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BAER RINGS OF GENERALIZED POWER SERIES*

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Abstract. We show that if R is a commutative ring and $(S, \leq)$ a strictly totally ordered monoid, then the ring $[[R^{S,\leq}]]$ of generalized power series is Baer if and only if R is Baer.

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A ring R is called *Baer* if the right annihilator of every nonempty subset of R is generated by an idempotent. Baer rings were studied in [**1**, **2**, **3**, **5**, **6**, **7**, **11**]. By [**5**, Theorem 3] the Baer condition is left-right symmetric. Semisimple artinian rings, domains and the rings of $n \times n$ upper triangular matrices over division rings are Baer, where $n = 1, 2, \ldots$.

A ring R is called a *right pp-ring* if each principal right ideal of R is projective, or equivalently, if the right annihilator of each element of R is generated by an idempotent. Baer rings are clearly right pp-rings. It was proved in [**9**] that if R is a commutative ring and $(S, \leq)$ a strictly totally ordered monoid then the ring $[[R^{S,\leq}]]$ of generalized power series is a pp-ring if and only if R is a pp-ring and every S-indexed subset C of the set $B(R)$ of all idempotents of R has a least upper bound in $B(R)$. In this paper we show that if R is a commutative ring and $(S, \leq)$ a strictly totally ordered monoid, then the ring $[[R^{S,\leq}]]$ of generalized power series is Baer if and only if R is Baer.

All rings considered here are associative with identity. Any concept and notation not defined here can be found in [**12**, **13**, **14**, **15**].

Let $(S, \leq)$ be an ordered set. Recall that $(S, \leq)$ is *artinian* if every strictly decreasing sequence of elements of S is finite, and that $(S, \leq)$ is *narrow* if every subset of pairwise order-incomparable elements of S is finite. Let S be a commutative monoid. Unless stated otherwise, the operation of S will be denoted additively, and the neutral element by 0. The following definition is due to P. Ribenboim. See [**12**, **13**, **14**, **15**].

Let $(S, \leq)$ be a strictly ordered monoid (that is, $(S, \leq)$ is an ordered monoid satisfying the condition that, if $s, s', t \in S$ and $s < s'$, then $s + t < s' + t$), and R a commutative ring. Let $A = [[R^{S,\leq}]]$ be the set of all maps $f : S \longrightarrow R$ such that $supp(f) = \{s \in S | f(s) \neq 0\}$ is artinian and narrow. With pointwise addition, A is an abelian additive group. For every $s \in S$ and $f, g \in A$, let $X_s(f, g) = \{(u, v) \in S \times S | s = u + v, f(u) \neq 0, g(v) \neq 0\}$. It follows from [**14**, 1.16] that $X_s(f, g)$ is finite. This fact allows us to define the operation of convolution

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$$(fg)(s) = \sum_{(u,v)\in X_s(f,g)} f(u)g(v).$$

With this operation, and pointwise addition, A becomes a commutative ring, called the *ring of generalized power series*. The elements of A are called generalized power series with coefficients in R and exponents in S.

For example, if $S = \mathbb{N} \cup \{0\}$ and $\leq$ is the usual order, then $[[R^{\mathbb{N}\cup\{0\},\leq}]] \cong R[[x]]$, the usual ring of power series. If S is a commutative monoid and $\leq$ is the trivial order, then $[[R^{S,\leq}]] = R[S]$, the monoid-ring of S over$R$. Further examples are given in [**10**, **13**].

We shall use the following notations introduced by Ribenboim in [**13**].

Let $f, f' \in A$. We say f is a *section of* f' (denoted $f \preceq f'$) if $s < s'$ for every $s \in supp(f)$ and every $s' \in supp(f' - f)$.

Let $r \in R$. Define a mapping $c_r \in A$ as follows:

$$c_r(0) = r, \qquad c_r(s) = 0, \qquad \text{for all } 0 \neq s \in S.$$

Let $s \in S$. Define a mapping $e_s \in A$ as follows:

$$e_s(s) = 1, \qquad e_s(t) = 0, \qquad \text{for all } s \neq t \in S.$$

Lemma 1. ([**8**, Lemma 3]) *If $f \preceq f'$, then $fc_r \preceq f'c_r$.*

Recall that a monoid S is *torsion-free* if the following property holds: if $s, t \in S$, if k is an integer, $k \geq 1$ and $ks = kt$, then $s = t$.

Lemma 2. ([**9**, Lemma 2.2]) *Let R be a reduced commutative ring and S a cancellative and torsion-free monoid. If $\phi^2 = \phi \in [[R^{S,\leq}]]$, then there exists an idempotent $e \in R$ such that $\phi = c_e$.*

Lemma 3. ([**4**]) *A ring R is a reduced right pp-ring if and only if R is a right pp-ring with every idempotent central.*

Lemma 4. *Let R be a commutative ring and S a cancellative and torsion-free monoid. Set $A = [[R^{S,\leq}]]$, the ring of generalized power series. If A is Baer, then R is Baer.*

Proof. Suppose that $\emptyset \neq X \subseteq R$. Then $C = \{c_x | x \in X\} \subseteq A$ and $C \neq \emptyset$. Since A is Baer, there exists an idempotent $\phi \in A$ such that $r_A(C) = \phi A$. Clearly A is a pp-ring. Thus, by Lemma 3, A is a reduced ring. Hence it is easy to see that R is reduced. Now, by Lemma 2, there exists an idempotent $e \in R$ such that $\phi = c_e$. For any $x \in X$, $xe = (c_x c_e)(0) = 0$, and so $e \in r_R(X)$. Now, suppose that $p \in r_R(X)$. Then $xp = 0$ for any $x \in X$. Thus $c_x c_p = 0$ for any $x \in X$. This means that $c_p \in r_A(C)$, and so $c_p = c_e f$, for some $f \in A$. Now, $p = c_p(0) = (c_e f)(0) = ef(0) \in eR$. Thus $r_R(X) = eR$, where e is an idempotent of R. Hence R is Baer.

Lemma 5. *Let R be a commutative ring and S a cancellative and torsion-free monoid such that $(S, \leq)$ is narrow. If R is Baer, then A is Baer.*

Proof. By [**13**, 3.3], there exists a compatible strict total order $\preceq'$ on S that is finer than $\leq$; (that is, $s \leq t$ implies $s \leq' t$ for all $s, t \in S$). Let $A' = [[R^{S,\leq'}]]$. Then A is a subring of A' by [**13**, 4.4]. Since $(S, \leq)$ is narrow, $A = A'$ by [**13**, 4.4], and so there is no loss of generality in assuming that $(S, \leq)$ is totally ordered. We may assume that $S \neq 0$.

It is enough to show that the right annihilator of every nonempty ideal of A is generated by an idempotent. Let L be an ideal of A. We shall show that $r(L) = \phi A$ for an idempotent $\phi^2 = \phi \in A$. For every $f \in A, f \neq 0$, $supp(f)$ is a nonempty well-ordered subset of S. We denote by $\pi(f)$ the smallest element of the support of f.

For every $s \in S$, set

$$I_s = \{f(s) | f \in L, \pi(f) = s\},$$

and $I = \cup_{s \in S} I_s$.

Since R is a Baer ring, there exists an idempotent $e^2 = e \in R$ such that $r(I) = eR$. We shall show that $r(L) = c_e A$.

Let $g \in L$. Suppose that $gc_e \neq 0$, and $\pi(gc_e) = t$. Then $(gc_e)(t) \neq 0$. Since $g(t)e = (gc_e)(t) \in I_t \subseteq I$, it follows that $g(t)e = (g(t)e)e = 0$, a contradiction. Thus, $gc_e = 0$, for every $g \in L$. This means that $c_e A \leq r(L)$.

Assume $0 \neq g \in r(L) - c_e A$. Set $\pi(g) = s$. For every $a \in I$, there exist $u \in S$, $f \in L$, such that $a = f(u)$, and $\pi(f) = u$. Since $g \in r(L), fg = 0$. Thus, by [**15**, 1.17], we have $f(u)g(s) = 0$. Hence $ag(s) = 0$. This means that $g(s) \in r(I) = eR$. Thus $g - c_{g(s)}e_s \in r(L) - c_e A$. Set $\pi(g - c_{g(s)}e_s) = t$. Then $(g - c_{g(s)}e_s)(t) \neq 0$. Since

$$(g - c_{g(s)}e_s)(s) = g(s) - g(s)e_s(s) = 0,$$

we have $s \neq t$. Thus $g(t) = (g - c_{g(s)}e_s)(t) \neq 0$,which implies that $s < t$.

Let α be an ordinal with cardinal greater than the cardinal $|S|$ of S, and Γ the set of all ordinals $\lambda < \alpha$. We shall show that for each $\lambda \in \Gamma$, there exists an element $f_\lambda \in A$ such that the following properties hold:

$$\begin{aligned} &f_\mu \preceq f_\nu \quad \text{and} \quad f_\mu \neq f_\nu \quad \text{when} \quad \mu < \nu, \\ &g - f_\mu c_e \in r(L), \\ &\pi(g - f_\mu c_e) < \pi(g - f_\nu c_e) \quad \text{when} \quad \mu < \nu, \\ &u < \pi(g - f_\mu c_e) \quad \text{for any} \quad u \in supp(f_\mu). \end{aligned}$$

First we set $f_1 = c_{g(s)}e_s$.

Let $\lambda \in \Gamma$ and assume that we have already found the elements $f_\mu \in A$, for every $\mu < \lambda$, satisfying the above properties (for ordinals $\mu < \nu < \lambda$). We shall construct an element $f_\lambda \in A$ such that the properties above are satisfied for $\mu < \nu \leq \lambda$.

Suppose that there exists an ordinal η such that $\lambda = \eta + 1$. If $g - f_\eta c_e = 0$, then $g = f_\eta c_e \in c_e A$, a contradiction. Thus $g - f_\eta c_e \neq 0$. Set $g_\eta = g - f_\eta c_e$, and $t_\eta = \pi(g_\eta)$. Let $f_\lambda : S \longrightarrow R$ be defined by

$$f_\lambda = f_\eta + c_{g_\eta(t_\eta)}e_{t_\eta}.$$

Then $f_\lambda \in A$. We show that $f_\eta \preceq f_\lambda$ and this implies that $f_\mu \preceq f_\lambda$ for any $\mu < \lambda$. Since $g_\eta(t_\eta) \neq 0$, it follows that

$$supp(f_\lambda - f_\eta) = supp(c_{g_\eta(t_\eta)}e_{t_\eta}) = \{t_\eta\}.$$

Suppose that $s \in supp(f_\eta)$. Then, by hypothesis,

$$s < \pi(g - f_\eta c_e) = t_\eta \in supp(f_\lambda - f_\eta).$$

Thus $f_\eta \preceq f_\lambda$. If $f_\eta = f_\lambda$, then $c_{g_\eta(t_\eta)}e_{t_\eta} = 0$, and so $g_\eta(t_\eta) = (c_{g_\eta(t_\eta)}e_{t_\eta})(t_\eta) = 0$, which is a contradiction. If $f_\mu = f_\lambda$, where $\mu < \eta$, then $f_\mu \preceq f_\eta \preceq f_\lambda = f_\mu$. Thus, by [**13**, 5.3], $f_\eta = f_\lambda$, also a contradiction. Hence $f_\mu \neq f_\nu$ when $\mu < \nu \leq \lambda$.

It is easy to see that $g_\lambda = g - f_\lambda c_e \in r(L)$.

For every $a \in I$, there exist $u \in S, f \in L$, such that $a = f(u)$, and $\pi(f) = u$. Since $g_\eta \in r(L)$, $fg_\eta = 0$. Thus, by [**15**, 1.17], we have $f(u)g_\eta(t_\eta) = 0$. Hence $ag_\eta(t_\eta) = 0$. This means that $g_\eta(t_\eta) \in r(I) = eR$. Denote $\pi(g - f_\lambda c_e) = t_\lambda$. Since

$$(g - f_\lambda c_e)(t_\eta) = ((g - f_\eta c_e - c_{g_\eta(t_\eta)}e_{t_\eta}c_e)(t_\eta)$$

$$= g_\eta(t_\eta) - g_\eta(t_\eta)ee_{t_\eta}(t_\eta) = 0,$$

it follows that $t_\lambda \neq t_\eta$. Thus

$$(g - f_\eta c_e)(t_\lambda) = (g - f_\eta c_e)(t_\lambda) - g_\eta(t_\eta)e_{t_\eta}(t_\lambda)e$$

$$= (g - f_\eta c_e - c_{g_\eta(t_\eta)}e_{t_\eta}c_e)(t_\lambda) = (g - f_\lambda c_e)(t_\lambda) \neq 0,$$

and so $t_\lambda \in supp(g - f_\eta c_e)$. Hence $t_\eta < t_\lambda$; that is, $\pi(g - f_\eta c_e) < \pi(g - f_\lambda c_e)$, which implies that $\pi(g - f_\mu c_e) < \pi(g - f_\lambda c_e)$, for any $\mu < \lambda$.

We now show that $u < \pi(g - f_\lambda c_e)$, for any $u \in supp(f_\lambda)$. It is clear that

$$supp(f_\lambda) = supp(f_\eta + c_{g_\eta(t_\eta)}e_{t_\eta}) \subseteq supp(f_\eta) \cup supp(c_{g_\eta(t_\eta)}e_{t_\eta}).$$

If $u \in supp(f_\eta)$, then $u < \pi(g - f_\eta c_e) < \pi(g - f_\lambda c_e)$. If $u \in supp(c_{g_\eta(t_\eta)}e_{t_\eta}) = \{t_\eta\}$,then $u = t_\eta = \pi(g - f_\eta c_e) < \pi(g - f_\lambda c_e)$.

Now let λ be a limit ordinal. For the family $\{f_\mu | \mu < \lambda\}$ of elements $f_\mu \in A$, it was proved, in [**13**, 5.4], that there exists an element $b = \preceq\text{-}sup(f_\mu)_{\mu<\lambda} \in A$ such that

(i) $f_\mu \preceq b$ for every $\mu < \lambda$:

(ii) if $b' \in A$ and $f_\mu \preceq b'$ for every $\mu < \lambda$, then $b \preceq b'$.

Let $f_\lambda = b = \preceq\text{-}sup(f_\mu)_{\mu<\lambda}$. By (i), we know that $f_\mu \preceq f_\lambda$, for every $\mu < \lambda$, and that $g_\lambda = g - f_\lambda c_e \in r(L)$. If $f_\mu = f_\lambda$, then $f_\mu \preceq f_{\mu+1} \preceq f_\lambda = f_\mu$, and thus $f_\mu = f_{\mu+1}$, a contradiction. Hence $f_\mu \neq f_\lambda$ for every $\mu < \lambda$.

For every $\mu < \lambda$,

$$g - f_\lambda c_e = g - f_\mu c_e - (f_\lambda - f_\mu)c_e.$$

Thus, by [**13**, 4.2], we have

$$\pi(g - f_\lambda c_e) \geq min\{\pi(g - f_\mu c_e), \pi((f_\lambda - f_\mu)c_e)\}. \qquad (*)$$

Let $\pi(g - f_\mu c_e) = t_\mu$. Since $f_\mu \preceq f_{\mu+1} \preceq f_\lambda$, by Lemma 1, we have

$$f_\mu c_e \preceq f_{\mu+1} c_e \preceq f_\lambda c_e.$$

If $f_\mu c_e = f_{\mu+1}c_e$, then $t_\mu = t_{\mu+1}$, a contradiction. Thus $f_\mu c_e \neq f_{\mu+1}c_e$. If $f_{\mu+1}c_e = f_\lambda c_e$, then

$$f_{\mu+1}c_e \preceq f_{\mu+2}c_e \preceq f_\lambda c_e = f_{\mu+1}c_e.$$

Thus, by [**13**, 5.3], $f_{\mu+1}c_e = f_{\mu+2}c_e$, and so $t_{\mu+1} = t_{\mu+2}$, which is a contradiction. Hence $f_{\mu+1}c_e \neq f_\lambda c_e$. Thus, by [**13**, 5.4], it follows that

$$\pi((f_\lambda - f_\mu)c_e) = \pi((f_{\mu+1} - f_\mu)c_e) = \pi((g - f_\mu c_e) - (g - f_{\mu+1}c_e))$$

$$\geq min\{\pi(g - f_\mu c_e), \pi(g - f_{\mu+1}c_e)\} = min\{t_\mu, t_{\mu+1}\} = t_\mu.$$

Thus, by $(*)$,

$$t_\lambda = \pi(g - f_\lambda c_e) \geq t_\mu.$$

Hence, $t_\mu \leq t_\lambda$ for all $\mu < \lambda$ so that $t_\mu < t_{\mu+1} \leq t_\lambda$ and $t_\mu \neq t_\lambda$. Thus $t_\mu < t_\lambda$.

We now show that $u < \pi(g - f_\lambda c_e)$, for any $u \in supp(f_\lambda)$. Since

$$supp(f_\lambda) = \cup_{\mu<\lambda} supp(f_\mu)$$

by [**13**, 5.4], there exists an ordinal $\mu < \lambda$ such that $u \in supp(f_\mu)$. Thus $u < t_\mu < t_\lambda$.

Now, we deduce that if $\mu < \nu, \mu, \nu \in \Gamma$ then $t_\mu < t_\nu$. Thus $|\{t_\lambda | \lambda \in \Gamma\}| = |\Gamma| > |S|$, and this is impossible.

Thus, we have $r(L) = c_e A$. Now the result follows.

THEOREM 6. *Let R be a commutative ring and S a cancellative and torsion-free monoid such that $(S, \leq)$ is narrow. Set $A = [[R^{S,\leq}]]$, the ring of generalized power series. Then A is Baer if and only if R is Baer.*

COROLLARY 7. *Let R be a commutative ring and $(S, \leq)$ a strictly totally ordered monoid. Then A is Baer if and only if R is Baer.*

Proof. By [**13**, 3.2], S is cancellative and torsion-free. Now the result follows from Theorem 6.

The following corollaries will give more examples of Baer rings.

COROLLARY 8. *Let $\mathbb{Q}^+ = \{a \in \mathbb{Q} | a \geq 0\}, \mathbb{R}^+ = \{a \in \mathbb{R} | a \geq 0\}$. Then the rings $[[\mathbb{Z}^{\mathbb{N}\cup\{0\},\leq}]], [[\mathbb{Z}^{\mathbb{Z},\leq}]], [[\mathbb{Z}^{\mathbb{Q}^+,\leq}]], [[\mathbb{Z}^{\mathbb{Q},\leq}]], [[\mathbb{Z}^{\mathbb{R}^+,\leq}]]$ and $[[\mathbb{Z}^{\mathbb{R},\leq}]]$ are Baer rings, where $\leq$ is the usual order.*

COROLLARY 9. *Let R be a commutative ring. Set $R((X)) = [[R^{\mathbb{Z},\leq}]]$, the ring of Laurent series over R where $\leq$ is the usual order on $\mathbb{Z}$. Then $R((X))$ is Baer if and only if R is Baer.*

NOTE. See [**16**, p. 335] for the definition of the ring of Laurent series over R.

It was shown in [**3**, Corollary 1.10] that for a reduced ring R, the ring $R((X))$ of Laurent series over R is Baer if and only if R is Baer. Since a commutative Baer ring is reduced, it is natural to ask if some of the results of this paper remain true in the more general case of R being reduced rather than commutative.

Corollary 10. *Let* $(S_1, \leq_1), \ldots\ldots, (S_n, \leq_n)$ *be totally strictly ordered monoids. Denote by* $(lex \leq)$ *and* $(revlex \leq)$ *the lexicographic order, the reverse lexicographic order, respectively, on the monoid* $S_1 \times \ldots \times S_n$. *Let* R *be a commutative ring. Then the following statements are equivalent.*

(1) *The ring* $[[R^{S_1 \times \ldots \times S_n, (lex \leq)}]]$ *is Baer.*
(2) *The ring* $[[R^{S_1 \times \ldots \times S_n, (revlex \leq)}]]$ *is Baer.*
(3) R *is Baer.*

Proof. (1)$\Longleftrightarrow$(3). It is easy to see that $(S_1 \times \ldots \times S_n, (lex \leq))$ is a totally strictly ordered monoid. Thus, by Corollary 7, $[[R^{S_1 \times \ldots \times S_n, (lex \leq)}]]$ is Baer if and only if R is Baer.

The proof of (2)$\Longleftrightarrow$(3) is similar.

Let R be a commutative ring, and consider the multiplicative monoid $\mathbb{N}_{\geq 1}$, endowed with the usual order $\leq$. Then $A = [[R^{\mathbb{N}_{\geq 1}, \leq}]]$ is the ring of arithmetical functions with values in R, endowed with the Dirichlet convolution

$$(fg)(n) = \sum_{d|n} f(d)g(n/d), \qquad \text{for each} \qquad n \geq 1.$$

Corollary 11. *Let* R *be a commutative ring. Then* $A = [[R^{\mathbb{N}_{\geq 1}, \leq}]]$ *is Baer if and only if* R *is Baer.*

Let $(S, \leq)$ be a strictly totally ordered monoid that is also artinian. For any $s \in S$, set $X_s = \{(u, v) | u + v = s, u, v \in S\}$. Then from [**16**, 4.1], it follows that X_s is a finite set. Let V be a free abelian additive group with the base consisting of elements of S. Then V is a coalgebra over $\mathbb{Z}$ with the comultiplication map and counit map as follows:

$$\Delta(s) = \sum_{(u,v) \in X_s} u \otimes v,$$

$$\epsilon(s) = \begin{cases} 1 & s = 0, \\ 0 & s \neq 0. \end{cases}$$

Then clearly $[[R^{S, \leq}]] \cong Hom(V, R)$, the dual algebra.

Corollary 12. *Let* R *be a commutative ring. Then, using the notations above, the dual algebra* $Hom(V, R)$ *is a Baer ring if and only if* R *is a Baer ring.*

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