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POINTWISE COMPACT SPACES

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1. **Introduction.** In 1962, J. M. G. Fell [5] indicated the important role played by certain topological spaces which, though locally compact in a specialized sense, do not, in general, satisfy even the weakest separation axiom. He called them "locally compact". These were called "punktal kompakt" by Flachsmeyer [6] and to avoid confusion, we shall call them *pointwise compact spaces.*

The purpose of the paper is to study these spaces in relation to the exponential law, the product of two K-spaces, and the product of two quotient maps. We begin with a characterization of K-spaces, which generalizes a known result for the Hausdorff case [8, p. 241]. We prove the exponential law for pointwise compact spaces. This theorem, which generalizes the original theorem of R. H. Fox, was stated by H. Poppe [10, p. 120], but his proof presupposes the theory of convergence spaces. Applying the exponential law we prove a product theorem for K-spaces, one of whose factors is pointwise compact. This generalizes the original theorem of Cohen [3, p. 79] and a more general version stated by Michael [9, p. 281]. Finally we obtain two results in quotient maps for pointwise compact spaces, generalizing previous results of Cohen [4, p. 220]. The terminology and facts used without specific reference are those of Kelley [8].

2. K-**Spaces.** Let $X=(X, \tau)$ be a topological space. The *k-extension* of τ is the family $k(\tau)$ of all subsets U of X such that $U \cap K$ is open in K for every compact subset K of X. It is clear that $k(\tau)$ is a topology on X which is larger than τ. Also, if K is a τ-compact subset of X then $\tau=k(\tau)$ on K, and therefore (X, τ) and $(X, k(\tau))$ have the same compact subsets. A topological space (X, τ) is a *K-space* if $k(\tau)=\tau$ [3, p. 79]. It is known that every locally compact space (X, τ) is a K-space. In fact, let $U \in k(\tau)$, $x \in U$, and let W be a compact neighbourhood of x. Since $(X-U) \cap W$ is closed in W and does not contain x, there is a neighbourhood V of x such that $(X-U) \cap W \cap V=\phi$. This proves that $U \in \tau$.

Let X, Y be topological spaces. A function $f: X \to Y$ is *k-continuous* if its restriction to each compact subset of X is continuous [2, p. 245]. Henceforth the family of all k-continuous functions on X to Y will be denoted by $C_k(X, Y)$, and the subfamily of all such functions which are continuous will be denoted by $C(X, Y)$. The following characterization is known when X is Hausdorff [8, exercise 7.K].

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2.1 Theorem. *A topological space* (X, τ) *is a K-space if and only if* $C_k(X, Y)=C(X, Y)$ *for every topological space* Y.

Proof. Let X be a K-space, and let Y be any topological space. Let $f\in C_k(X, Y)$. If U is open in Y, then, for every compact subset K of X, $f^{-1}(U)\cap K$ is open in K, and therefore $f^{-1}(U)\in k(\tau)=\tau$. Consequently, $f\in C(X, Y)$.

Suppose that $C_k(X, Y)=C(X, Y)$ for every topological space Y. Let $f:(X, \tau)\to (X, k(\tau))$ be the identity map. Since $\tau=k(\tau)$ on every compact subset of X, f is k-continuous. Thus f is continuous, so that τ is larger than $k(\tau)$.

3. **Exponential law.** A topological space is called *pointwise compact* if, for every point, the neighbourhood filter has a base consisting of compat neighbourhoods. A locally compact space which is Hausdorff or regular is pointwise compact [8, p. 146]. There are pointwise compact spaces which are neither Hausdorff nor regular [5, p. 475]: Let $X=[0, 1]\cup\{2\}$ and let a subset U of X be called open if $U\cap[0, 1]$ is open in $[0, 1]$, with respect to the usual topology, and, in the case $2\in U$, U contains the open interval $(0, \varepsilon)$ for some $\varepsilon>0$. Then X is a topological space which is pointwise compact and non-Hausdorff. Since X is T_1, X is non-regular. There are locally compact spaces which are not pointwise compact: Let Q^* be the one-point compactification of the rational line; it is clear that the compact space Q^* is not pointwise compact. Thus a topological space may be locally compact Hausdorff, locally compact regular, pointwise compact, or locally compact. These classes are in the order of inclusion, and the examples show that the inclusions are proper.

Let X, Y, Z be non-empty sets. The map $\omega:(f, y)\to f(y)$ on $Z^Y\times Y$ to Z is called the *evaluation map*. An element f of $Z^{X\times Y}$ determines the function $\tilde{f}:x\to f(x, \circ)$ on X to Z^Y. The map $\mu:f\to\tilde{f}$ is a bijection of $Z^{X\times Y}$ onto $(Z^Y)^X$, called the *exponential map*. The restrictions of these maps to subsets will be denoted by the same symbols.

The following lemma, called the *partial exponential law* for the compact open topology τ_c, is essentially the lemma 1 of R. H. Fox [7, p. 430].

3.1 Lemma. *If* X, Y *and* Z *are topological spaces, then* $\mu(C(X\times Y, Z))\subseteq C(X, (C(Y, Z), \tau_c))$.

The following theorem and its corollary generalize the theorem 1 of R. H. Fox [7, p. 430], the theorem 2 of R. Arens [1, p. 482], respectively:

3.2 Theorem. *Let* X, Y *and* Z *be topological spaces. If* Y *is pointwise compact, then* $\mu(C(X\times Y, Z))=C(X, (C(Y, Z), \tau_c))$.

Proof. Because of 3.1, it remains to show that if $f\in Z^{X\times Y}$ is such that $\tilde{f}:X\to (C(Y, Z), \tau_c)$ is continuous, then f is continuous. Let W be an open subset of Z; it must be shown that $f^{-1}(W)$ is open. Let $(x_0, y_0)\in f^{-1}(W)$. Since $\tilde{f}(x_0)\in C(Y, Z)$ and Y is pointwise compact, there is a compact neighbourhood K of y_0 such that $\tilde{f}(x_0)(K)\subseteq W$. Then $[K, W]=\{h:h\in C(Y, Z)$ and $h(K)\subseteq W\}$ is a neighbourhood of

$\tilde{f}(x_0)$ in $(C(Y, Z), \tau_c)$. Since $\tilde{f}$ is continuous, there is a neighbourhood U of x_0 such that $\tilde{f}(U)\subseteq [K, W]$. Then $U\times K$ is a neighbourhood of (x_0, y_0) contained in $f^{-1}(W)$.

COROLLARY. *If H is a family of continuous functions on a pointwise compact space Y to a topological space Z, then τ_c is jointly continuous on H.*

Proof. Let $\omega: H\times Y\to Z$ be the evaluation map. Since $\tilde{\omega}: H\to(C(Y, Z), \tau_c)$ is the inclusion map (because $\tilde{\omega}(f)=\omega(f, \circ)=f$), it is continuous. By the theorem, ω is continuous.

4. Product theorem.

4.1 THEOREM. *If X is a K-space and Y is a pointwise compact space, then $X\times Y$ is a K-space.*

Proof. Let Z be an arbitrary topological space; by 2.1, it suffices to show that $C_k(X\times Y, Z)=C(X\times Y, Z)$. Let $f\in C_k(X\times Y, Z)$ and let K' be a compact subset of Y. Then, for all $x\in X$, $f\mid(\{x\}\times K')$ is continuous. Since $j(y)=(x, y)(y\in K')$ is a homeomorphism of K' onto $\{x\}\times K'$, the restriction $f(x, \circ)\mid K'=(f\mid(\{x\}\times K'))\circ j$ is continuous. We have shown that, for all $x\in X$, $\tilde{f}(x)=f(x, \circ)$ is k-continuous; that is, $\tilde{f}$ maps X into $C_k(Y, Z)$. But since Y is a K-space, $C_k(Y, Z)=C(Y, Z)$, so $\tilde{f}$ maps X into $C(Y, Z)$.

We will show that $\tilde{f}\in C_k(X, (C(Y, Z), \tau_c))$. Let K be an arbitrary compact subset of X and let Q be an open subset of $(C(Y, Z), \tau_c)$. It must be shown that $(\tilde{f}\mid K)^{-1}(Q)=\tilde{f}^{-1}(Q)\cap K$ is open in K. We may suppose that Q is of the form $Q=[K', U]=\{h: h\in C(Y, Z) \text{ and } h(K')\subseteq U\}$, where K' is a compact subset of Y and U is an open subset of Z. Let $x_0\in\tilde{f}^{-1}(Q)\cap K$, so that $x_0\in K$ and $f(x_0, \circ)\in Q$. Because of the form of Q, $\{x_0\}\times K'\subseteq f^{-1}(U)$, and therefore $\{x_0\}\times K'\subseteq f^{-1}(U)\cap(K\times K')$. Since $f\in C_k(X\times Y, Z)$, $f^{-1}(U)\cap(K\times K')$ is open in $K\times K'$. By the theorem of Wallace [8, p. 142], there is a neighbourhood N of x_0 in K such that $N\times K'\subseteq f^{-1}(U)\cap(K\times K')\subseteq f^{-1}(U)$. Let $x\in N$. Then, for any $y\in K'$, we have $\tilde{f}(x)(y)=f(x, y)\in U$; therefore $\tilde{f}(x)(K')\subseteq U$. But, as shown in the first paragraph, $\tilde{f}(x)\in C(Y, Z)$, therefore $\tilde{f}(x)\in Q$. Thus $x\in\tilde{f}^{-1}(Q)\cap K$ for all $x\in N$, that is, $N\subseteq\tilde{f}^{-1}(Q)\cap K$, proving that $\tilde{f}^{-1}(Q)\cap K$ is open in K.

Since X is a K-space, we have $\tilde{f}\in C_k(X, (C(Y, Z), \tau_c))=C(X, (C(Y, Z), \tau_c))$. Then, since Y is pointwise compact, 3.2 implies that $f\in C(X\times Y, Z)$, and the proof is complete.

COROLLARY 1. ([3, p. 79]). *Let X, Y be K-spaces whose compact sets are regular, one of which (say Y) is locally compact. Then $X\times Y$ is a K-space.*

In this result of Cohen, local compactness is understood to mean the existence for each point of a neighbourhood with compact closure. Thus the hypothesis implies the regularity of Y, and in particular, the pointwise compactness of Y.

COROLLARY 2. *If X is a K-space and Y is a locally compact space which is regular or Hausdorff, then $X\times Y$ is a K-space.*

5. **Quotient maps.** A surjection $f:X\to Y$ of a topological space X onto a topological space Y is a *quotient map* if a subset V of Y is open in Y if and only if $f^{-1}(V)$ is open in X. A subset U of X is saturated (with respect to f) if $U=f^{-1}(f(U))$. Thus a surjection $f:X\to Y$ is a quotient map if and only if it is continuous and the image of every saturated open subset of X is open. It is easily verified that the composition of two quotient maps is a quotient map. The following theorem and its corollary generalize the theorems 1.4, 1.5, respectively, of Cohen [4, p. 220].

5.1 THEOREM. *If X is a pointwise compact space, then the Cartesian product $1_X\times f$ is a quotient map whenever f is a quotient map.*

Proof. Let $f: Y\to Z$, $h=1_X\times f$, and let W be an open subset of $X\times Y$ which is saturated with respect to h. Since h is a continuous surjection, it remains to show that $h(W)$ is open in $X\times Z$. Let $(x_0, z_0)\in h(W)$, and let $y_0\in Y$ be such that $f(y_0)=z_0$ and $(x_0, y_0)\in W$. Let $N=\{x:x\in X$ and $(x, y_0)\in W\}$, so that N is a neighbourhood of x_0. Since X is pointwise compact, there is a compact neighbourhood U of x_0 contained in N.

Let $V=\{y:y\in Y$ and $U\times\{y\}\subseteq W\}$. We will show that V is an open subset of Y which is saturated with respect to f. In fact, let $y\in V$, so that $U\times\{y\}\subseteq W$. By the theorem of Wallace, there is a neighbourhood M of y such that $U\times M\subseteq W$. Then $M\subseteq V$, so V is open. Since $U\times V\subseteq W$, $U\times f^{-1}(f(V))=h^{-1}(h(U\times V))\subseteq h^{-1}(h(W))=W$, and therefore $f^{-1}(f(V))\subseteq V$, so V is saturated with respect to f.

Since f is a quotient map, $f(V)$ is open in Z, so $\mathring{U}\times f(V)$ is open in $X\times Z$. Since $y_0\in V$ and $U\times V\subseteq W$, $(x_0, z_0)\in\mathring{U}\times f(V)=h(\mathring{U}\times V)\subseteq h(W)$, proving that $h(W)$ is open in $X\times Z$.

COROLLARY. *If $f_1:X_1\to Y_1, f_2:X_2\to Y_2$ are quotient maps and X_1, Y_2 are pointwise compact, then $f_1\times f_2$ is a quotient map.*

Proof. Since $f_1\times f_2=(f_1\times 1_{Y_2})\circ(1_{X_1}\times f_2)$, and $f_1\times 1_{Y_2}$, $1_{X_1}\times f_2$ are quotient maps, $f_1\times f_2$ is a quotient map.

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