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GAME MODEL FOR ONLINE AND OFFLINE RETAILERS UNDER BUY-ONLINE AND PICK-UP-IN-STORE MODE WITH DELIVERY COST AND RANDOM DEMAND

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Abstract

Online retailers are increasingly adding buy-online and pick-up-in-store (BOPS) modes to order fulfilment. In this paper, we study a system of BOPS by developing a stochastic Nash equilibrium model with incentive compatibility constraints, where the online retailer seeks optimal online sale prices and an optimal delivery schedule in an order cycle, and the offline retailer pursues a maximal rate of sharing the profit owing to the consignment from the online retailer. By an expectation method and optimality conditions, the equilibrium model is first transformed into a system of constrained nonlinear equations. Then, by a case study and sensitivity analysis, the model is validated and the following practical insights are revealed. (I) Our method can reliably provide an equilibrium strategy for the online and offline retailers under BOPS mode, including the optimal online selling price, the optimal delivery schedule, the optimal inventory and the optimal allocation of profits. (II) Different model parameters, such as operational cost, price sensitivity coefficient, cross-sale factor, opportunity loss ratio and loss ratio of unsold goods, generate distinct impacts on the equilibrium solution and the profits of the BOPS system. (III) Optimization of the delivery schedule can generate greater consumer surplus, and makes the offline retailer share less sale profit from the online retailer, even if the total profit of the BOPS system becomes higher. (IV) Inventory subsidy is an indispensable factor to improve the applicability of the game model in BOPS mode.

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1. Introduction

1.1. Background With explosion of mobile apps and e-commerce, consumers are free to trade online and offline, and these technologies have fundamentally changed the omni-channel business. Since the consumers can search for information in physical

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stores and online platforms at the same time, they can use online channels for actual purchases at a lower price and take goods offline [1, 23]. Many studies have shown that the omni-channel business is the future of the retail industry [21, 25].

Buy-online and pick-up-in-store (BOPS) is one of the typical omni-channel retailing formats. As an integration of online and offline channels, BOPS can enrich customer choices and improve operational efficiency. Actually, for the online retailers, BOPS service allows them to use offline inventory delivery for their online sales, such that delivery fulfilment time reduces and customer loyalty increases [12]. It was reported [9] that 42% of retailers have added the BOPS option to their sales systems. For the offline retailers, BOPS can offer the opportunity to cross-sell and cross-promote products, which can bring them additional profits [3]. For customers, the BOPS service reduces waiting time compared with courier service.

1.2. Literature review Despite wider application of BOPS services in practice, theoretical research on how to quantitatively improve their cost efficiency is still far from meeting the operational needs, especially those from the perspective of offline retailers and consumers.

Ofek et al. [22] studied how the competing retailers operate dual channels, and examined how pricing strategies and physical store assistance levels change as a result of the additional internet outlet. It suggested that when the decision to open an internet channel is endogenized, there exists an asymmetric equilibrium where only one retailer elects to operate an online arm but earns lower profits than its bricks-only rival. In their analysis, the demand of consumers was assumed to be fixed, and it is also not a consignment mode in a physical store.

Gallino and Moreno [8] analysed the impact of BOPS strategy by using a proprietary data set. They concluded that BOPS implementation results in lower online sales, higher store sales, higher store traffic and the additional store sales can be generated by the cross-selling effect and the channel-shift effect. Note that their developed model [8] is not a game model for the online and offline retailers.

Chen et al. [4] constructed a random Nash equilibrium model to optimize strategy of the online and offline retailers under the BOPS model, where the demand of consumers was supposed to be a continuously random variable, and inventory or shortage cost was considered. Owing to the uncertainty of demand, it was shown that inventory or shortage cost can seriously affect the profits of online retailers and offline retailers. Insufficient store inventory often hinders cross-channel fulfilment and increases the likelihood of losing customers, while excessive inventory increases its cost. However, shipment cost and optimization of delivery plan were not taken into account in this model.

Cao et al. [2] developed an analytical framework to study the impact of BOPS mode on the demand allocations and profitability of a retailer who sells products to customers through multiple distribution channels. It was concluded that BOPS can help the retailer tap new customer segments and generate additional demand, but may also hurt the retailer by cannibalizing existing channels and increasing operating costs. This analytical framework was not studied by a game model done by Chen et al. [4].

Kim et al. [15] used a scenario-based factor survey approach to study how perceived perceptions of innovation and perceived risk of online shopping affect consumers' intentions to use BOPS, but no optimization model was constructed to make optimal decisions for the online or offline retailers such that their profits are maximized.

Jin et al. [13] developed a theoretical model in which one physical store adopting BOPS uses a recommended service area to fulfil orders from both determined (online) and casual (offline) customers in one order cycle; they derived the optimal decisions on the product price and recommended service radius for the retailer adopting BOPS. However, cross-selling benefits and commodity shipment costs were not reflected in this model.

Liu et al. [20] explored whether it is always beneficial for companies to introduce BOPS on the basis of dual channel, and analysed the impact of traditional-consumer proportion and degree of consumers' service sensitivity. The results suggested that whether corporations adopt the BOPS mode or not depends on the size of BOPS-consumer and consumers' degree of service sensitivity, and corporations should not only consider the production and service cost but also have a precise understanding and orientation of consumers' service sensitivity when making the price and service strategies. However, cross-selling benefits and commodity transport costs were not reflected in this model. The profit distribution of the online and offline retailers, cross-selling revenue and various costs of commodity transaction were not also taken into account.

Shi et al. [26] studied a BOPS strategy for the online retailer in the face of both informed and uninformed consumers, where the retailer sells the product over two periods: the informed consumers make pre-orders with unknown valuations in the first period and then pick up the pre-orders in store with realized valuations in the second period. Thresholds of the unit production cost and demand uncertainty were given to answer whether the BOPS strategy with pre-orders is beneficial or not. We note that no cross-selling effect and no practical constraint were considered in this paper [26].

In summary, by utilizing the online and offline retailers' pursuit of profit, the well-structured channel supply chain under the BOPS mode can be cost effective and meet customer needs. Therefore, the retailers have strategic implications for the design of supply chain networks in the BOPS mode, including the time and quantity of inventory, the mode and the shipment frequency [5]. In short, there is widespread consensus on the roles of BOPS mode: it provides customers with real-time information about in-store inventory availability and introduces a new shopping mode that can add convenience to the customers. The former effect (information effect) helps attract customers by letting customers know about inventory availability, but it is a double-edged sword, because it disappoints the customers who are willing to visit the store when inventory is not available. The latter effect (convenience) is suitable for the customers to use the store pick-up function, which can attract the customers to the store and may even open up new sources of demand.

However, in the existing research results on the BOPS mode, there are still the following deficiencies.

(1) There are few game models to describe the competition relation between the online and offline retailers in the BOPS mode.
(2) No integrated optimization model has been constructed to offer an optimal strategy for the selling, inventory, delivery and profit allocation for the online and offline retailers in BOPS.
(3) The demand is often assumed to be fixed, rather than a continuously random variable.

1.3. Research intention of this paper From the above literature review, it is clear that development of an integrated stochastic equilibrium model is valuable to the BOPS mode, which can offer the online and the offline retailers an optimal strategy of selling, inventory, delivery and profit allocation. In this paper, we attempt to develop such a model to answer how the online retailer first determines an optimal selling price online under random demand, and then chooses an optimal delivery schedule (the delivery frequency and the quantity of single delivery), and how the offline retailer shares the sale profit with the online retailer owing to consignment.

Compared with the stochastic model of Chen et al. [4], we are more concerned how to optimize the delivery schedule, since it is a distinct feature of consignment strategy under the BOPS mode. Additionally, apart from the existing results, it is necessary to take into account the overstocking cost and understocking loss in the case that the demand is uncertain.

In essence, the problem of optimal decision making in the BOPS mode is formulated by a stochastic Nash equilibrium model. By the aid of this model, we will expound the following issues.

(i) How to determine the optimal strategy of selling, inventory, delivery and profit allocation by mathematical modelling?
(ii) What are the impacts of model parameters on the optimal decisions and the profits of the online and offline retailers?

The rest of this paper is organized as follows. In the next section, a stochastic Nash equilibrium model is constructed. Section 3 is devoted to analysis of the model's properties. Case study and sensitivity analysis are conducted in Section 4. Conclusions and directions in future research are given in Section 5.

2. Equilibrium model for online and offline retailers

In this section, we present an equilibrium model for the online and offline retailers under BOPS mode.

2.1. Problem description and notation We first make the following settings to specify the handled problem in this paper.

- In the system of BOPS, there are an online retailer (OL) and an offline retailer (OF).

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- To achieve a cooperation between OL and OF, a mechanism of incentive compatibility is adopted.
- The online retailer first orders products from suppliers at a given cost and then consigns a part of the products to the offline retailer.
- The offline retailer has unlimited stocking capacity for the consignment quantity. She sells the goods for the online retailer to share the profit generated by the consignment sale.
- The consigned goods is complementary to the other goods sold in the physical store (the offline retailer). Therefore, for the offline retailer, there are additional sales generated by cross-sale.
- By the principle of "economic lot size", we assume that inventory consumption is uniform.
- Smaller delivery frequency is beneficial to the online retailer, but it can increase inventory cost of the offline retailer. Since only one planning period is considered, loss caused by unsold goods is taken into account. For this, the inventory subsidy must be taken into account, which is paid by the online retailer to the offline retailer. In other words, the unsold goods do not enter the next planning period.
- In the planning period, the goods are not perishable.

The following is the notation used in this paper.

Parameters

T_0: a fixed planning period, which can be regarded as a unit time.
OL: the online retailer.
OF: the offline retailer.
$D(p)$: the demand for the consignment goods per unit time.
$y(p)$: the expected demand at the offline retailer per unit time.
$m(D)$: the cross-sale quantity generated by consignment per unit time.
a: the primary demand per unit time.
β: the price sensitivity.
ϵ: a random scaling factor.
u: the probability density function factor.
$f(x)$: the probability density function of ϵ.
$F(x)$: the cumulative distribution function of ϵ.
$E(\cdot)$: the expectation of a random variable.
c_{ol}: the unit cost of ordering the consignment goods by the online retailer.
c_{of}: the unit cost of selling the consignment goods by the offline retailer.
h: the unit cost of handling the consignment goods by the offline retailer per unit time.
c_d: the fixed cost of a single delivery by the online retailer.
c: the unit shipment cost of the consignment goods by the online retailer.

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α: the opportunity loss ratio.
γ: the loss ratio of unmarketable products.
δ: $\delta \in \{0, 1\}$. Here $\delta = 1$ if and only if the online retailer offers inventory subsidy to the offline retailer.
L: the online retailer's expected loss.
π_{ol}: the online retailer's expected profit.
π_{of}: the offline retailer's expected profit.
π: the total profit.
v: the cross-sale factor.
Q: the total shipment quantity during the planning period; $Q = Nd$.
z: the stocking factor.

Decision variables

r: the rate of sale-profit sharing.
p: the selling price.
N: the delivery frequency per unit time.
d: the single-delivery quantity.

2.2. Demand function Suppose that the demand for the consignment goods is random and price-dependent during a single selling season [1, 19]. A popular model for such a demand per unit time is specified by

$$D(p) = y(p)\epsilon, \tag{2.1}$$

where p is the selling price of the online retailer and ϵ is a random factor with expected value $E[\epsilon] = 1$. Particularly, we suppose that the support set of ϵ is an interval $[A, B] \subset \mathbf{R}$ $(B > A \geq 0)$, and the relation between the demand and the price is

$$y(p) = ap^{-\beta}, \tag{2.2}$$

where $\beta > 1$ is called a price sensitivity parameter and a is the primary demand [4]. In practice, different values of β are used to reflect the potential feature of the goods. For luxury goods, β is relatively large compared with daily necessities. Clearly, the expected demand of the consignment goods per unit time is $y(p)$.

With the above definition of the demand, we are going to construct an equilibrium model for the optimal decision making of the online and offline retailers.

2.3. Optimization model for online retailers The online retailer maximizes the profit by optimizing the online sale price, the delivery quantity in a single shipment and the delivery frequency during the planning period.

Denote by p the online sale price of consignment goods, d the delivery quantities and N the delivery frequency, respectively. By a take-it-or-leave-it consignment contract, let r $(0 \leq r \leq 1)$ be the revenue rate of the offline retailer sharing with the online retailer. Due to uncertainty of the demand, the profit is associated with possible

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cost of over-storage or loss of under-stocking. If the demand D is defined by (2.1) and (2.2), then the cost of over-storage is

$$L_1 = N\Big(\delta \frac{hT_0}{N} + \gamma c_{ol}\Big)\Big(d - D(p)\frac{T_0}{N}\Big)^+, \tag{2.3}$$

where the second and third factors in the right-hand side of equation (2.3) are the inventory subsidy and loss of unsold goods, respectively. Then, the loss generated by the shortage of the consignment goods is referred to as

$$L_2 = N\alpha(1-r)(p - c_{ol})\Big(D(p)\frac{T_0}{N} - d\Big)^+,$$

where $(\cdot)^+$ is defined by

$$(v)^+ = \begin{cases} 0, & v \le 0, \\ v, & v > 0 \end{cases}$$

and $\alpha(1-r)(p - c_{ol})$ is the unit opportunity loss. Besides these costs, the online retailer needs to pay the shipment cost $L_3 = N(c_d + cd)$. Thus, for the online retailer, the total stochastic profit from selling the consignment goods is written as

$$\pi_{ol}(p, N, d) = N(1-r)(p - c_{ol}) \min\Big\{d, D(p)\frac{T_0}{N}\Big\} - (L_1 + L_2 + L_3).$$

Let F and f be the cumulative distribution function and the probability density function of the random parameter ϵ in D with support set $[A, B]$, respectively. We call $z = dN/T_0 y(p)$ the online retailer's stocking factor, and define a function $\Lambda : [A,B] \to R$, given by

$$\Lambda(z) = \int_A^z (z - x) f(x)\, dx. \tag{2.4}$$

Consequently, $\Lambda(z) = \int_A^z F(x)\, dx$, and $l(z) = z - \Lambda(z)$ is positive and increasing in z (see Propositions 2.1 and 2.2 in [4]). The following proposition gives the expected total cost or loss of the online retailer.

Proposition 2.1. *Suppose that the demand D is defined by (2.1) and (2.2). Let Λ be defined by (2.4). Then, the expected total cost or loss of the online retailer is*

$$L(p, z, N; r) = T_0 y(p)\Big[\Big(\delta \frac{hT_0}{N} + \gamma c_{ol}\Big)\Lambda(z) + \alpha(1-r)(p - c_{ol})\{l(B) - l(z)\} + cz\Big] + Nc_d.$$

Proof. From the definition of D,

$$\begin{aligned} E\Big[\Big(d - D(p)\frac{T_0}{N}\Big)^+\Big] &= E\Big[y(p)\frac{T_0}{N}(z - \epsilon)^+\Big] \\ &= y(p)\frac{T_0}{N} E[(\epsilon - z)^+] \\ &= y(p)\frac{T_0}{N} \int_A^z (z - x) f(x)\, dx \\ &= y(p)\frac{T_0}{N}\Lambda(z) \end{aligned}$$

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and

$$\begin{aligned} E\Big[\Big(D(p)\frac{T_0}{N}-d\Big)^+\Big] &= y(p)\frac{T_0}{N}E[(\epsilon-z)^+] \\ &= y(p)\frac{T_0}{N}\int_z^B (x-z)f(x)\,dx \\ &= y(p)\frac{T_0}{N}\Big((x-z)F(x)|_z^B-\int_z^B F(x)\,dx\Big) \\ &= y(p)\frac{T_0}{N}\Big((B-z)F(B)-\int_z^B F(x)\,dx\Big) \\ &= y(p)\frac{T_0}{N}\Big(B-\int_A^B F(x)\,dx-\Big(z-\int_A^z F(x)\,dx\Big)\Big) \\ &= y(p)\frac{T_0}{N}\{l(B)-l(z)\}. \end{aligned}$$

Thus, the expected total cost or loss of the online retailer reads

$$L(p,z,N;r)=T_0y(p)\Big[\Big(\delta\frac{hT_0}{N}+\gamma c_{ol}\Big)\Lambda(z)+\alpha(1-r)(p-c_{ol})\{l(B)-l(z)\}+cz\Big]+Nc_d.$$

The desired result has now been proved. □

In the case that the online retailer is risk-neutral, the expected profit of the online retailer is

$$\begin{aligned} \pi_{ol}(p,z,N;r) &= N(1-r)(p-c_{ol})E[\min\{d,D(p)(T_0/N)\}]-E[L] \\ &= T_0y(p)[(1-r)(p-c_{ol})\{(1+\alpha)l(z)-l(B)\} \\ &\quad -(\delta(hT_0/N)+\gamma c_{ol})\Lambda(z)-cz]-Nc_d. \end{aligned} \tag{2.5}$$

Since the profit of the online retailer from selling unit consignment goods is $(1-r)(p-c_{ol})-c$, we require that

$$(1-r)(p-c_{ol})\geq c \tag{2.6}$$

as one of the necessary conditions that the online retailer is willing to choose the consignment mode.

2.4. Optimization model for offline retailers Under the assumptions in this paper, the offline retailer attempts to maximize her own profit by choosing a sale-profit share r as large as possible, with the given decisions (p,d,N) (or (p,z,N)) of the online retailer. The profit of the offline retailer consists of two parts. One part is the share of the sale profit of consignment goods from the online retailer. The other one is from the profit brought by cross-sale in virtue of the consignment goods. We denote the first part of the profit as π^1_{of}. Then,

$$\begin{aligned} \pi^1_{of} &= N\{r(p-c_{ol})-c_{of}\}\min\Big\{d,D(p)\frac{T_0}{N}\Big\}-\frac{1}{2}N\Big(\frac{hT_0}{N}\Big)\min\Big\{d,D(p)\frac{T_0}{N}\Big\} \\ &\quad -Nh\frac{T_0}{N}\Big(d-D(p)\frac{T_0}{N}\Big)^+ + Nh\delta\frac{T_0}{N}\Big(d-D(p)\frac{T_0}{N}\Big)^+, \end{aligned} \tag{2.7}$$

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where the first term in the right-hand side of equation (2.7) is the profit from the sale of consigned goods, the second term is the inventory cost of uniformly sold consignment goods, the third term is the inventory cost of unsold consignment goods in the physical store and the last term is the inventory subsidy for unsold consigned goods paid by the online retailer to the offline retailer.

Suppose that the cross-sale quantity generated by consignment per unit time is $m(D) = kD$, where $k > 0$ is a given constant. Denote by p_0 the net profit of unit goods in the cross-sale. Then, the second part of the offline retailer's profit reads

$$\pi_{of}^2 = p_0 T_0 m(D) = k p_0 T_0 D.$$

Consequently, in the case that the online retailer is risk-neutral, the expected total profit of the offline retailer is

$$\begin{aligned}\pi_{of}(r; p, z, N) &= E[\pi_{of}^1 + \pi_{of}^2] \\ &= T_0 y(p)\Big[\Big\{r(p - c_{ol}) - c_{of} - \frac{hT_0}{2N}\Big\} l(z) - (1-\delta)\frac{hT_0}{N}\Lambda(z) + p_0 k\Big]. \end{aligned} \tag{2.8}$$

Since the expected shared profit of the offline retailer from selling consignment goods is $E(\pi_{of}^1)$, we require that $E(\pi_{of}^1) \geq 0$ as one of necessary conditions, such that the offline retailer is willing to choose the consignment mode and pay for all those services. We call it the participation constraint, for which the following condition is satisfied:

$$r(p - c_{ol}) l(z) \geq c_{of} + \frac{hT_0}{2N} l(z) + (1-\delta)\frac{hT_0}{N}\Lambda(z). \tag{2.9}$$

2.5. Incentive compatibility constraints From the profit functions (2.5) and (2.8), it follows that the following inequalities always hold:

$$\frac{d\pi_{of}(r; p, z, N)}{dr} > 0, \quad \frac{d\pi_{ol}(r; p, z, N)}{dr} < 0,$$

which are in line with the fact that the online and offline retailers have opposite preferences for the profit-sharing rate r. If a consignment contract fails to effectively supervise and restrict the profit-sharing rate, the offline retailer may harm the interest of the online retailer, known as the "agency problem". To ensure that the online retailer is willing to choose the consignment mode, incentive compatibility constraints are often regarded as one of the necessary conditions, such that the online retailer accepts a suitable division of the pie with the offline retailer [7, 14, 17]. Specifically, to ensure a successful consignment mode addressed in this paper, an incentive compatibility constraint is proposed, which requires that the online or offline retailers have the same minimum marginal profit. Mathematically, for any r,

$$\min\Big\{\frac{\partial \pi_{of}}{\partial p}(r), \frac{\partial \pi_{of}}{\partial N}(r), \frac{\partial \pi_{of}}{\partial z}(r)\Big\} = \min\Big\{\frac{\partial \pi_{ol}}{\partial p}(r), \frac{\partial \pi_{ol}}{\partial N}(r), \frac{\partial \pi_{ol}}{\partial z}(r)\Big\}. \tag{2.10}$$

REMARK 2.2. Compared with the model of Chen et al. [4], the incentive compatibility constraint (2.10) is a new constraint to ensure a successful consignment mode between the online and offline retailers. By this constraint, how the offline retailer determines a profit-sharing rate r as big as possible should take account of the profit (decision making) of the online retailer. For the online retailer, the added constraint (2.10) makes a smaller set of strategies such that the profit of the online retailer can be shared by the offline retailer.

REMARK 2.3. Clearly, if

$$\frac{\partial \pi_{of}}{\partial p}(r) = \frac{\partial \pi_{ol}}{\partial p}(r), \quad \frac{\partial \pi_{of}}{\partial N}(r) = \frac{\partial \pi_{ol}}{\partial N}(r), \quad \frac{\partial \pi_{of}}{\partial z}(r) = \frac{\partial \pi_{ol}}{\partial z}(r), \tag{2.11}$$

then (2.10) holds. From this viewpoint, (2.11) can be called strong incentive compatibility constraints. However, for the equilibrium model in this paper, we will prove in Theorem 3.2 that (2.10) is equivalent to

$$\min\left\{\frac{\partial \pi_{of}}{\partial p}(r), \frac{\partial \pi_{of}}{\partial z}(r)\right\} = 0.$$

2.6. Nash equilibrium model between online and offline retailers With the above preparation, the retailing system under BOPS mode is involved with the solution of the following Nash equilibrium model by integrating (2.5), (2.8) and all the relevant constraints:

$$\begin{cases} \max\limits_{(p,z,N;r)} \pi_{ol} = T_0 y(p)\Big[(1-r)(p-c_{ol})\{(1+\alpha)l(z) - \alpha l(B)\} \\ \qquad\qquad -\Big(\delta \dfrac{hT_0}{N} + \gamma c_{ol}\Big)\Lambda(z) - cz\Big] - Nc_d, \\ \text{subject to } (1-r)(p-c_{ol}) \geq c, \text{ and} \\ \qquad\qquad N \geq 0, \quad z \in [A,B]; \\ \max\limits_{(r;p,z,N)} \pi_{of} = T_0 y(p)\Big[\Big(r(p-c_{ol}) - c_{of} - \dfrac{hT_0}{2N}\Big)l(z) - (1-\delta)\dfrac{hT_0}{N}\Lambda(z) + p_0 k\Big], \\ \text{subject to (2.10) holds, and} \\ \qquad\qquad r(p-c_{ol})l(z) \geq c_{of} + \dfrac{hT_0}{2N} l(z) + (1-\delta)\dfrac{hT_0}{N}\Lambda(z). \end{cases} \tag{2.12}$$

Since $\delta = 1$, $\Lambda(B) = B - 1$ and $l(B) = 1$, the model (2.12) can be rewritten as

$$\begin{cases} \max\limits_{(p,z,N;r)} \pi_{ol} = T_0 y(p)\Big[(1-r)(p-c_{ol})\{(1+\alpha)l(z) - \alpha\} - \Big(\dfrac{hT_0}{N} + \gamma c_{ol}\Big)\Lambda(z) - cz\Big] \\ \qquad\qquad -Nc_d, \\ \text{subject to } (1-r)(p-c_{ol}) \geq c, \text{ and} \\ \qquad\qquad N \geq 0, \quad z \in [A,B]; \\ \max\limits_{(r;p,z,N)} \pi_{of} = T_0 y(p)\Big[\Big(r(p-c_{ol}) - c_{of} - \dfrac{hT_0}{2N}\Big)l(z) + p_0 k\Big], \\ \text{subject to } r \geq \dfrac{c_{of} + hT_0/2N}{p - c_{ol}}, \text{ and (2.10) holds.} \end{cases} \tag{2.13}$$

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Remark 2.4. Since this paper does not concern the choice of single-channel retailing mode, the model (2.13) provides a way to formulate the BOPS system, rather than an ordinary noncooperative Nash game. In this BOPS system, the players (the offline and online retailers) are not allowed to choose respective single-channel retailing modes. Actually, to ensure a successful BOPS mode, the model (2.13) contains incentive compatibility and participation constraints, given by (2.9) and (2.10).

Remark 2.5. Apart from the existing models available in the literature, the model (2.13) is a modified Nash equilibrium problem taking account of an optimal delivery, apart from the randomness of demands as in [4]. Therefore, the model (2.13) is more applicable in practice than those BOPS models without delivery cost.

Remark 2.6. Owing to the complexity of the model (2.13), it is impossible to directly get its analytical solution. In the literature, random algorithms are often developed to find its approximate solution [16, 24]. In this paper, we intend to first analyse the analytical properties of the model (2.13). Then, an efficient algorithm will be developed to seek for an equilibrium solution.

Remark 2.7. Since the delivery (or ordering) frequency means the delivered (ordered) number per unit time, it is often regarded to be any real number [10]. In other words, N in the model (2.13) is a real continuous decision variable, and T_0/N is a time span between two successive deliveries. Similarly, since the delivered quantity of goods is usually calculated by weight or volume of goods, d in the model (2.13) is also a real continuous variable.

3. Properties of model and solution method

In this section, we will study the analytical properties of the model (2.13) such that a solution method can be proposed.

3.1. Equilibrium conditions

We first prove the following results.

Theorem 3.1. *For a given r, let $(p^*(r), z^*(r), N^*(r))$ be an optimal decision of the online retailer. Then, $(p^*(r), z^*(r), N^*(r))$ satisfies the following system of nonlinear equations.*

$$\begin{cases} p = \dfrac{\beta}{\beta-1}\Big(c_{ol} + \dfrac{(hT_0/N + \gamma c_{ol})\Lambda(z) + cz}{(1-r)\{(1+\alpha)l(z) - \alpha\}}\Big), \\ 1 - F(z) = \dfrac{c + hT_0/N + \gamma c_{ol}}{(\alpha+1)(1-r)(p - c_{ol}) + hT_0/N + \gamma c_{ol}}, \\ N = T_0\sqrt{\dfrac{ap^{-\beta}h\Lambda(z)}{c_d}}. \end{cases} \tag{3.1}$$

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PROOF. We first prove that the solution $(p^*(r), z^*(r), N^*(r))$ satisfying equation (3.1) is in the feasible region. Since $l(z) \le 1$ and $z - l(z) = \Lambda(z) \ge 0$,

$$\begin{aligned}(1-r)(p^* - c_{ol}) &= \frac{1-r}{\beta - 1}c_{ol} + \frac{(hT_0/N^* + \gamma c_{ol})\Lambda(z^*) + cz^*}{(1+\alpha)l(z^*) - \alpha} \\ &> \frac{cz^*}{(1+\alpha)l(z^*) - \alpha} > c.\end{aligned}$$

Thus, from

$$\begin{aligned}0 &< \frac{c + hT_0/N^* + \gamma c_{ol}}{(\alpha+1)(1-r)(p^* - c_{ol}) + hT_0/N^* + \gamma c_{ol}} \\ &< \frac{(1-r)(p^* - c_{ol}) + hT_0/N^* + \gamma c_{ol}}{(\alpha+1)(1-r)(p^* - c_{ol}) + hT_0/N^* + \gamma c_{ol}} < 1,\end{aligned}$$

it follows that $1 > F(z^*) > 0$. From the definition of the cumulative distribution function F, it is clear that $z \in (A, B)$. Then,

$$N^* = \sqrt{\frac{ap^{*-\beta}h\Lambda(z)}{c_d}}T_0 \ge 0.$$

Next, we prove that at the point $(p^*(r), z^*(r), N^*(r))$, the gradient of the online retailer's profit in (2.13) is 0, that is, $\nabla\pi_{ol}(p^*(r), z^*(r), N^*(r)) = 0$. In fact, from the definitions of $y(p)$ and $\pi_{ol}(p, z, N; r)$, it follows that for a given r,

$$\begin{cases}\dfrac{dy(p)}{dp} = -\dfrac{\beta}{p}(ap^{-\beta}) = -\dfrac{\beta}{p}y(p), \\ \dfrac{\partial\pi_{ol}(p,z,N;r)}{\partial p} = -\dfrac{\beta}{p}y(p)T_0[(1-r)(p - c_{ol})\{(1+\alpha)l(z) - \alpha\} \\ \qquad -(hT_0/N + \gamma c_{ol})\Lambda(z) - cz] + y(p)T_0[(1-r)\{(1+\alpha)l(z) - \alpha\}] = 0, \\ \dfrac{\partial\pi_{ol}(p,z,N;r)}{\partial z} = T_0y(p)[(1+\alpha)(1-r)(p - c_{ol})(1 - F(z)) \\ \qquad -(hT_0/N + \gamma c_{ol})F(z) - c] = 0, \\ \dfrac{\partial\pi_{ol}(p,z,N;r)}{\partial N} = T_0y(p)(hT_0/N^2)\Lambda(z) - c_d = 0.\end{cases} \tag{3.2}$$

Since $y(p)T_0 > 0$,

$$\begin{cases}p = \dfrac{\beta}{\beta-1}\left(c_{ol} + \dfrac{(hT_0/N^* + \gamma c_{ol})\Lambda(z^*) + cz^*}{(1-r)(1+\alpha)l(z^*) - \alpha}\right) \equiv p^*(r), \\ F(z) = 1 - \dfrac{c + hT_0/N^* + \gamma c_{ol}}{(\alpha+1)(1-r)(p^* - c_{ol}) + hT_0/N^* + \gamma c_{ol}} \equiv F(z^*(r)), \\ N = T_0\sqrt{\dfrac{ap^{*-\beta}h\Lambda(z)}{c_d}} \equiv N^*(r).\end{cases}$$

Therefore, for any given r, $(p^*(r), z^*(r), N^*(r))$ satisfying equation (3.1) are the first-order optimality conditions of optimal decision of the online retailer. □

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THEOREM 3.2. *Under the participation constraint* (2.9), *the incentive compatibility constraint* (2.10) *is equivalent to*

$$\Big(c_{of} + \frac{hT_0}{2N}\Big)pl(z) \geq \beta p_0 k(p - c_{ol}),$$

while the profit-sharing rate satisfies

$$r \in \Big\{\frac{c_{of} + hT_0/2N}{p - c_{ol}}, \frac{(c_{of} + hT_0/2N)l(z) - p_0 k}{(p - c_{ol} - p/\beta)l(z)}\Big\}.$$

PROOF. Since (3.2) holds for any given r, the incentive compatibility constraint (2.10) is equivalent to

$$\min\Big\{\frac{\partial \pi_{of}}{\partial p}(r), \frac{\partial \pi_{of}}{\partial N}(r), \frac{\partial \pi_{of}}{\partial z}(r)\Big\} = 0. \tag{3.3}$$

By the participation constraint (2.9),

$$\frac{\partial \pi_{of}}{\partial N}(r) = T_0 y(p)\frac{hT_0}{2N^2}l(z) > 0,$$
$$\frac{\partial \pi_{of}}{\partial z}(r) = T_0 y(p)\Big[\Big(r(p - c_{ol}) - c_{of} - \frac{hT_0}{2N}\Big)(1 - F(z))\Big] \geq 0,$$

where $\partial \pi_{of}(r)/\partial z = 0$ if and only if

$$r = \frac{c_{of} + hT_0/2N}{p - c_{ol}}. \tag{3.4}$$

Thus, in the case that

$$r \neq \frac{c_{of} + hT_0/2N}{p - c_{ol}},$$

equation (3.3) is equivalent to

$$\frac{\partial \pi_{of}}{\partial p}(r) = 0.$$

By direct calculation,

$$\frac{\partial \pi_{of}}{\partial p}(r) = -\frac{\beta}{p}T_0 y(p)\Big[\Big\{r\Big(p - c_{ol} - \frac{p}{\beta}\Big) - c_{of} - \frac{hT_0}{2N}\Big\}l(z) + p_0 k\Big] = 0.$$

It follows that

$$r = \frac{(c_{of} + hT_0/2N)l(z) - p_0 k}{(p - c_{ol} - p/\beta)l(z)}. \tag{3.5}$$

On the basis of the above analysis, we conclude that when (3.4) holds, the constraint (2.10) is equivalent to $\partial \pi_{of}(r)/\partial p \geq 0$, that is,

$$\frac{\beta}{p}T_0 y(p)\Big[\Big\{\frac{c_{of} + hT_0/2N}{p - c_{ol}}\Big(p - c_{ol} - \frac{p}{\beta}\Big) - c_{of} - \frac{hT_0}{2N}\Big\}l(z) + p_0 k\Big] \geq 0.$$

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Consequently,

$$\Big(c_{of} + \frac{hT_0}{2N}\Big) pl(z) \geq \beta p_0 k(p - c_{ol}),$$

which makes the online retailer accept that the offline retailer shares the profit with a profit-sharing rate specified by (3.4). If (3.4) does not hold, then the constraint (2.10) implies that the profit-sharing rate of the offline retailer is specified by (3.5). With the the participation constraint (2.9),

$$\frac{(c_{of} + hT_0/2N)l(z) - p_0 k}{(p - c_{ol} - p/\beta)l(z)} \geq \frac{c_{of} + hT_0/2N}{p - c_{ol}}.$$

Since

$$p - c_{ol} - \frac{p}{\beta} = \frac{(hT_0/N + \gamma c_{ol})\Lambda(z) + cz}{(1 - r)\{(1 + \alpha)l(z) - \alpha\}} > 0$$

holds for any (p, z, N) satisfying (2.6) and the first equation in (3.1),

$$\Big\{\Big(c_{of} + \frac{hT_0}{2N}\Big) - \frac{p_0 k}{l(z)}\Big\}(p - c_{ol}) \geq \Big(c_{of} + \frac{hT_0}{2N}\Big)\Big(p - c_{ol} - \frac{p}{\beta}\Big),$$

which also yields

$$(c_{of} + hT_0/2N)pl(z) \geq \beta p_0 k(p - c_{ol}).$$

The desired result has now been proved. □

For any given decision of the online retailer, the offline retailer will choose an optimal sharing rate r from the total sale profit such that the expected profit $\pi_{of}(r; z, p, N)$ is maximized. The following result provides the optimal optimal sharing rate r corresponding to a strategy of the online retailer.

Theorem 3.3. *In the model (2.13), let r be the optimal profit-sharing rate of the offline retailer corresponding to a given strategy (p, z, N) of the online retailer. Then,*

$$r(p, z, N) = \frac{(c_{of} + hT_0/2N)l(z) - p_0 k}{(p - c_{ol} - p/\beta)l(z)}. \tag{3.6}$$

Proof. Since r given by (3.6) is a feasible solution of the optimization model of the offline retailer, it satisfies the incentive compatibility constraint (2.10). By Theorem 3.2,

$$(c_{of} + hT_0/2N)pl(z) \geq \beta p_0 k(p - c_{ol}).$$

Since

$$p - c_{ol} - \frac{p}{\beta} = \frac{(hT_0/N + \gamma c_{ol})\Lambda(z) + cz}{(1 - r)\{(1 + \alpha)l(z) - \alpha\}} > 0$$

holds for any (p, z, N) satisfying (2.6) and the first equality in (3.1),

$$\begin{aligned} &\frac{(c_{of} + hT_0/2N)l(z) - p_0 k}{(p - c_{ol} - p/\beta)l(z)} - \frac{c_{of} + hT_0/2N}{p - c_{ol}} \\ &\quad = \frac{(c_{of} + hT_0/2N)pl(z) - \beta p_0 k(p - c_{ol})}{\beta(p - c_{ol} - p/\beta)l(z)(p - c_{ol})} \geq 0. \end{aligned} \tag{3.7}$$

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It yields

$$r(p,z,N) \geq \frac{c_{of} + hT_0/2N}{p - c_{ol}}.$$

On the other hand, from $d\pi_{of}/dr = T_0 y(p) l(z)(p - c_{ol}) \geq 0$, it follows that π_{of} is nondecreasing in r. From (3.7) and the result in Theorem 3.2, we conclude that for the given strategy (p,z,N) of the online retailer,

$$\pi_{of}\left[\frac{(c_{of} + hT_0/2N)l(z) - p_0 k}{(p - c_{ol} - p/\beta)l(z)}\right] \geq \pi_{of}\left(\frac{c_{of} + hT_0/2N}{p - c_{ol}}\right),$$

that is, the optimal profit-sharing rate r in the model (2.13) is specified by (3.6). This completes the proof. □

Combining the results in Theorems 3.1 and 3.3, we obtain the following result.

Theorem 3.4. *Let (p^*, z^*, N^*, r^*) be an equilibrium solution of the model (2.13). Then, (p^*, z^*, N^*, r^*) solves the following system of constrained nonlinear equations.*

$$\begin{cases} p = \dfrac{\beta}{\beta - 1}\left[c_{ol} + \dfrac{(hT_0/N + \gamma c_{ol})\Lambda(z) + cz}{(1-r)((1+\alpha)l(z) - \alpha)}\right], \\ 1 - F(z) = \dfrac{c + hT_0/N + \gamma c_{ol}}{(\alpha+1)(1-r)(p - c_{ol}) + hT_0/N + \gamma c_{ol}}, \\ N = \sqrt{\dfrac{T_0^2 a p^{-\beta} h \Lambda(z)}{c_d}}, \\ r = \dfrac{(c_{of} + hT_0/2N)l(z) - p_0 k}{(p - c_{ol} - p/\beta)l(z)}, \\ \left(c_{of} + \dfrac{hT_0}{2N}\right) p l(z) \geq \beta p_0 k (p - c_{ol}). \end{cases} \tag{3.8}$$

3.2. Solution method Based on the properties of the game model, we further transform the system of constrained nonlinear equations in (3.8) into a constrained optimization problem, such that it can be solved by off-the-shelf optimization algorithms.

We denote

$$\begin{aligned} h_1(p,z,N,r) &= p(r) - \frac{\beta}{\beta - 1}\left[c_{ol} + \frac{(hT_0/N + \gamma c_{ol})\Lambda(z) + cz}{(1-r)\{(1+\alpha)l(z) - \alpha\}}\right], \\ h_2(p,z,N,r) &= F(z) - 1 + \frac{c + hT_0/N + \gamma c_{ol}}{(\alpha+1)(1-r)(p - c_{ol}) + hT_0/N + \gamma c_{ol}}, \\ h_3(p,z,N,r) &= N^2 - \frac{T_0^2 a p^{-\beta} h \Lambda(z)}{c_d}, \\ h_4(p,z,N,r) &= r - \frac{(c_{of} + hT_0/2N)l(z) - p_0 k}{(p - c_{ol} - p/\beta)l(z)}. \end{aligned}$$

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To facilitate solution of the constrained nonlinear equations in (3.8), it is rewritten as

$$\begin{aligned} h_i(p, z, N, r) = 0, \quad i = 1, 2, 3, 4, \\ \text{subject to } (c_{of} + hT_0/2N)pl(z) \geq \beta p_0 k(p - c_{ol}). \end{aligned} \tag{3.9}$$

Further, we denote $x = (p, z, N, r)^T$ and transform (3.9) into a constrained optimization problem as follows [27]:

$$\begin{aligned} \min f(x) = \tfrac{1}{2}\{h_1^2(x) + h_2^2(x) + h_3^2(x) + h_4^2(x)\}, \\ \text{subject to } (c_{of} + hT_0/2N)pl(z) \geq \beta p_0 k(p - c_{ol}). \end{aligned} \tag{3.10}$$

Clearly, if x^* is a feasible solution of Problem (3.10) such that $f(x^*) = 0$, then $h_i(x^*) = 0$ for all $i = 1, 2, 3, 4$, and x^* is the solution of the constrained system of nonlinear equations (3.8). With this thought, any powerful algorithm for solving a smooth constrained optimization problem can be used to find an equilibrium point of the original problem (3.8) [6, 11, 18], other than the random algorithms used in the literature [16, 24].

4. Case study and sensitivity analysis

In this section, we will validate the developed game model by a case study, and explore underlying managerial implications by sensitivity analysis of this model.

4.1. Case study Before the case study, we first construct a class of probability density functions for the random variable ϵ in the demand function (2.1). As done by Chen et al. [4], we also use the following cubic function as a model of the density function:

$$f_u(x) = kx(x - u)(x - w).$$

To facilitate our case study, we suppose that the support set of the stochastic disturbance ϵ of demand in (2.1) is the interval $[0, 2]$, that is, $A = 0$ and $B = 2$. Then, f_u satisfies

$$\begin{cases} f_u(x) \geq 0, \quad x \in [0, 2], \\ \int_0^2 f_u(x)\,dx = F(2) = 1, \\ \int_0^2 x f_u(x)\,dx = E[\epsilon] = 1. \end{cases}$$

In this case, the density function is

$$f_u(x) = \frac{u - 2}{4u^2/3 - 16u/5 + 8/5} \cdot x(x - u)\left(x - \frac{2u - 18/5}{u - 2}\right),$$

where $u \geq 2$ is called a shape factor of the density function. Without loss of generality, we choose $u = 3$ in this paper. Then, the density function of ϵ is

$$f_3(x) = \frac{x}{4}\left(x - \frac{12}{5}\right)(x - 3), \quad x \in [0, 2].$$

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To conduct the case study, we choose the parameters in the model (2.13) as follows.

$$\begin{array}{lll} u = 3, & T_0 = 30\,(\text{days}), & a = 10000, \\ \alpha = 0.4, & \beta = 1.6, & \gamma = 0, \ \ \delta = 1, \\ h = 2(\text{CNY/RMB}), & c_{ol} = 30(\text{CNY/RMB}), & c_{of} = 8(\text{CNY/RMB}), \\ c_d = 100(\text{CNY/RMB}), & c = 2(\text{CNY/RMB}), & v = 5(\text{CNY/RMB}). \end{array} \tag{4.1}$$

For the given values of parameters in (4.1), we implement any solver of unconstrained optimization problems, such as FMINUNC in the Matlab platform, to solve (3.10). All computer codes are written in Matlab (R2018a), and the numerical experiments are carried out on a laptop with an Intel Core i5 1.60 GHz processor and with 8 GB of RAM under Windows 10. Within a total elapsed CPU time of less than 0.2 seconds, we get a solution of Problem (3.10) as follows:

$$\begin{array}{lll} x^* = (121.98, 1.60, & 7.16, 0.45), & d^* = 30.73, \\ \pi_{ol} = 4881.06, & \pi_{of} = 4609.52, & \pi = 9490.58, \end{array} \tag{4.2}$$

where π is the total profit given by $\pi = \pi_{ol} + \pi_{of}$. The corresponding minimal value of the objective function in the model (3.10) is

$$f(x^*) = \tfrac{1}{2}\{h_1^2(x^*) + h_2^2(x^*) + h_3^2(x^*) + h_4^2(x^*)\} = 6.33 \times 10^{-13}.$$

As an equilibrium point of the original game model (2.13), the numerical results in (4.2) demonstrate that for the online retailer, the optimal selling price is 121.98 CNY/RMB, the single-delivery quantity is 30.73 and the delivery frequency is 7.16. For the offline retailer, the maximal profit-sharing rate is 0.45. Furthermore, at this equilibrium point, the total profit earned by the online and offline retailers is 9490.58 CNY/RMB.

These results verify that the proposed method in this paper can reliably provide the optimal strategy of selling, inventory, delivery and profit allocation for the two players under BOPS mode.

4.2. Sensitivity analysis of cost coefficients To explore the underlying managerial implications in our game model, we now study what are the impacts of operational costs (c_{of}, h, c_d, c) on the equilibrium point and the profits by sensitivity analysis.

In practice, the service cost and inventory cost are mainly borne by the offline retailer, while the shipment cost is completely borne by the online retailer. With this consideration, the following three scenarios are designed for the subsequent analysis:

$$\begin{array}{l} \text{Scenario 1: } (c_{of}, h, c_d, c) = (2, 1, 400, 4)(\text{CNY/RMB}), \\ \text{Scenario 2: } (c_{of}, h, c_d, c) = (5, 2, 250, 3)(\text{CNY/RMB}), \\ \text{Scenario 3: } (c_{of}, h, c_d, c) = (8, 3, 150, 2)(\text{CNY/RMB}). \end{array}$$

Clearly, the above three scenarios stand for the following types of markets with distinct operational costs.

Scenario 1 : (underdeveloped area) lower service cost and inventory cost with higher shipment cost.

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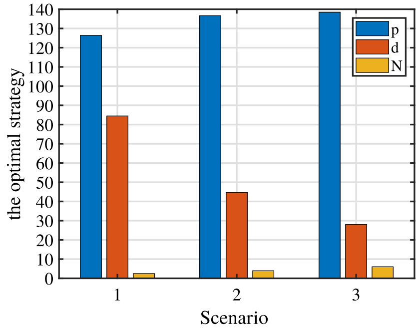
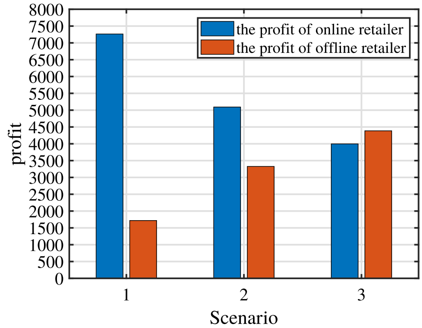


(a) On the equilibrium solution



(b) On the profits

FIGURE 1. Effects of operational costs.

Scenario 2 : (developing area) medium service cost and inventory cost with medium shipment cost.

Scenario 3 : (developed area) higher service cost and inventory cost with lower shipment cost.

The impacts of distinct operational costs on the profits and on the equilibrium solution are depicted in Figure 1, from which we note the following results.

(1) The increasing shipment cost results in a smaller delivery frequency and an increasing single shipment quantity (see the yellow and orange-red parts in Figure 1(a). The shipment costs, including the fixed and the unit shipment cost, seriously affect the optimal delivery schedule. As the unit service and holding cost paid by the offline retailers increases, the optimal selling price also increases with different degrees in line with the three types of markets (see the blue parts in Figure 1(a)).

(2) Compared with developing areas, the online retailers in the developing and underdeveloped areas are more profitable, although they need to pay for higher shipment cost (see Figure 1(b)). Similarly, with the highest service and holding cost in Scenario 3, the profit of the offline retailer is the highest. These results imply that the proposed models in this paper can keenly capture uncertainty of the operational costs.

4.3. Sensitivity analysis of price sensitivity coefficient We next study the impact of the price sensitivity coefficient on the allocation of profits and the equilibrium solution.

We change the value of the price sensitivity coefficient β in the model (2.13) from 1.15 to 1.85 by steps of 0.05; the corresponding equilibrium solutions are shown in Figure 2.

From Figure 2, the following is clear.

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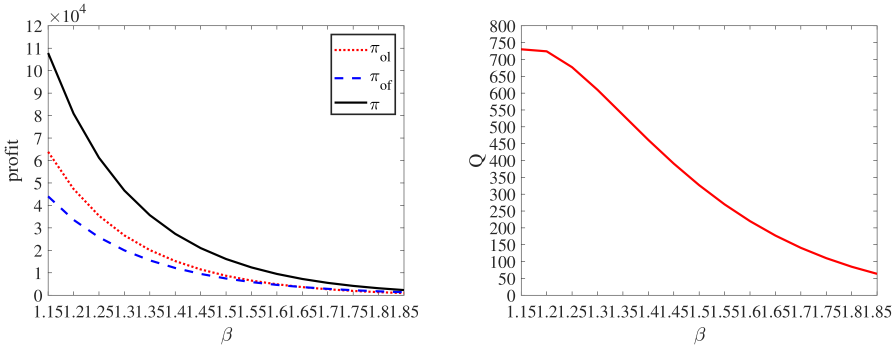

(a) On the profits (b) On the total shipment quantity

FIGURE 2. Impacts of price sensitivity coefficient.

(1) As the price sensitivity coefficient rises, the profits of the online and offline retailers and the total shipment quantity go down (see Figure 2). It is easily seen that the consignment goods (luxury goods) with higher price sensitivity coefficient bring less profits to both of the online and offline retailers, but it seems that the profit of the offline retailer is less sensitive to the price sensitivity coefficient than that of the online retailer (see Figure 2(a)).
(2) Consignment goods with lower price sensitivity coefficient and higher sales volume seem to be more suitable to adopt the BOPS mode (see Figures 2(a) and 2(b)).

4.4. Sensitivity analysis of cross-sale factor We are in a position to investigate the impacts of the cross-sale factor v. Since $v = p_0k$ represents the profit from the unit cross-sale generated by the unit consignment goods, we explore its relation with the profit-sharing rate and its impacts on the price. We change v from 0 to 10 by a step size of 0.5. The numerical results are plotted in Figure 3.

From Figure 3, we conclude the following results.

(1) The profit-sharing rate, r, is sensitive to the cross-sale factor (see Figure 3(a)). With an increasing unit profit of cross-sale, the profit-sharing rate decreases. In other words, by the BOPS mode, both of the offline and the online retailers can share the revenue generated by the cross-sale.
(2) When the cross-sale becomes more profitable, the optimal selling price of the online retailer deceases, which surely brings dividends to the consumers (see Figure 3(b)). Therefore, choosing a suitable type of goods with a higher cross-sale factor in the BOPS mode can create a tripartite win–win situation.

4.5. Sensitivity analysis of opportunity loss ratio We come to answer what are the influences of opportunity loss ratio on the equilibrium solution, which are related with shortage of the consignment goods. To conduct our analysis, the opportunity

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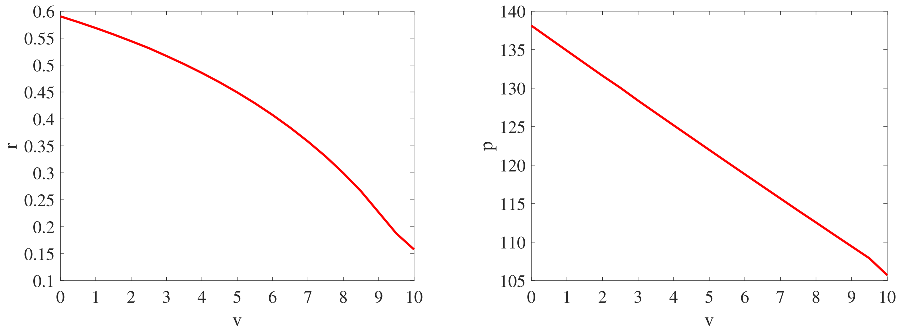

(a) On the sales-profit sharing (b) On the price

Figure 3. Impacts of cross-sale factor.

loss ratio is taken from 0 to 1 by a step size of 0.1. For these different choices, the numerical results are presented in Figure 4, where the following is presented.

(1) An increasing opportunity loss ratio results in a higher selling price, higher delivery frequency, greater quantity in single delivery, greater total order volume and smaller rate of sharing the sale profit (see Figure 4(a)–(e)). We conclude that the opportunity loss ratio affects the optimal equilibrium point in the BOPS mode, though the value of each equilibrium solution differs within 10% ($\alpha = 0$ and $\alpha = 1$).
(2) The profits of the online and offline retailers are only a little sensitive to the opportunity loss ratio (see Figure 4(f)). In other words, the Nash game strategy [4] in the BOPS mode can hedge the impact of opportunity loss on profits, which can be earned by dynamically changing the equilibrium points for the varying opportunity loss ratios.

4.6. Sensitivity analysis of loss ratio of unsold goods In practice, unsold goods are a loss to retailers. Lower loss ratio of unsold goods often allows the retailers to get greater profit of a BOPS system. Therefore, it is interesting to answer what are the impacts of the loss ratio of unsold goods on the equilibrium.

We change the loss ratio γ of unsold goods in the model (2.13) from 0 to 1 with a step size of 0.1. Numerical results are presented in Figure 5.

From Figure 5, we conclude the following results.

(1) As the loss ratio γ increases, the selling price increases and the sales-profit sharing rate decreases, which indicates that the loss of unsold goods is borne together by the online retailer, offline retailers and customers (see Figures 5(a) and 5(b)).
(2) With an increasing loss ratio γ, both the delivery frequency and the delivery quantities decrease. The total shipment quantity Q also decreases during the planning period. It reveals that higher loss of unsold goods has negative impacts to the equilibrium strategy (see Figures 5(c) and 5(d)).

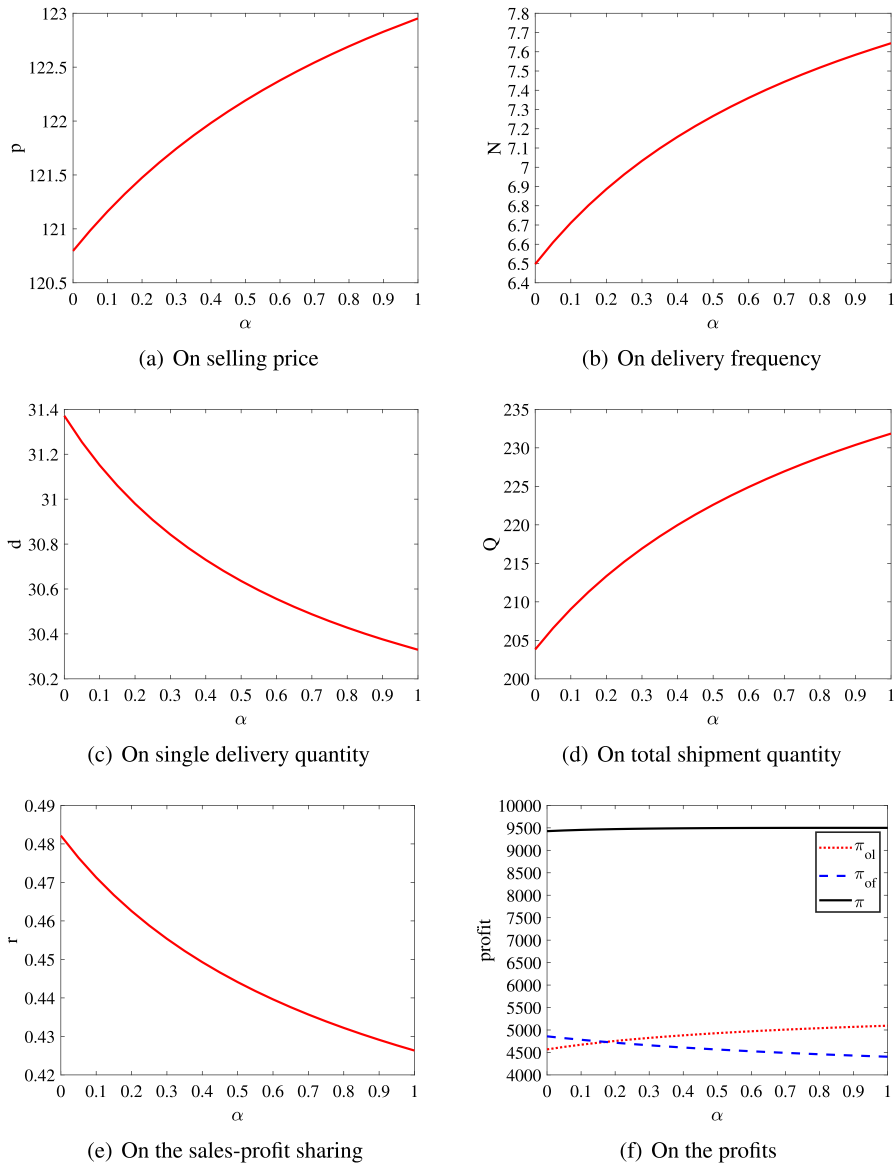

FIGURE 4. Impact of opportunity loss ratio.

(3) An increasing loss ratio γ results in a linear reduction of the retailers' profit. Therefore, it should be profitable to reduce the loss rate γ of unsold goods by recycling or reusing them (see Figure 5(e)).

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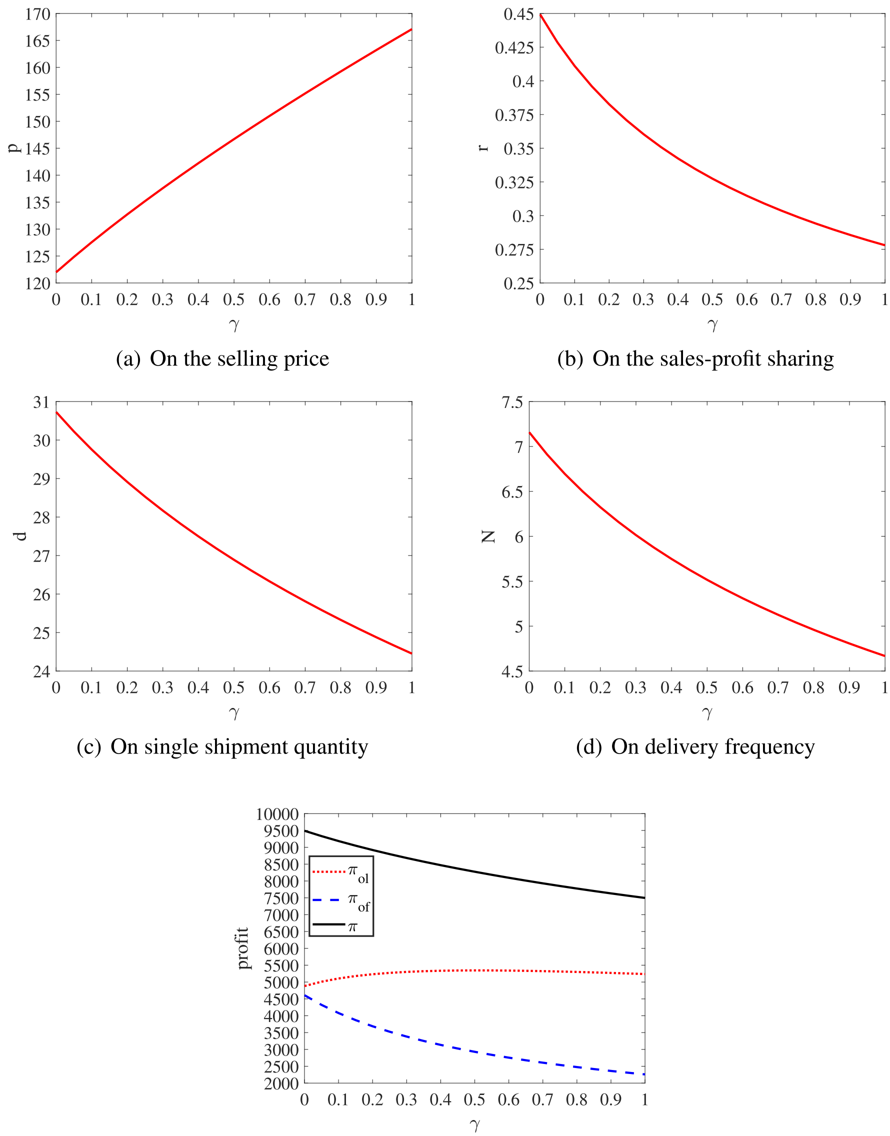

(a) On the selling price

(b) On the sales-profit sharing

(c) On single shipment quantity

(d) On delivery frequency

(e) On the profits

FIGURE 5. Impact of the loss ratio of unsold goods.

4.7. Comparison between two models without and with delivery cost

Compared with the model of Chen et al. [4], our model concerns the delivery cost in the BOPS mode. So, an interesting issue is to answer what are the differences between the two similar models. Since no delivery schedule is regarded in [4], we can think that its delivery frequency $N^* = 1$. Additionally, since the model in [4] is not

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TABLE 1. Comparison between the BOPS systems with or without delivery cost.

Model	NS	$f(x^*)$	π	π_{ol}	π_{of}	p	d	r
Model_{us}	$N^* = 5.00$	3.83×10^{-14}	7340.5	5439.9	1900.6	179.71	22.66	0.24
$\text{Model}_{\text{Chen}}$	$N = 1$	5.34×10^{-16}	5398.3	1160.3	4238.0	490.51	20.10	0.75

associated with any inventory subsidy, we have $\delta = 0$. Thus,

$$\frac{\partial \pi_{ol}(p, z, N; r)}{\partial N} = T_0 y(p) \delta \frac{hT_0}{N^2} \Lambda(z) - c_d = -c_d < 0.$$

Clearly, smaller delivery frequency is beneficial to the online retailer, but it may increase inventory pressure and cause loss to the offline retailer. Thus, introduction of inventory subsidy in the game model is helpful to the cooperation between the online and offline retailers. We intend to address how the equilibrium point depends on the delivery schedule, as well as what are the differences between the two models with or without inventory subsidy.

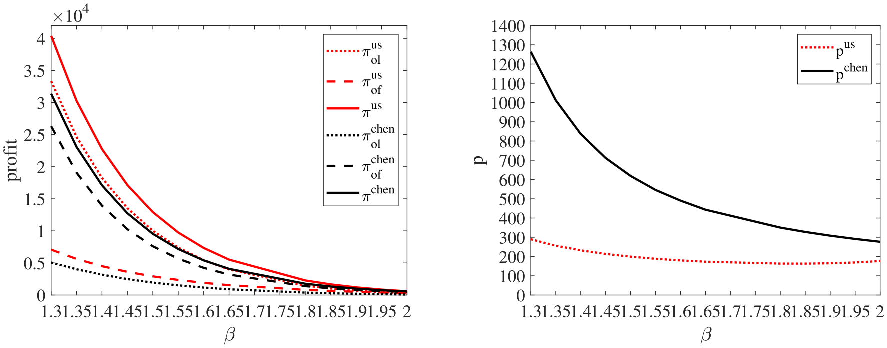
For brevity, we denote "the model in this paper" and "the model in [4]" by "Model_{us}" and "$\text{Model}_{\text{Chen}}$", respectively. Also, "NS" represents the number of shipments. Since the unsold goods are regarded to be worthless without any profit, we take the loss ratios of the unsold products $\gamma = 1$ and the opportunity loss $\alpha = 1$ in $\text{Model}_{\text{Chen}}$ and Model_{us}, respectively. Additionally, the shipment cost for these unsold goods should also be added to $\text{Model}_{\text{Chen}}$. The other model parameters are chosen to be the same as in (4.1).

Clearly, unlike $\text{Model}_{\text{Chen}}$, Model_{us} is concerned with the optimal delivery schedule to maximize the total profit of the BOPS system, as well as the online retailer's needs to pay the inventory subsidies to the offline retailer for the unsold goods in Model_{us}. By solving Model_{us} and $\text{Model}_{\text{Chen}}$, we get the numerical results in Table 1.

Table 1 demonstrates the following results.

(1) By Model_{us}, the profit of the online retailer is 5439.9, much higher than that in $\text{Model}_{\text{Chen}}$. However, at the equilibrium point obtained by Model_{us}, the optimal selling price of products is 179.71 for the online retailer, lower than that by $\text{Model}_{\text{Chen}}$. In other words, Model_{us} is beneficial to bring about greater consumer surplus. Additionally, the big difference of the equilibrium solutions between Model_{us} and $\text{Model}_{\text{Chen}}$ indicates that optimization to the delivery schedule in Model_{us} greatly affects the optimal strategy of the game model in the BOPS mode. Thus, Model_{us} is more applicable to incorporate the delivery schedule into the game model of the BOPS system than $\text{Model}_{\text{Chen}}$, where there is only a shipment.

(2) By Model_{us}, the profit of the offline retailer is 1900.6, lower than that by $\text{Model}_{\text{Chen}}$. Therefore, even if the total profit of the BOPS system in Model_{us} becomes higher ($\pi = 7340.5$), which is a 1942.2 CNY/RMB increase over that in $\text{Model}_{\text{Chen}}$, the offline retailer can only share less sale profit from the online retailer by optimizing delivery schedule and inventory subsidies. Thus, we



(a) Profits of the BOPS system (b) Optimal selling prices

FIGURE 6. Robustness to varying price sensitivity coefficients.

can conclude that Model_{us} is a more applicable model on BOPS modes than $\text{Model}_{\text{Chen}}$.

To further show robustness of Model_{us} to the uncertain price sensitivity coefficient, we compute the equilibrium points of the BOPS system by changing the value of the price sensitivity coefficient, β, from 1.3 to 2 with a step size of 0.05. In Figure 6, we show the numerical results with regard to the two models Model_{us} and $\text{Model}_{\text{Chen}}$.

Figure 6 indicates that Model_{us} is better than $\text{Model}_{\text{Chen}}$. Its main advantages can be stated as follows.

(1) The total profits of the BOPS system by Model_{us} are always higher than those by $\text{Model}_{\text{Chen}}$ for the varying price sensitivity coefficients. The profit of the offline retailer in Model_{us} is less sensitive to the change of the price sensitivity coefficient than that in $\text{Model}_{\text{Chen}}$ (see Figure 6(a)).

(2) For the varying price sensitivity coefficients, the optimal selling prices at the equilibrium obtained by Model_{us} are more robust, always lower than those by $\text{Model}_{\text{Chen}}$ (see Figure 6(b)). Clearly, lower price is helpful to generate greater consumer surplus such that more customers are attracted by the developed BOPS mode.

5. Conclusions

In this paper, we have established a new stochastic Nash equilibrium model of BOPS mode, where the online retailer can obtain an optimal online sale price and an optimal delivery schedule in an order cycle, while the offline retailer can get a maximal sharing rate of the profit from the online retailer. Compared to the existing models in the literature, our model is more applicable in practice.

Based on analytical properties of the model, the complicated model is transformed into a smooth constrained optimization problem, such that any off-from-shelf efficient algorithm is used to find its solution.

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By a case study and sensitivity analysis, the developed game model has been validated, and a number of practical managerial implications have been revealed with the help of this model as follows.

(1) The proposed integrated Nash game model in this paper can reliably provide an equilibrium strategy for the online and offline retailers under BOPS mode, including the optimal online selling price, the optimal delivery schedule, the optimal inventory and the optimal allocation of profits. By this model, due to existence of cross-selling profit, the offline retailer cares more about whether the consignment cooperation is successful or not, rather than a sufficiently greater rate of sharing the profit.
(2) Different model parameters, such as operational cost, price sensitivity coefficient, cross-sale factor, opportunity loss ratio and loss ratio of unsold goods, generate distinct impacts on the equilibrium solution and the profits of the BOPS system. Specifically, (a) the online retailers need to pay higher shipment cost in the underdeveloped area but it is more profitable. Similarly, the profit of the offline retailer in the developed area is the highest despite of higher unit service and holding cost. (b) By the Nash game strategy, the opportunity loss ratio may affect the equilibrium point in the BOPS mode, but only generates little change in the profits. (c) By the BOPS mode, both the offline and online retailers can share revenue generated by cross-selling, and choosing of a suitable type of goods with higher cross-sale factor can create a tripartite win–win situation. (d) The profit of the offline retailers is less sensitive to the price sensitivity coefficient and the cross-sale factor, compared with that of the online retailers. Consignment goods with lower price sensitivity coefficient are more suitable to adopt the BOPS mode.
(3) Optimization of delivery schedule plays a critical role in developing an efficient game model for the BOPS mode. It is beneficial to bring greater consumer surplus, and make the offline retailer share less sale profit from the online retailer, even if the total profit of the BOPS system becomes higher.

In future research, it would be an interesting issue to develop a deterministic global optimization algorithm to solve the unconstrained optimization (3.10), since any local optimization method can not ensure that the obtained value of the objective function is zero at the equilibrium. On the other hand, it is valuable to develop a new model to study the interaction between a single online retailer and multiple heterogeneous offline retailers under the BOPS mode. It is also significant to extend the proposed method to the case of perishable goods.

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