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Finite groups which are the product of two nilpotent subgroups

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Suppose $G = AB$ where G is a finite group and A and B are nilpotent subgroups. It is proved that the derived length of G modulo its Frattini subgroup is at most the sum of the classes of A and B. An upper bound for the derived length of G in terms of the derived lengths of A and B also is obtained.

1. Introduction

Suppose G is a finite group which is the product of two nilpotent subgroups A and B. That G must be solvable was proved by Kegel [5]. It has been conjectured that $d(G)$, the derived length of G, is at most $c(A) + c(B)$ where $c(A)$ and $c(B)$ denote the class of A and B, respectively. This conjecture has been verified only in two special cases:

(1) when A and B are both abelian (Itô [4]), and

(2) when A and B have relatively prime orders (Hall and Higman [3]).

One of the principal theorems of the present paper is that if $D(G)$ is the Frattini subgroup of G, then $d\big(G/D(G)\big) \le c(A) + c(B)$. As a result, the problem of finding some upper bound for $d(G)$ in terms of $c(A)$ and $c(B)$ is reduced to finding a bound on $d\big(D(G)\big)$. But $D(G)$ is nilpotent, and, if P and Q are Sylow p-subgroups of A and B, respectively, a theorem of Wielandt [6] implies that PQ is a Sylow p-subgroup of G. Hence, if $f(x, y)$ were some function such that whenever $S = PQ$, where

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S is a p-group and P and Q are subgroups, it followed that $d(A) \le f(c(P), c(Q))$, then it would be true that $d(G) \le f(c(A), c(B)) + c(A) + c(B)$. Thus one consequence of this paper is that the original problem is reduced to a problem concerning p-groups. Unfortunately, $G/D(G)$ is abelian if G is a p-group, and so the results of this paper are trivial for p-groups.

It is also possible to bound $d(G/D(G))$ in terms of other invariants of A and B . For example, if the Sylow 2-subgroups of both A and B have class at most 2 , then $d(G/D(G)) \le d(A)d(B) + 1$. Under a similar hypothesis, the nilpotent length of G is at most $d(A) + d(B)$. In comparing these results with the previous one, it should be remembered that, in general, $d(A)$ is much smaller than $c(A)$ (to be more specific, $d(A) \le 1 + \log_2(c(A))$). I conjecture that the requirement on the Sylow 2-subgroups of A and B in these theorems is unnecessary.

2. Preliminaries

All groups considered in this paper are finite and solvable. $F(G)$ and $D(G)$ denote the Fitting and Frattini subgroups of G , respectively. G' is the derived group of G and $G^{(n)}$ is defined inductively by $G^{(0)} = G$ and $G^{(n+1)} = (G^{(n)})'$. $d(G)$ and $l(G)$ are the derived length and nilpotent length, respectively, of G . If G is nilpotent, $c(G)$ is the class of G . If p is a prime, then $l_p(G)$ is the p-length of G and $O_p(G)$ and $O_{p'}(G)$ are the largest normal p-subgroup and p'-subgroup of G , respectively. $O_{pp'}(G)$ is defined by $O_{p'}(G/O_p(G)) = O_{pp'}(G)/O_p(G)$. If A and B are subsets of G , then $[A, B]$ is the subgroup of G generated by all elements of the form $x^{-1}y^{-1}xy$ where $x \in A$ and $y \in B$. If C is a third subset, then $[A, B, C] = [[A, B], C]$. $C_G(A)$ is the centralizer of A in G and $Z(G)$ is the center of G . $|S|$ denotes the number of elements in the set S .

If n is a positive integer, F_n denotes the collection of all groups G satisfying $d(G/D(G)) \le n$. It is an easy exercise to verify

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that the group G belongs to F_n if, and only if, $G^{(n-1)}$ is nilpotent. It is immediate from this that F_n is a saturated formation in the sense of Gaschütz [1]. N_n is the collection of all groups G of nilpotent length at most n. N_n is also a saturated formation.

LEMMA 1. *Suppose G is a group and F is a saturated formation such that $G \notin F$ but F contains every proper homomorphic image of G. Then $D(G) = 1$ and G has only one minimal normal subgroup.*

This follows directly from the definition of a saturated formation.

LEMMA 2. *Let G be a group, p a prime, and P a Sylow p-subgroup of G. Assume that $O_{p'}(G) = 1$ and that either $p > 2$ or $c(P) \le 2$. Then $d\big(P/O_p(G)\big) \le d(P) - 1$.*

This follows from [3, Theorem 3.2.1] if $p > 3$, from [3, Theorem 3.2.2] if $p = 3$, and from [3, Lemma 1.2.3] if $p = 2$.

LEMMA 3. *Let P be a Sylow 2-subgroup of G and assume that $c(P) = 3$. Then $d(P) = 2$ and $l_2(G) \le 2$.*

Proof. Without loss of generality, we may assume that $O_{2'}(G) = 1$. P is not abelian and $P^{(2)} \subseteq [P, P, P, P] = 1$. Hence $d(P) = 2$. Let V be $O_2(G)/D\big(O_2(G)\big)$ written additively. Since $[y, x, x, x] = 1$ for all x and y in P, it follows that if we represent $G/O_2(G)$ as a linear group operating on V, then $G/O_2(G)$ satisfies the hypothesis of Theorem 3.1 of [2]. Hence $l_2\big(G/O_2(G)\big) \le 1$ by that theorem. Lemma 3 now follows.

THEOREM 1. *Assume that A and B are proper nilpotent subgroups of the group G such that $G = AB$. Assume that $D(G) = 1$ and that G has only one minimal normal subgroup M. Let p be a prime dividing $|M|$. Then $M = O_p(G) = F(G) \ne G$ and one of A and B is a Sylow p-subgroup of G while the other is a Hall p'-subgroup of G.*

Proof. Since G is solvable, M is an elementary abelian p-group. Due to the uniqueness of M, $F(G) = O_p(G) \supseteq M$. Let J and K be the

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Hall p'-subgroups of A and B, respectively. Let P and Q be the Sylow p-subgroups of A and B, respectively. Then by a theorem of Wielandt [6], PQ is a Sylow p-subgroup of G and JK is a Hall p'-subgroup of G. Since $D(G) = 1$, there is a maximal subgroup H which does not contain M. Then $MH = G$, and, since M is abelian, $M \cap H$ is normal in MH. Hence $M \cap H = 1$, and so H is a complement to M in G. Then $|H| = |G/M|$ which implies that H contains a Hall p'-subgroup of G. Replacing H by a conjugate if necessary, we may assume that H contains JK.

Since $C_H(M)$ is normal in $MH = G$ and $M \cap H = 1$, we must have $C_H(M) = 1$. Hence $C_G(M) = MC_H(M) = M$. Next, $M \cap Z(F(G))$ is a non-identity normal subgroup of G. The minimality of M implies that $M \subseteq Z(F(G))$. Then $F(G) \subseteq C_G(M) = M$. Hence $M = F(G)$. If $G = M$, then G can have no proper non-identity subgroup which would imply that $|G| = p$. Since A and B are both proper and $G = AB$, this is impossible. Thus $G \neq M$.

Since $M \subseteq PQ$ and $[PQ, J \cap K] = 1$, $J \cap K \subseteq C_G(M) = M$. Therefore $J \cap K = 1$. Now let $R = O_{p'}(H)$. H is isomorphic to G/M and $M = O_p(G)$. Hence $O_p(H) = 1$. It follows from this that $C_H(R) = Z(R)$. Now $C_M(R)$ is normal in $MH = G$ and M is a minimal normal subgroup of G. Thus $C_M(R)$ is either 1 or M. But $C_M(R) = M$ would imply that $R \subseteq C_G(M) = M$, an impossibility. Thus $C_M(R) = 1$. Since $C_G(R) \subseteq C_H(R)M = Z(R)M$ and $C_M(R) = 1$, we must have $C_G(R) = Z(R)$. But $R \subseteq JK$ and $[JK, P \cap Q] = 1$. Hence $P \cap Q = 1$, which implies that $A \cap B = 1$. Therefore $|G| = |A||B|$.

Now let $A_1 = AM \cap H$ and $B_1 = BM \cap H$. Since $AM = M(AM \cap H)$ and $A_1 \cap M = H \cap M = 1$, A_1 is isomorphic to AM/M which is isomorphic to the nilpotent group $A/(A \cap M)$. Similarly, B_1 is isomorphic to $B/(B \cap M)$. Clearly J and K are Hall p-subgroups of A_1 and B_1, respectively. Thus if L is a p-subgroup of $A_1 \cap B_1$, then $[R, L] \subseteq [JK, L] = 1$.

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Since $C_G(R)$ is a p'-group, it follows that $A_1 \cap B_1$ is a p'-group. Then $A_1 \cap B_1 \subseteq J \cap K = 1$. Hence

$$|A_1B_1| = |A_1||B_1| = |A||B|/(|A \cap M||B \cap M|)$$
$$= |G|/|(A \cap M)(B \cap M)| \geq |G|/|M| = |H| .$$

Since $A_1B_1 \subseteq H$, this implies that $A_1B_1 = H$ and $M = (A \cap M)(B \cap M)$.

Now $A \cap M \subseteq C_M(J)$. Therefore $C_M(J) = (A \cap M)\big(B \cap M \cap C_M(J)\big)$. But $C_M(J) \cap B \cap M \subseteq C_M(JK) \subseteq C_M(R) = 1$. Thus $C_M(J) = A \cap M$. Similarly $C_M(K) = B \cap M$.

Suppose now that $[J, K] = 1$. Then $C_M(J)$ and $C_M(K)$ are both normalized by JK. Since $C_M(J) \cap C_M(K) = 1$, $C_M(J) = \big[C_M(J), K\big] \subseteq [M, K]$. Since $M = [M, K] \times C_M(K)$ and $M = C_M(J) \times C_M(K)$, we must have $[M, K] = C_M(J)$. Similarly $[M, J] = C_M(K)$. Since P normalizes J and Q normalizes K, it follows that PQ normalizes both $C_M(J)$ and $C_M(K)$. Thus $C_M(J)$ and $C_M(K)$ are normal in $(JP)(KQ) = AB = G$. Due to the minimality of M, one of $C_M(J)$ and $C_M(K)$ is 1. Assume, say, that $C_M(J) = 1$. Then $C_M(K) = M$. Hence $K \subseteq C_G(M) = M$ and so $K = 1$. Then $P \subseteq C_G(J) = C_G(JK) \subseteq C_G(R) = Z(R)$ which implies that $P = 1$. It now follows that A is a Hall p'-subgroup and B is a Sylow p-subgroup.

We now assume that $[J, K] \neq 1$ and derive a contradiction. Let $T = [J, K]$. Since T is a p'-group; $[M, T] \neq 1$. From Maschke's Theorem, there is a subgroup N in M such that N is a minimal normal subgroup in MJK and $[N, T] \neq 1$. If $x \in N$, then $x = yz$ for some $y \in C_M(J)$ and $z \in C_M(K)$. Then $N \supseteq [x, J] = [z, J]$. It follows from this that $Nz \in C_{M/N}(JK) = C_M(JK)N/N = N/N$. Hence $z \in C_N(K)$. Similarly, $y \in C_N(J)$. It now follows that $N = C_N(J) \times C_N(K)$.

Suppose that $C_N(J) = 1$. Then $[N, K] = 1$. In that case

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$[N, K, J] = 1$ and $[N, J, K] = [N, K] = 1$. The three subgroups lemma yields $[T, N] = 1$. Hence $C_N(J) \neq 1$. Similarly $C_N(K) \neq 1$. Then both $[N, J]$ and $[N, K]$ are proper subgroups of N .

Let S be a maximal subgroup of N containing $[N, J]$. Then J normalizes S and so $(S \cap C_N(K))^{MKJ} = (S \cap C_N(K))^J \subseteq S$. Due to the minimality of N , we must have $S \cap C_N(K) = 1$. Since $|N/S| = p$, we must have $|C_N(K)| = p$. Similarly $|C_N(J)| = p$ and so $|N| = p^2$.

Then $|[N, J]| = p$. Since the automorphism group of a group of order p is abelian, $J/C_J(N)$ must be abelian. Similarly $K/C_K(N)$ is abelian. Now let $U = JK/C_{JK}(N)$, $J_1 = JC_{JK}(N)/C_{JK}(N)$, and $K_1 = KC_{JK}(N)/C_{JK}(N)$. Then J_1 and K_1 are abelian and $U = J_1K_1$. A theorem of Itô [4] implies that there is a non-identity normal subgroup of U contained in either J_1 or K_1 . Assume then that L is a non-identity normal subgroup of U and $L \subseteq J_1$. Then $C_N(L)$ is normal in MJK and $C_N(L) \neq N$. The minimality of N implies that $C_N(L) = 1$ which contradicts $C_N(L) \supseteq C_N(J_1) = C_N(J) \neq 1$. This contradiction finishes the proof of the theorem.

3. The main theorems

For the rest of the paper, we assume that A and B are nilpotent subgroups of the group G such that $G = AB$.

THEOREM 2. $d(G/D(G)) \leq c(A) + c(B)$.

Proof. Let G be a minimal counter-example and let $n = c(A) + c(B)$. If N is a non-identity normal subgroup of G , then due to the minimality of G we must have $d((G/N)/D(G/N)) \leq n$. Hence $G/N \in F_n$ but $G \notin F_n$. Since the theorem is certainly true if G is nilpotent, A and B must be proper. Applying Lemma 1 and Theorem 1, we find that $(|A|, |B|) = 1$ and $D(G) = 1$. The theorem now follows from [3, Theorem 1.2.4].

THEOREM 3. *Assume that the Sylow 2-subgroups of both* A *and* B

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have class at most 2 . *Then*

$$d\big(G/D(G)\big) \le d(A)d(B) + 1 .$$

Proof. Let G be a minimal counter-example and let $n = d(A)d(B) + 1$. Then F_n does not contain G but does contain every proper homomorphic image of G . By Lemma 1, $D(G) = 1$ and G has exactly one minimal normal subgroup M . A and B must be proper and so the hypothesis of Theorem 1 is satisfied. Thus, without loss of generality, we may assume that A is a Sylow p-subgroup of G , that $M = O_p(G) = F(G)$, and that B is a Hall p'-subgroup of G . Lemma 2 implies that $d(A/M) = d(A) - 1$. Then

$$d\big((G/M)/D(G/M)\big) \le \big(d(A)-1\big)d(B) + 1 .$$

Hence, if $m = n - d(B)$ and $H = G/M$, then $H \in F_m$. As pointed out earlier, $H \in F_m$ if, and only if, $H^{(m-1)}$ is nilpotent. Thus $d\big(H/F(H)\big) \le m - 1$. Since $O_p(H) = 1$, $d\big(F(H)\big) \le d(B)$. Thus $d(H) \le m - 1 + d(B)$. This implies that $d(G) \le d(H) + 1 \le n$.

THEOREM 4. *Assume that the Sylow* 2*-subgroups of both* A *and* B *have class at most* 3 . *Then* $l(G) \le d(A) + d(B)$.

Proof. Let G be a minimal counter-example and let $r = d(A) + d(B)$. Then N_r contains every proper homomorphic image of G but does not contain G . Therefore, as in the proof of Theorem 3, we may assume that $F(G) = O_p(G)$, that A is a Sylow p-subgroup of G , and that B is a Hall p'-subgroup of G . Let $H = G/F(G)$. Then $l(G) = l(H) + 1 \le d(B) + d\big(A/F(G)\big) + 1$. Since G is a counter-example, we must have $d\big(A/F(G)\big) = d(A)$. It follows from Lemma 2 that $p = 2$ and $c(A) = 3$. Then Lemma 3 implies that $l_2(G) \le 2$. From this, we obtain $l(G) \le 4$. Since $l(G) > d(A) + d(B)$, we must have $d(B) = 1$. Since $F(H)$ is a $2'$-group and $C_H\big(F(H)\big) \subseteq F(H)$, it follows that $H/F(H)$ is a 2-group. Therefore $l(H) \le 2$. Then $l(G) \le 3 = d(A) + d(B)$ and the theorem is proved.

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