & \cdots & \frac{1}{c_1} & \cdots & 1 & \frac{1}{c_t} & \cdots & \frac{1}{c_t} \\ -1 & 0 & 1 & -\frac{1}{c_0} & -\frac{1}{c_0} & \cdots & -\frac{1}{c_0} & 1 & -\frac{1}{c_1} & \cdots & -\frac{1}{c_1} & \cdots & 1 & -\frac{1}{c_t} & \cdots & -\frac{1}{c_t} \end{matrix}.$$

Notice that, as $p \geq 1$, all equations given by the last two rows of this matrix, are already included in the Deuber-Hindman system. Moreover, $c_i > 0$ for $1 \leq i \leq t$. Let the two leftmost columns be $\mathbf{a_1}, \mathbf{a_2}$ and let $\mathbf{a_j^{J_i}}$ denote the column $(\alpha_j^{J_i}, \frac{1}{c_i}, -\frac{1}{c_i})^T$. This matrix is finite. By assumption this matrix has the columns property, hence some columns have to add up to the zero-vector. Thus there exist subsets $K_i \subseteq \{0, 1, \ldots, r_i\}$ and a subset $I \subseteq \{1, 2\}$ such that $\sum_{i \in I} \mathbf{a_i} + \sum_{i=0}^t \sum_{l \in K_i} \mathbf{a_l^{J_i}} = \mathbf{0}$. By considering the second row of the matrix it follows either that $0 \notin K_i$ for every i, $1 \leq i \leq t$, or that $\bigcup_{i=0}^t K_i = \{\alpha_0^{J_l}\}$ for some l. Now the third row of the matrix decides the situation and gives that $0 \in K_l$ for some l and $I = \{1, 2\}$. This implies immediately $\alpha_0^{J_l} = 0$.

Now proceed by induction, where the next element appearing in the (m_l, p_l, c_l)-set, say $\alpha_j^{J_l}$ takes over the role of $\alpha_0^{J_l}$. Notice that, by the richness of the Deuber-Hindman system we again obtain a finite matrix which is similar to the matrix A. Finally, we get $\alpha_i^{J_j} = 0$ for all possible values of i, j which implies that the Deuber-Hindman system is maximal and finishes the proof of Theorem 6. ∎

Notice that the same argument could be applied to the system describing an (m, p, c)-set. Therefore, (m, p, c)-sets are in this sense maximal:

COROLLARY 2. *Let m, p, c be positive integers. Then the system*

$$\left\langle cx_i + \sum_{j=i+1}^{m} \lambda_j x_j = cx_{(c,i,\lambda_{i+1},\lambda_{i+2},\ldots,\lambda_m)} \mid i = 0, 1, \ldots, m, -p \leq \lambda_j \leq p \right\rangle$$

describing (m, p, c)-sets, is maximal.

REFERENCES

[De 73] W. Deuber, *Partitionen und lineare Gleichungssysteme*, Math. Z. **133**(1973), 109–123.

[De 89] ———, *Developments based on Rado's dissertation "Studien zur Kombinatorik"*, in: Proceedings 12th Brithish Combinatorial Conference, Norwich, 1989, 52–74.

[DH 87] W. Deuber and N. Hindman, *Partitions and sums of (m, p, c)-sets*, J. Combin. Th. (A) (1987), 300–302.

[GRS 80] R. L. Graham, B. L. Rothschild and J. H. Spencer, *Ramsey theory*, John Wiley, New York, 1980.

[Hin 74] N. Hindman, *Finite sums from sequences within cells of a partition of N*, J. Combin. Th. (A) **17**(1974), 1–11.

[Ra 33] R. Rado, *Studien zur Kombinatorik*, Math. Z. **36**(1933), 424–480.

[Sch 16] I. Schur, *Über die Kongruenz* $x^m + y^m \equiv z^m \pmod{p}$, Jber. Dt. Math. - Verein, 1916, 114–117.
[vdW 27] B. L. van der Waerden, *Beweis einer Baudetschen Vermutung*, Nieuw Arch. Wisk. **15**(1927), 212–216.

Fakultät für Mathematik
SFB 343
Universität Bielefeld
Postfach 8640
W-4800 Bielefeld 1
Germany

Départment de Mathématiques
Université Claude Bernard
Lyon 1 - 43 Bd - du 11 Novembre
1918, 69622 Villeurbanne
France