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On the topological stable rank of certain transformation group C^*-algebras

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Abstract. We consider the crossed product or transformation group C^*-algebras arising from actions of the group of integers on a totally disconnected compact metrizable space. Under a mild hypothesis, we give a necessary and sufficient dynamical condition for the invertibles in such a C^*-algebra to be dense. We also examine the property of residual finiteness for such C^*-algebras.

1. *Introduction*

Throughout this paper, we let X denote a compact, totally disconnected metrizable space. Let $C(X)$ denote the C^*-algebra of continuous complex-valued functions on X. We say that a subset E of X is clopen if it is both closed and open, and we use χ_E to denote the characteristic function of E. We define a partition of X to be a finite collection of pairwise disjoint clopen subsets whose union is all of X. Given a partition $\mathscr{P}$ of X, we let $C(\mathscr{P})$ denote the finite-dimensional C^*-subalgebra of $C(X)$ which is generated by $\{\chi_E \mid E \in \mathscr{P}\}$. Since the clopen sets generate the topology of X, $\bigcup C(\mathscr{P})$ is dense in $C(X)$, where the union is taken over all partitions $\mathscr{P}$.

We let ϕ be a homeomorphism of X. We obtain a *-automorphism, also denoted by ϕ, of $C(X)$ by defining $\phi(f) = f \circ \phi^{-1}$, for each f in $C(X)$. We will consider the crossed product C^*-algebra $C(X)x_\phi\mathbb{Z}$ obtained from the action of the group of integers on $C(X)$ generated by ϕ. (See 7.6.5 of Pedersen [**4**].) It is generated as a C^*-algebra by $C(X)$ and a unitary operator, which we denote by $u_{X,\phi}$ or by u when no confusion will arise, satisfying $ufu^* = \phi(f)$, for all f in $C(X)$.

We refer the reader to Blackadar [**1**] for standard facts regarding K-theory for C^*-algebras.

First of all, the K-theory of $C(X)$ is computable since it is an AF C^*-algebra (Effros [**3**] or Blackadar [**1**]). Specifically, $K_1(C(X)) = 0$ and $K_0(C(X))$ is isomorphic to $C(X, \mathbb{Z})$, the group of continuous $\mathbb{Z}$-valued functions on X, with positive cone the non-negative functions. Secondly, we may apply the Pimsner–Voiculescu six-term exact sequence [**6**] to completely determine the K-theory of $C(X)x_\phi\mathbb{Z}$ (also see Blackadar [**1**]). We summarize the results in the following theorem.

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THEOREM 1.1. (i) *Let* i *denote the inclusion of* $C(X)$ *in* $C(X)x_\phi\mathbb{Z}$. *Then* $i_*: K_0(C(X)) \to K_0(C(X)x_\phi\mathbb{Z})$ *is surjective and its kernel is the image of* id-ϕ_*, *considered as an endomorphism of* $K_0(C(X))$.
(ii) *If* X *contains no nontrivial clopen* ϕ*-invariant subsets, then* $K_1(C(X)x_\phi\mathbb{Z}) \cong \mathbb{Z}$ *and is generated by the class of* u.

A subset of X is said to be minimal for ϕ if it is a minimal non-empty, closed, ϕ-invariant set. We let S^1 denote the circle and M_m denote the C^*-algebra of m by m matrices.

In this paper, we wish to determine the topological stable rank of the C^*-algebra $C(X)x_\phi\mathbb{Z}$ (denoted tsr $(C(X)x_\phi\mathbb{Z})$). We refer the reader to Rieffel [**8**] for a complete treatment of the topic, but we note that from 7.4 of [**8**], tsr $(C(X)x_\phi\mathbb{Z})$ is either 1 or 2. Also from [**8**], we see that our original problem reduces to the question of whether or not the invertible elements of $C(X)x_\phi\mathbb{Z}$ are dense.

We first consider the case when X itself is minimal for ϕ in § 2. We show that given $a_1, a_2, \ldots, a_n$ in $C(X)x_\phi\mathbb{Z}$ and $\varepsilon > 0$, there is a unital C^*-subalgebra A of $C(X)x_\phi\mathbb{Z}$ and $a_1', a_2', \ldots, a_n'$ in A such that $\|a_i - a_i'\| < \varepsilon$, for $i = 1, \ldots, n$, and A is $*$-isomorphic to

$$[C(S^1) \otimes M_{J_1}] \oplus M_{J_2} \oplus \cdots \oplus M_{J_K}$$

for some integers $J_1, J_2, \ldots, J_K$.

This result is of interest in its own right (compare with the definition of *AF*-algebra), but as a corollary we show that the invertibles are dense in $C(X)x_\phi\mathbb{Z}$, when X is minimal.

In § 3, we turn to the general case. (Actually, we impose the very mild restriction that X have no nontrivial clopen ϕ-invariant subsets.) We obtain a necessary and sufficient condition on (X, ϕ) for the invertibles in $C(X)x_\phi\mathbb{Z}$ to be dense; namely, that there is exactly one minimal set for ϕ. We also show that these conditions are actually equivalent to $C(X)x_\phi\mathbb{Z}$ having the cancellation property for finitely generated projective modules (see Rieffel [**8**]).

Finally, in § 4, we examine the question of finiteness and residual finiteness of $C(X)x_\phi\mathbb{Z}$. Using the work of Pimsner [**5**], we produce a necessary and sufficient condition on (X, ϕ) for the C^*-algebra $C(X)x_\phi\mathbb{Z}$ to be residually finite. We also produce a somewhat surprising example of a residually finite $C(X)x_\phi\mathbb{Z}$ where the invertibles are not dense.

2. *The minimal case*

Here, we deal with the case when X is minimal for ϕ.

THEOREM 2.1. *Suppose that* X *is miminal for* ϕ. *Given a partition* $\mathcal{P}$ *of* X *and an* $\varepsilon > 0$, *there is a unital* C^**-subalgebra,* $A \subseteq C(X)x_\phi\mathbb{Z}$, *which is* $*$*-isomorphic to*

$$[C(S^1) \otimes M_{J_1}] \oplus M_{J_2} \oplus \cdots \oplus M_{J_K},$$

(*for some integers* $J_1, J_2, \ldots, J_K$) *and such that* $C(\mathcal{P}) \subseteq A$ *and there is a unitary* u' *in* A *such that* $\|u - u'\| < \varepsilon$.

The proof is lengthy, so we defer it for the moment while we point out the following consequences.

COROLLARY 2.2. *Suppose that X is minimal for ϕ. Given $a_1, a_2, \ldots, a_n \in C(X)x_\phi\mathbb{Z}$, there is a unital C^*-subalgebra $A \subseteq C(X)x_\phi\mathbb{Z}$ of the same form as in* 2.1 *and $a'_1, a'_2, \ldots, a'_n \in A$ such that $\|a_i - a'_i\| < \varepsilon$, for all $i = 1, 2, \ldots, n$.*

The proof of 2.2 is a trivial consequence of 2.1.

COROLLARY 2.3. *If X is minimal for ϕ, then the invertible elements of $C(X)x_\phi\mathbb{Z}$ are dense.*

Proof. Given an element a in $C(X)x_\phi\mathbb{Z}$ and $\varepsilon < 0$, we find A and a' as in 2.2, with $\|a - a'\| < \varepsilon/2$. From Proposition 1.7, Theorem 3.3 and Theorem 5.2 of Rieffel [**8**], we may find an invertible element, a'', of A such that $\|a'' - a'\| < \varepsilon/2$. Thus, $\|a - a''\| < \varepsilon$ and a'' is invertible. □

For a C^*-algebra B and a positive integer n, we let $GL_n(B)$ denote the invertible elements of $M_n \otimes B$. Also, $GL_n^0(B)$ denotes the connected component of the identity in $GL_n(B)$.

COROLLARY 2.4. *If X is minimal for ϕ, then for all positive integers n, the natural map*

$$GL_n(C(X)x_\phi\mathbb{Z})/GL_n^0(C(X)x_\phi\mathbb{Z}) \to K_1(C(X)x_\phi\mathbb{Z})$$

is an isomorphism.

Proof. As we noted in Theorem 1.1, $K_1(C(X)x_\phi\mathbb{Z})$ is generated by the class of u. Therefore, the natural map above is surjective for all n.

We wish to show that the natural map

$$GL_n(B)/GL_n^0(B) \to K_1(B)$$

is injective, for $B = C(X)x_\phi\mathbb{Z}$. Let a be an element of $GL_n(C(X)x_\phi\mathbb{Z})$. It is well-known that the set of invertible elements in a Banach algebra is open. Together with Proposition 2.1, this implies that we may find a unital C^*-subalgebra $A \subset C(X)x_\phi\mathbb{Z}$, as in 2.1, and an element a' in $GL_n(A)$ such that a and a' are homotopic in $GL_n(C(X)x_\phi\mathbb{Z})$. Thus, it suffices to consider the case $B = A$, with A of the form in 2.1.

Since A is a direct sum of matrix algebras over $\mathbb{C}$ and $C(S^1)$, we see that it is sufficient to consider the cases $B = \mathbb{C}$ and $B = C(S^1)$. For the former, $GL_n(\mathbb{C}) = GL_n^0(\mathbb{C})$ and so the result is trivial. As for $B = C(S^1)$, we may view elements of $GL_n(C(S^1))$ as continuous functions from S^1 into $GL_n(\mathbb{C})$ and in this way identify $GL_n(C(S^1))/GL_n^0(C(S^1))$ with $\pi_1(GL_n(\mathbb{C}))$. In (10.17) on p. 204 of Whitehead [**11**], it is shown that $\pi_1(GL_n(C)) \simeq \mathbb{Z}$ and that the path $f(t) = e^{2\pi i t}p + (I_n - p)$, for $t \in [0, 1]$, where p is any rank one projection in M_n, is a representative of a generator of the group. From this we conclude that the natural map is an isomorphism in the case $B = C(S^1)$. □

Proof of Theorem 2.1. If X is finite, then $C(X)x_\phi\mathbb{Z} \cong C(S^1) \otimes M_J$, where J is the cardinality of X. So the result is trivial in this case.

Let us consider the case when X is infinite. Any finite orbit would be a closed invariant subset of X and so we see that the action is free. Thus, $C(X)x_\phi\mathbb{Z}$ is simple. (See Theorem 5.15 of Zeller-Meier [12].) It will be useful to have a concrete representation of $C(X)x_\phi\mathbb{Z}$.

We fix a ϕ-invariant probability measure on X (such a measure always exists—see Theorem 1, p. 37 of Corfeld et al. [2]), and consider the Hilbert space $L^2(X)$, suppressing the measure in our notation. If Y is a clopen subset of X, there is a natural decomposition $L^2(X) = L^2(Y) \oplus L^2(X-Y)$.

We define a covariant representation of our C^*-dynamical system as follows. Let $C(X)$ act on $L^2(X)$ by multiplication:

$$(f\xi)(x) = f(x)\xi(x), \quad \text{for } f \in C(X),\ \xi \in L^2(X) \quad \text{and } x \in X.$$

We also define a unitary operator u on $L^2(X)$ by

$$(u\xi)(x) = \xi(\phi^{-1}(x)), \quad \text{for } \xi \in L^2(X) \quad \text{and } x \in X.$$

Then $C(X)x_\phi\mathbb{Z}$ is the C^*-algebra generated by $C(X)$ and u. (See 7.6.4 of Pedersen [4]. We use the fact that $C(X)x_\phi\mathbb{Z}$ is simple to conclude that this representation is faithful.)

We begin with the partition $\mathcal{P}$ and $\varepsilon > 0$. Choose an integer $N > 0$ such that $\pi/N < \varepsilon$.

Choose a point x_0 in X arbitrarily. The points $x_0, \phi(x_0), \ldots, \phi^N(x_0)$ are all distinct so we may choose a clopen neighbourhood Y of x_0 satisfying

(i) $Y, \phi(Y), \ldots, \phi^N(Y)$ are pairwise disjoint, and
(ii) For each $n = 0, 1, \ldots, N$, the set $\phi^n(Y)$ is contained in an element of $\mathcal{P}$. (2.1)

We will construct an approximation to ϕ. This technique was first developed by Versik [9, 10] and may be viewed as an analogue of the Rokhlin Lemma [2] in a topological setting.

We define a function $\lambda : Y \to \mathbb{Z}$ by

$$\lambda(y) = \inf\{n \geq 1 \mid \phi^n(y) \in Y\}, \quad \text{for } y \in Y.$$

The minimality of ϕ implies that the positive itterates of any point are dense and since Y is open, this is well-defined. It is straightforward to see that λ is upper (lower) semi-continuous because Y is open (closed). Thus, λ is continuous. As Y is compact, so is $\lambda(Y)$. Therefore, $\lambda(Y)$ is finite; say $\lambda(Y) = \{J_1, J_2, \ldots, J_K\}$. Notice that by (2.1), each J_k is greater than or equal to N.

We define clopen sets $Y(k,j)$, for each $k = 1, 2, \ldots, K$, and each $j = 1, \ldots, J_k$, by

$$Y(k,j) = \phi^j(\lambda^{-1}(J_k)).$$

It is immediate from the definitions that we have

$$\phi(Y(k,j)) = Y(k, j+1), \quad \text{for } 1 \leq j < J_k, \tag{2.2}$$

$$\bigcup_{k=1}^{K} Y(k, J_k) = Y, \quad \text{and} \tag{2.3}$$

$$\phi\left(\bigcup_{k=1}^{K} Y(k, J_k)\right) = \bigcup_{k=1}^{K} Y(k, 1) = \phi(Y). \tag{2.4}$$

It follows from (2.2) and (2.4) that the union of all $Y(k, J)$ is invariant under ϕ. This union is also closed and therefore must be all of X. Thus, $\{Y(k,j) \mid 1 \le k \le K, 1 \le j \le J_k\}$ is a partition of X. It is clear that by dividing up the individual towers, $\{Y(k,j) \mid 1 \le j \le J_k\}$, we may make this partition finer than $\mathscr{P}$. (This will increase the value of K, but does not affect any of the properties (2.2)-(2.4)). We do this so that $C(\mathscr{P}) \subset C(\mathscr{P}_0)$, where

$$\mathscr{P}_0 = \{Y(k,j) \mid 1 \le j \le J_k,\ 1 \le k \le K\}.$$

We now define a finite dimensional unital C^*-subalgebra $A_0 \subset C(X)x_\phi\mathbb{Z}$. In fact, A_0 will be *-isomorphic to

$$M_{J_1} \oplus M_{J_2} \oplus \cdots \oplus M_{J_K}.$$

To do this, it suffices to define a system of matrix units by

$$e_{ij}^{(k)} = \chi_{Y(k,i)} u^{i-j},\ 1 \le k \le K,\ 1 \le i, j \le J_k.$$

It is straightforward to verify that, for fixed k, $\{e_{ij}^{(k)} \mid 1 \le i, j \le J_k\}$ forms a complete system of matrix units for M_{J_k} and that the projections

$$p_k = \sum_{i=1}^{J_k} e_{ii}^{(k)}, \qquad 1 \le k \le K,$$

form a partition of unity in $C(X)x_\phi\mathbb{Z}$. Also notice that

$$C(\mathscr{P}_0) = \operatorname{span}\{e_{ii}^{(k)} \mid 1 \le k \le K,\ 1 \le i \le J_k\},$$

so that $C(\mathscr{P}) \subset A_0$. For any positive integer n, M_n is generated (as a C^*-algebra) by the diagonal matrices and the single nilpotent matrix consisting of ones immediately below the diagonal and zeros elsewhere. In our case, this allows us to observe that A_0 is the C^*-subalgebra of $C(X)x_\phi\mathbb{Z}$ generated by $C(\mathscr{P}_0)$ and $u\chi_{X-Y}$.

Let us return to the dynamical situation and define homeomorphisms ϕ_0 and ψ_0 of X by

$$\psi_0(x) = \begin{cases} \phi(x) & \text{for } x \notin Y = \bigcup_k Y(k, J_k) \\ \phi^{1-J_k}(x) & \text{for } x \in Y(k, J_k) \end{cases}$$

$$\phi_0 = \psi_0^{-1} \circ \phi.$$

Observe that ψ_0 is the disjoint union of K periodic homeomorphisms, and that ϕ_0 is the identity off of Y. (Its restriction to Y is the first return map of ϕ with respect to Y.)

We define unitary operators u_0 and v_0 on $L^2(X)$ by

$$(u_0\xi)(x) = \xi(\phi_0^{-1}(x)),$$

$$(v_0\xi)(x) = \xi(\psi_0^{-1}(x)),$$

for $\xi \in L^2(X)$ and $x \in X$. It is easily checked that

$$v_0 = \sum_{k=1}^{K} \left[\sum_{i=2}^{J_k} e_{i,i-1}^{(k)} + e_{1,J_k}^{(k)} \right],$$

so that $v_0 \in A_0$, and that $v_0 u_0 = u$ so that $u_0 \in C(X)x_\phi\mathbb{Z}$. Since $\psi_0 = \phi$ off of Y, $v_0 = u$ and $u_0 = I$ on $L^2(X - Y)$.

We now let $Z = Y(1, J_1)$ and repeat the procedure beginning by defining $\lambda': Z \to \mathbb{Z}$ and obtaining $K', J'_1, \ldots, J'_{K'}$ and clopen $Z(k, j)$ which satisfy conditions analogous to (2.2)-(2.4). We also insist that the partition $\mathscr{P}_1 = \{Z(k,j) \mid 1 \le k \le K', 1 \le j \le J'_k\}$ is finer than $\mathscr{P}_0$ and that $C(\mathscr{P}_1)$ contains χ_Y.

As before, we obtain a finite dimensional C^*-subalgebra, $A_1 \subset C(X)x_\phi\mathbb{Z}$. (We will not need to write down matrix units for A_1.) As before for A_0, we may describe A_1 as the C^*-subalgebra generated by $C(\mathscr{P}_1)$ and $u\chi_{X-Z}$. Since $\mathscr{P}_1 \supset \mathscr{P}_0$ and $Z \subset Y$, it is clear that $A_0 \subset A_1$.

Define homeomorphisms, ϕ_1 and ψ_1, of X and unitaries, u_1 and v_1, on $L^2(X)$ in an analogous way so that $u = v_1u_1$, $v_1 \in A_1$, $u_1 \in C(X)x_\phi\mathbb{Z}$ and $v_1 = u$ and $u_1 = I$ on $L^2(X - Z)$.

The unitary operator u_1 is the identity except on

$$L^2(Z) = L^2(Y(1, J_1)) = e^{(1)}_{J_1,J_1}L^2(X)$$

so the C^*-algebra generated by A_0 and u_1 will be of the desired form (as in (2.1)) and will contain $C(\mathscr{P})$ as desired. (This will be shown later.) However, it will not have an approximation to u. Now $v_1u_1 = u$, but v_1 is in A_1 and not in A_0. We will apply Berg's technique (see Lemma 1 of Versik [9]) to produce a unitary z in A_1 such that zv_0z^* approximates v_1. We will do this with sufficient care so that z commutes with u_1 and with $C(\mathscr{P})$. Then the C^*-algebra generated by zA_0z^* and u_1 will have all the desired properties.

Consider the unitary operator $v_1v_0^* \in A_1$. From the conditions above, $v_1v_0^* = I$ on $L^2(X - \phi(Y))$. Since it is contained in a finite dimensional algebra, its spectrum is finite and we may find $w \in A_1$, which is a unitary operator on $L^2(\phi(Y))$ such that $w^N = v_1v_0^*$ and such that $\|w - I\| < \pi/N \le \varepsilon$.

Recalling that u carries $L^2(\phi^j(Y))$ isometrically onto $L^2(\phi^{j+1}(Y))$ for $0 \le j < N$, define a unitary operator $z \in A_1$ by

$$z = \begin{cases} u^jw^{N-j}u^{-j} & \text{on } L^2(\phi^{j+1}(Y)) \quad 0 \le j < N \\ I & \text{on } L^2(X - (\phi(Y) \cup \cdots \cup \phi^N(Y))). \end{cases} \tag{2.5}$$

We consider the operator $zv_0 - v_1z$. For $0 \le j < N$, this operator carries $L^2(\phi^j(Y))$ into $L^2(\phi^{j+1}(Y))$. First, let us consider the operator on $L^2(Y)$, where $z = I$. Now v_0 carries $L^2(Y)$ onto $L^2(\phi(Y))$, where $z = w^N = v_1v_0^*$, by 2.5. Therefore,

$$(zv_0 - v_1z)|L^2(Y) = (v_1v_0^*v_0 - v_1)|L^2(Y) = 0.$$

Next, let us take $1 \le j < N$ and consider $zv_0 - v_1z|L^2(\phi^j(Y))$. Recall that $z|L^2(\phi^j(Y)) = u^{j-1}w^{N-j+1}u^{1-j}$. Also note that v_0 carries $L^2(\phi^j(Y))$ onto $L^2(\phi^{j+1}(Y))$ where $z = u^jw^{N-j}u^{-j}$. Finally, on $L^2(\phi^j(Y))$, we have $v_0 = u = v_1$. Putting this together, we obtain

$$\begin{aligned}(zv_0 - v_1z)|L^2(\phi^j(Y)) &= u^jw^{N-j}u^{-j}u - uu^{j-1}w^{N-j+1}u^{1-j} \\ &= u^jw^{N-j}(I - w)u^{1-j}.\end{aligned}$$

From this we see that

$$\|(zv_0 - v_1z)|L^2(\phi^j(Y))\| = \|I - w\| < \pi/N < \varepsilon.$$

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Thirdly, it is immediate from the definitions that on

$L^2(X-(Y\cup\phi(Y)\cup\cdots\cup\phi^N(Y)))$, $v_0=u=v_1$ and $z=I$, so $zv_0-v_1z=0$.

Altogether, we conclude that $\|zv_0-v_1z\|<\varepsilon$. Then we have

$$\|zv_0z^*u_1-v_1u_1\|=\|zv_0-v_1z\|<\varepsilon. \tag{2.6}$$

As we noted earlier, $C(\mathcal{P})\subset A_0$. We now wish to show that z commutes with each element of $C(\mathcal{P})$. This follows immediately from the following two facts. First, z leaves invariant $L^2(\phi^j(Y))$, for each $j=0,1,\ldots,N$, and $z=I$ on $L^2(X-(Y\cup\cdots\cup\phi^N(Y)))$. Secondly, Y was chosen so that $\phi^j(Y)$ is contained in a single element of $\mathcal{P}$, for each $j=0,1,\ldots,N$. (This means that each $f\in C(\mathcal{P})$ acts as a multiple of the identity on each $L^2(\phi^j(Y))$.)

Let A be the C^*-algebra generated by zA_0z^* and u_1. We claim that A satisfies the properties of Proposition 2.1. First, $C(\mathcal{P})\subset A_0$ and z commutes with $C(\mathcal{P})$, so $C(\mathcal{P})\subset A$. Secondly, $v_0\in A_0$, so $u'=(zv_0z^*)u_1\in A$ and $\|u'-u\|<\varepsilon$, from (2.6). All that remains to be shown is that

$$A\simeq[M_{J_1}\otimes C(S^1)]\oplus M_{J_2}\oplus\cdots\oplus M_{J_K}.$$

Since $z=I$ on $L^2(Y)$ (2.5) and $u_1=I$ on $L^2(X-Y)$, z and u_1 commute, so A is unitarily equivalent (via z) to B, the C^*-algebra generated by A_0 and u_1. Let

$$\hat{u}=\sum_{j=1}^{J_1} e_{jJ_1}^{(1)}\,u_1\,e_{J_1j}^{(1)}+\sum_{k>1}p_k.$$

Then $\hat{u}$ is a unitary operator in B. It is clear that $\hat{u}$ and A_0 generate B (as a C^*-algebra). Moreover, $\hat{u}$ commutes with A_0 and $\hat{u}p_k=p_k\hat{u}=p_k$ for all $k>1$. All that needs to be shown now is that $sp(\hat{u})=sp(u_1)=S^1$. Recall that $u=v_1u_1$, so that in $K_1(C(X)x_\phi\mathbb{Z})$, $[u]=[v_1]+[u_1]$. As we noted in § 1, $[u]\neq 0$, while $[v_1]=0$ since $v_1\in A_1$, which is a finite dimensional C^*-algebra. Thus, $[u_1]\neq 0$ and so $sp(u_1)$ must be the entire circle.

This completes the proof of Theorem 2.1. □

3. *The general case*

We now examine the general case. We will make the simplifying assumption that X has no non-trivial clopen ϕ-invariant subsets. (Note that if $Y\subseteq X$ is clopen and ϕ-invariant, then

$$C(X)x_\phi\mathbb{Z}\cong[C(Y)x_\phi\mathbb{Z}]\oplus[C(X-Y)x_\phi\mathbb{Z}].)$$

Recall that a C^*-algebra A is said to have the cancellation property if, whenever U, V and W are finitely generated projective left A-modules such that $U\oplus W\cong V\oplus W$, then we have $U\cong V$. In terms of projections, this says that whenever projections p and q in A (or $M_n\otimes A$) determine the same element in $K_0(A)$, then they are actually unitarily equivalent. We refer the reader to Rieffel [**8**] for a more complete treatment.

THEOREM 3.1. *Suppose that X has no non-trivial, clopen, ϕ-invariant subsets. The following are equivalent.*

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(i) *The invertible elements in $C(X)x_\phi\mathbb{Z}$ are dense.*
(ii) *The C^*-algebra $C(X)x_\phi\mathbb{Z}$ has the cancellation property.*
(iii) *There is a unique minimal set for ϕ.*

Remark. Regarding condition (iii), a standard Zorn's Lemma argument shows that there is always at least one minimal set.

Proof. The implication (i)⇒(ii) is valid for all C^*-algebras. (See Blackadar [**1**].)

Let us now show that (iii)⇒(i). Let us denote the unique minimal set by Y. Consider the following short exact sequence (see Zeller-Meier [**12**]).

$$0 \to C_0(X-Y)x_\phi\mathbb{Z} \xrightarrow{i} C(X)x_\phi\mathbb{Z} \xrightarrow{q} C(Y)x_\phi\mathbb{Z} \to 0.$$

For brevity, we shall denote these C^*-algebras by I, A and B, respectively.

If x is any point of X, the set of accumulation points of $\{\phi^n(x) \mid n \in \mathbb{Z}\}$ is closed and ϕ-invariant and therefore must contain Y. We conclude that Y satisfies the condition that $\bigcup_{n\in\mathbb{Z}} \phi^n(W) = X$, whenever W is a clopen set containing Y. So by Theorem 2.2 of Poon [7], I is an AF-algebra. We conclude that the invertible elements in the unitalized C^*-algebra $I^\sim$ are dense. From Corollary 2.3, we also know that the invertible elements of B are dense. From Corollary 2.4, we see that every invertible in B has the form $u^k_{Y,\phi}v$, where $k \in \mathbb{Z}$ and $v \in GL^0_1(B)$, Clearly, $u_{Y,\phi}$ lifts to $u_{X,\phi} \in A$ and it is well-known that every element of $GL^0_1(B)$ may be lifted to something in $GL^0_1(A)$ (3.4.4 of Blackadar [**1**]). We conclude that every invertible in B may be lifted to an invertible in A.

We are now ready to show that the invertibles in A are dense. Let $a \in A$ and $\varepsilon > 0$. There is an invertible $b \in B$ such that $\|q(a) - b\| < \varepsilon/3$ and there is an invertible $c \in A$ such that $q(c) = b$. Since $\|q(a) - q(c)\| < \varepsilon/3$, we may find $d \in I$ such that $\|a - c - d\| < 2\varepsilon/3$. We may find an invertible $e \in I^\sim$ such that $\|(1 + c^{-1}d) - e\| < \varepsilon/3\|c\|$. Then ce is invertible and

$$\|a - ce\| \le \|a - c(1 + c^{-1}d)\| + \|c\|\,\|(1 + c^{-1}d) - e\| < 2\varepsilon/3 + \varepsilon/3 = \varepsilon.$$

Finally, we consider (ii)⇒(iii). Suppose that (iii) is false. That is, there are (at least) two distinct minimal sets, Y_1 and Y_2, for ϕ. Since these are distinct, $Y_1 \cap Y_2$ will be a proper subset of either Y_1 or Y_2. Moreover, it is closed and ϕ-invariant so, by the minimality of Y_1 and Y_2, must be empty. We consider the short exact sequence

$$0 \to C_0(X - Y_1 - Y_2)x_\phi\mathbb{Z} \xrightarrow{i} C(X)x_\phi\mathbb{Z} \xrightarrow{q} C(Y_1 \cup Y_2)x_\phi\mathbb{Z} \to 0.$$

Again, we denote these C^*-algebras by I, A and B, respectively, and note that $C(Y_1 \cup Y_2)x_\phi\mathbb{Z}$ is $*$-isomorphic to $[C(Y_1)x_\phi\mathbb{Z}] \oplus [C(Y_2)x_\phi\mathbb{Z}]$. Thus, $K_1(B) \cong \mathbb{Z} \oplus \mathbb{Z}$, and is generated by the classes of

$$u_1 = u_{Y_1 \cup Y_2,\phi}\chi_{Y_1} + \chi_{Y_2} \quad \text{and} \quad u_2 = u_{Y_1 \cup Y_2,\phi}\chi_{Y_2} + \chi_{Y_1}.$$

It is clear that $q_*: K_1(A) \to K_1(B)$ takes $[u_{X,\phi}]$ to $[u_{Y_1 \cup Y_2,\phi}] = [u_1u_2]$. Recall that $[u_{X,\phi}]$ generates $K_1(A)$ so we see that $[u_1]$ is not in the image of q_* and so $\partial[u_1] \ne 0$, where ∂ denotes the index map $K_1(B) \to K_0(I)$. Let us compute $\partial[u_1]$. Let E be any clopen set in X containing Y_1 and disjoint from Y_2. Let $F = X - (E \cup \phi(E))$. Note that since $\phi(Y_2) = Y_2$, $\phi(E)$ is also disjoint from Y_2 so $Y_2 \subseteq F$. Let $a = u_{X,\phi}\chi_E + \chi_F$.

Then a is a partial isometry in A with $a^*a=\chi_{E\cup F}$ and $aa^*=\chi_{\phi(E)\cup F}$. Also note that $q(a)=u_1$. So, $\partial[u_1]=[\chi_{\phi(E)-E}]-[\chi_{E-\phi(E)}]$. Denote the projections by p_1 and p_2, respectively. Note that they are both elements of I. As we have already observed $[p_1]-[p_2]=\partial[v]\neq 0$ in $K_0(I)$. However, in $K_0(A)$, $[i(p_i)]-[i(p_2)]=i_*\partial[u_1]=0$. Suppose that there is a unitary w in A such that $wp_1w^*=p_2$. Then wp_1 is a partial isometry in I (since $p_1\in I$) with $(wp_1)(wp_1)^*=p_2$ and $(wp_1)^*(wp_1)=p_1$. This would imply $[p_2]=[p_1]$ in $K_0(I)$. We conclude that while p_1 and p_2 determine the same class in $K_0(A)$, they are not unitarily equivalent. Therefore, A fails to have the cancellation property. □

4. *Residual finiteness*

In this section, we examine the relationship between the results of § 3 and the property of residual finiteness of the C^*-algebra $C(X)x_\phi\mathbb{Z}$. We take a moment to discuss, in general terms, why this is relevant. There are two canonical types of examples of unital C^*-algebras in which the invertibles are not dense: $C(M)$, where M is a compact metrizable space whose dimension is at least two (see Rieffel [**8**]) and any C^*-algebra containing a non-unitary isometry. Recall that a C^*-algebra, A, is called finite if every isometry in A is actually a unitary. Let us take this second phenomenon one step further. A unital C^*-algebra, A, is called residually finite if every quotient of A is finite. If A has a quotient, A/I, which is not finite, then the invertibles in A/I are not dense and it follows that the invertibles in A are not dense. We summarize by saying that there are two obstructions to the invertibles in a C^*-algebra being dense: a topological one and finiteness one.

In our situation, X is zero-dimensional and the group $\mathbb{Z}$ has a one-dimensional dual. The C^*-algebra $C(X)x_\phi\mathbb{Z}$ is a 'non-commutative $X\times\hat{\mathbb{Z}}\cong X\times S^1$', which is one-dimensional. Thus, one would not expect 'topological' obstructions to the invertibles being dense, only finiteness. However, we shall produce an example of a $C(X)x_\phi\mathbb{Z}$ which is residually finite and yet the invertibles are not dense.

We begin with a necessary and sufficient dynamical condition for $C(X)x_\phi\mathbb{Z}$ to be residually finite. This is a more or less straightforward consequence of a result of M. Pimsner.

We begin by making some definitions. Fix a metric d on X. For a positive real number ε, and points x and y in X, an ε-chain from x to y is a finite sequence of points $(x_1, x_2, \dots, x_n)$ in X such that $x_1=x$, $x_n=y$ and $d(\phi(x_i), x_{i+1})<\varepsilon$, for $i=1, 2, \dots, n-1$. A point x in X is chain recurrent for ϕ if, for every $\varepsilon>0$, there is an ε-chain from x to x (of length at least two).

We remark that a point of x may be chain recurrent for ϕ in X, but not when considered as a point in some ϕ-invariant closed subset of X, since the ε-chains may not lie in the subset.

For any x in X, we let $\omega^+(x)$ and $\omega^-(x)$ denote the accumulation points of $\{\phi^n(x)\mid n\geq 0\}$ and $\{\phi^n(x)\mid n\leq 0\}$, respectively. These are both closed and ϕ-invariant.

THEOREM 4.1. (Pimsner [**5**].) *The C^*-algebra $C(X)x_\phi\mathbb{Z}$ is finite if and only if each point of X is chain recurrent for ϕ.*

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Pimsner actually showed much more; namely that these conditions are equivalent to $C(X)x_\phi\mathbb{Z}$ being AF-embeddable.

THEOREM 4.2. *The following are equivalent.*

(i) *$C(X)x_\phi\mathbb{Z}$ is residually finite.*

(ii) *For every x in X and every $\varepsilon>0$, there is an ε-chain, $(x_1, x_2, \ldots, x_n)$, from x to x, with each $x_i \in \{\phi^n(x) \mid n \in \mathbb{Z}\}$.*

(iii) *For each x in X, $\omega^+(x) \cap \omega^-(x)$ is not empty.*

Proof. (i)$\Rightarrow$(iii). We prove the contrapositive. Suppose there is an x in X with $\omega^+(x) \cap \omega^-(x)$ empty. Let $Y = \{\phi^n(x) \mid n \in \mathbb{Z}\} \cup \omega^+(x) \cup \omega^-(x)$ and let $E = \{\phi^n(x) \mid n \geq 0\} \cup \omega^+(x)$. Then $C(Y)x_\phi\mathbb{Z}$ is a quotient of $C(X)x_\phi\mathbb{Z}$ which contains $u_{Y,\phi}\chi_E + \chi_{Y-E}$ which is a non-unitary isometry. The proof of (iii)$\Rightarrow$(ii) is straightforward, so we omit it.

(ii)$\Rightarrow$(i). We need to show that for any ideal I in $C(X)x_\phi\mathbb{Z}$, the quotient $C(X)x_\phi\mathbb{Z}/I$ is finite. We begin with the following special case. Suppose Y a closed ϕ-invariant subset of X. We consider $I = C_0(X-Y)x_\phi\mathbb{Z}$. In this case, $C(X)x_\phi\mathbb{Z}/I \cong C(Y)x_\phi\mathbb{Z}$ and we will apply 4.1. Let x be any point in Y. Since Y is ϕ-invariant, the entire orbit of x under ϕ is in Y. Therefore, by (ii), we can find, for any $\varepsilon>0$, an ε-chain from x to x within Y. By 4.1, $C(Y)x_\phi\mathbb{Z}$ is finite.

As for the general ideal I, let us suppose that the quotient $C(X)x_\phi\mathbb{Z}/I$, which we denote B, is not finite. Then we may find an irreducible representation π of B (and also of $C(X)x_\phi\mathbb{Z}$) on the Hilbert space $\mathcal{H}$ such that $\pi(B)$ is not finite. By analyzing $\pi \mid C(X)$, it can be shown that $\mathcal{H}$ is isomorphic to $L^2(X, \mu)$, for some (possibly infinite) ϕ-invariant measure μ on X and $C(X)$ is represented on $L^2(X, \mu)$ as multiplication operators. Since $\pi(B)$ is not finite, $\mathcal{H}$ cannot be finite-dimensional. Thus, $Y = \text{support}\ (\mu)$ is closed, ϕ-invariant and infinite. Since π is irreducible, Y must contain a point whose orbit under ϕ is infinite. We conclude that $\pi(B)$ is actually *-isomorphic to $C(Y)x_\phi\mathbb{Z}$, which we have already seen is finite. This contradiction establishes the result. □

We conclude by giving an example of X and ϕ such that $C(X)x_\phi\mathbb{Z}$ is residually finite but the invertibles are not dense.

We provide a picture of X in figure 1. Each box is a clopen set. The points x_0 and x_1 are fixed by ϕ. Also, ϕ is such that $\phi^{-1}(E_1) \subseteq E$, $\phi(E_1) = E_2$, $\phi(E_2) \subseteq F$,

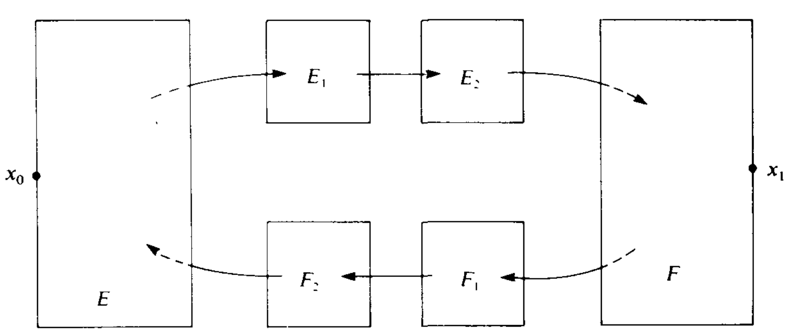

FIGURE 1

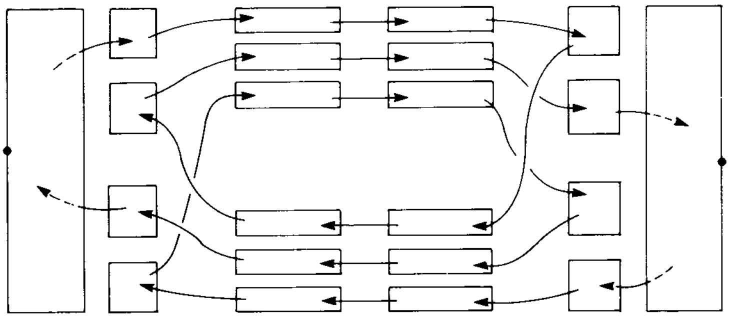

FIGURE 2

$\phi^{-1}(F_1) \subseteq F$, $\phi(F_1) = F_2$ and $\phi(F_2) \subseteq E$. We also show a 'finer' picture of X in figure 2, with arrows indicating ϕ.

From these figures, one can continue inductively to give a rigorous definition of ϕ and X. It is clear that there are no clopen ϕ-invariant subsets of X, while $\{x_0\}$ and $\{x_1\}$ are each minimal sets for ϕ. Thus, the invertibles in $C(X)x_\phi\mathbb{Z}$ are not dense. It can also be seen that, for any x in X, $\omega^+(x) \cap \omega^-(x)$ contains either x_0 or x_1 (or both) and so $C(X)x_\phi\mathbb{Z}$ is residually finite.

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