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IDEALS AND SUBALGEBRAS OF A FUNCTION ALGEBRA

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Introduction. Let X be a compact Hausdorff space and $C(X)$ the set of all continuous complex-valued functions on X. A *function algebra* A *on* X is a uniformly closed, point separating subalgebra of $C(X)$ which contains the constants. Equipped with the sup-norm, A becomes a Banach algebra. We let M_A denote the *maximal ideal space* and S_A the *Shilov boundary*.

The set of *finite, regular complex Borel measures on* X will be denoted by $M(X)$. We define

$$A^{\perp} = \{\mu \in M(X) : \int f d\mu = 0 \text{ for all } f \in A\}$$

and call $A^{\perp}$ the set of *annihilating measures* for A.

Suppose A and B are function algebras on X with $B \subset A$ and assume that there is a nonzero ideal J of A contained in B which has countable hull with respect to A. In §2 we determine M_B and S_B given M_A and S_A. We show in §3 that if $A^{\perp}$ contains no nonzero completely singular annihilating measure (see definition in §1), then neither does $B^{\perp}$. In §4 examples are given which show that in certain directions the results of §2 and §3 are sharp.

1. Definitions and preliminaries. We give M_A the weak-star topology induced from A^*, the dual space of A. If $f \in A$, then $\hat{f} \in C(M_A)$ is the *Gelfand transform* of f. If $\Phi \in M_A$, then there is a non-void set of probability measures $M_\Phi(A) \subset M(X)$ such that $\alpha \in M_\Phi(A)$ satisfies $\Phi(f) = \int f d\alpha$ for all $f \in A$. We call $M_\Phi(A)$ the set of *representing measures* for Φ. We say that $\mu \in A^{\perp}$ is *completely singular* if for every $\Phi \in M_A$, we have $\mu \perp \alpha$ for all $\alpha \in M_\Phi(A)$. Let supp μ denote the *support* of $\mu \in M(X)$.

Let E be a closed set in X. If $A|E = C(E)$ and if there is $F \in A$ such that $F|E = 1$ and $|F| < 1$ on $X \backslash E$, then E is a *peak interpolation set*. It follows that $\mu|E = 0$ if $\mu \in A^{\perp}$. We say F *peaks* on E.

We will need to use the abstract F. and M. Riesz theorem (see [**13**] for this result and historical background) and the fact that S_A is either uncountable or $A = C(X) = C(S_A)$ [**13**, p. 119].

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2. Maximal ideal space and Shilov boundary. If A is a function algebra on X and $S \subset A$, let the *hull of S with respect to A* be given by $\text{hull}_A S = \{\Phi \in M_A : \Phi(f) = 0 \text{ for all } f \in S\}$. Let B be a function algebra on X with $B \subset A$. Define the *restriction map* $r: M_A \to M_B$ by $r(\Phi) = \Phi|B$. Then r is a continuous map of M_A into M_B.

2.1 PROPOSITION. *Let A and B be function algebras on X with $B \subset A$. If there is a nonzero ideal J of A contained in B with* $\text{hull}_A J$ *countable, then* $M_B = r(M_A)$.

Proof. If $\theta \in M_B \backslash \text{hull}_B J$, then there is $f \in J$ such that $\theta(f) = 1$. Define $\Phi \in M_A$ by $\Phi(g) = \theta(gf)$ for $g \in A$. Then $r(\Phi) = \theta$ and $\Phi \in M_A \backslash \text{hull}_A J$. Therefore, r maps M_A onto $(M_B \backslash \text{hull}_B J) \cup r\ (\text{hull}_A J)$. Since $r(\text{hull}_A J) \subset \text{hull}_B J$, we have

$$(M_B \backslash \text{hull}_B J) \cup r(\text{hull}_A J) = M_B \backslash (\text{hull}_B J \backslash r(\text{hull}_A J)).$$

We show $\text{hull}_B J = r(\text{hull}_A J)$. The canonical map $c_A : A \to A/J$ induces a homeomorphism $c_A^* : M_{A/J} \to \text{hull}_A J$ defined by $c_A^*(\Phi)(f) = \Phi(f + J)$ for $f \in A$ and $\Phi \in M_{A/J}$ [**13**, p. 27]. Similarly $M_{B/J}$ and $\text{hull}_B J$ are homeomorphic by the map c_B^*. We note that $M_{A/J}$ is countable.

Let $i: B/J \to A/J$ be the injection map and let $r' : M_{A/J} \to M_{B/J}$ be defined by $r'(\Phi)(F) = \Phi(i(F))$ where $\Phi \in M_{A/J}$ and $F \in B/J$. Let $\hat{}: B/J \to (B/J)\hat{}$ be the Gelfand transform of B/J. Then $(B/J)\hat{}$ is a point separating subalgebra of $C(M_{B/J})$ which contains the constants. It follows that $(B/J)\hat{}$ has a Shilov boundary $S_{B/J}$ and that $r'(M_{A/J}) \supset S_{B/J}$ [**11**, p. 147]. Consequently, $S_{B/J}$ is countable.

Also, $S_{B/J}$ is compact and so $(B/J)\hat{}$ is uniformly dense in $C(S_{B/J})$ [**13**, p. 119]. Hence, each $\Phi \in M_{B/J}$ extends to a multiplicative linear functional on $C(S_{B/J})$. In particular, $M_{B/J} = S_{B/J}$, and so $r'(M_{A/J}) = M_{B/J}$.

Noting that $r = c_B^* \circ r' \circ (c_A^*)^{-1}$, we see that $r(\text{hull}_A J) = \text{hull}_B J$ as desired.

If $B \subset A$, then we have $B \subset C(M_A)$. If B also separates points on M_A, then r is a 1-1 map of M_A into M_B. Since r is continuous, M_A is homeomorphic to $r(M_A)$. In this case, we write $M_A \subset M_B$ and thereby identify a point of M_A with its restriction to B.

2.2 COROLLARY. *If A and B satisfy the conditions of the proposition and, in addition,* $M_A \subset M_B$, *then* $M_B = M_A$.

Proof. In this case, r is 1-1.

As above, let A and B be function algebras on X with $B \subset A$ and suppose that B contains a nonzero ideal J of A. We always have $S_A \supset S_B \supset \overline{S_A \backslash \text{hull}_A J}$ [**13**, p. 44].

2.3. PROPOSITION. *Let A and B be function algebras on X with $B \subset A$ and* $M_A \subset M_B$. *If there is a nonzero ideal J of A contained in B with* $\text{hull}_A J$ *countable, then* $S_A = S_B$.

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Proof. By Corollary 2.2, $M_A = M_B$. Let $E = S_A \backslash S_B$. Then E is open in S_A. Moreover, since $S_B \supset S_A \backslash \text{hull}_A J$, it follows that $E \subset \text{hull}_A J$. If $z \in E$ is isolated in E, then z is isolated in S_A. Hence, z is a peak point for A. Using the peak function for z and the fact that $b\hat{f}(M_A) \subset f(S_A)$ for any $f \in A$ ([**13**, p. 10; $b\hat{f}(M_A)$ is the topological boundary of $\hat{f}(M_A)$ in $\mathbf{C}$), we conclude that z is isolated in $M_A = M_B$. By Shilov's idempotent theorem, z is a peak point for B, and so, $z \in S_B$. Therefore, $\bar{E}$ contains no isolated points. But $\bar{E} \subset \text{hull}_A J$, and $\bar{E}$ cannot be both countable and perfect [**7**, p. 87]. Therefore, $E = \phi$.

Simple examples show that $M_B \supset M_A$ is necessary in Proposition 2.3.

3. Annihilating measures. We consider a function algebra A on X which has no nonzero completely singular annihilating measures in $A^\perp$, the set of annihilating measures for A supported on X. One well-known example of such an algebra is $R(X)$, the uniform closure on $X \subset \mathbf{C}$ of the rational functions with poles off X.

Let E be a compact set in S^2, the Riemann sphere, and let $A_E = \{f \in C(S^2) : f$ is analytic on $S^2 \backslash E\}$. If A_E contains a nonconstant function, then $M_{A_E} = S^2$ [**3**, p. 28], and $A_E{}^\perp$ has no nonzero completely singular elements [**3**, p. 63, Exercise 1(c)].

3.1 Proposition. *Let A and B be function algebras on X with $B \subset A$. Suppose there is a nonzero ideal J of A contained in B with* $\text{hull}_A J$ *countable. If there are no nonzero completely singular measures in $A^\perp$, then there are none in $B^\perp$.*

Proof. If $\mu \in B^\perp$, then by the abstract F. and M. Riesz theorem, $\mu = \mu_a + \mu_s$ where μ_a and $\mu_s \in B^\perp$, $\mu_a \perp \mu_s$, and μ_s is completely singular but μ_a is not. Let $j \in J$. Then $j\mu_s \in A^\perp$. Since any representing measure for a point in M_A is a representing measure for a point in M_B, it follows that $j\mu_s$ is a completely singular measure in $A^\perp$. Hence, $j\mu_s = 0$ for all $j \in J$. Therefore, supp $\mu_s \subset \text{hull}_A J \cap X$. The countability of $\text{hull}_A J$ implies that $B|\text{supp}\mu_s$ is uniformly dense in $C(\text{supp }\mu_s)$. Consequently, $\mu_s = 0$.

Example 4.1 will show that the countability of $\text{hull}_A J$ is necessary in Proposition 3.1.

3.2 Proposition. *Let A and B be function algebras on X with $B \subset A$. Suppose there is a nonzero ideal J of A contained in B with* $\text{hull}_A J$ *countable. Assume there is $j \in J$ such that $\{z \in X : j(z) = 0\} = \text{hull}_A J \cap X$. Let $K = \text{hull}_A J \cap X$. If $\mu \in B^\perp$, then there is $\nu \in A^\perp$ with $\nu|K = 0$ and $\mu_0 \in M(K)$ such that*

$$\mu = (1/j)\nu + \mu_0.$$

Moreover, if $\nu = 0$, then $\mu = 0$.

Proof. If $\mu \in B^\perp$, then by the abstract F. and M. Riesz theorem, $\mu = \mu_a + \mu_s$ where μ_a and $\mu_s \in B^\perp$, $\mu_a \perp \mu_s$, and μ_s is completely singular but μ_a is not. Just as in the proof of Proposition 3.1 we conclude that $\mu_s = 0$. Also,

$j\mu_a \in A^\perp$. If $\nu = j\mu_a$, then $\nu|K = 0$. By the assumption that

$$K = \{z \in X : j(z) = 0\},$$

we conclude that $\mu = \mu_a = (1/j)\nu + \mu_0$ where $\mu_0 \in M(K)$.

If $\nu = 0$, then supp $\mu \subset \text{hull}_A J \cap X$. Again, as in the proof of Proposition 3.1, it follows that $\mu = 0$.

In Example 4.2 we show there are cases where μ_0 is not the zero measure. The assumptions of Proposition 3.1 do not necessarily provide $j \in J$ such that $j|X$ vanishes precisely on K. This is seen by the following example. Let E be a set with no interior in S^2 such that A_E separates points on S^2. Let $z_0 \in S^2 \backslash E$ and $J = \{f \in A_E : f(z_0) = 0\}$. Then $\text{hull}_{A_E} J = \{z_0\}$, but any $j \in J$ also vanishes somewhere on E [**13**, p. 41].

Glicksberg [**5**] gives a description of the closed ideals contained in a function algebra having no nonzero completely singular annihilating measures.

4. Examples. If X and Y are compact Hausdorff spaces and $f : X \to Y$ is a continuous map, then given $\mu \in M(X)$, there is $\nu = \mu \circ f^{-1} \in M(Y)$ defined by $\nu(K) = \mu(f^{-1}(K))$ for $K \subset Y$.

4.1 *Example.* There is a compact Hausdorff space X' and function algebras A' and A_0 on X' with $A_0 \subset A'$ which have the following properties. There is a nonzero ideal J of A' contained in A_0 with $\text{hull}_{A'} J$ uncountable, and $A_0{}^\perp$ contains nonzero completely singular measures while $A'^\perp$ does not.

Proof. Let X be an uncountable compact metric space and suppose A is a function algebra on X. Pelczynski [**10**] has shown that there is a Cantor set $E \subset X$ which is a peak interpolation set for A. We also suppose that $A^\perp$ contains no nonzero completely singular measures.

Let Y be an uncountable compact metric space and let B be a function algebra on Y such that $B^\perp$ contains nonzero completely singular measures ([**5**, p. 113, Footnote 6] together with [**8**, p. 281, Example 5]). There is a continuous map p of E onto Y [**7**, p. 127]. Let X' be the set obtained from X by identifying the points of E which are identified by p. Give X' the quotient topology and let $q : X \to X'$ be the quotient map. It follows easily that X' is compact Hausdorff.

Let $A' = \{f \in C(X') : f \circ q \in A\}$. We show that A' is a function algebra on X' and that $E' = q(E)$ is a peak interpolation set for A'. Let $h \in C(E')$. Since $A|E = C(E)$, there is $H \in A$ with $H|E = h \circ q$. Let

$$H \circ q^{-1} = \begin{cases} H \circ q^{-1} & \text{on } X' \backslash E' \\ h & \text{on } E'. \end{cases}$$

Then $H \circ q^{-1}$ is continuous and belongs to A'. If $F \in A$ peaks on E, then

$$F \circ q^{-1} = \begin{cases} F \circ q^{-1} & \text{on } X' \backslash E' \\ 1 & \text{on } E' \end{cases}$$

peaks on E'. Hence, E' is a peak interpolation set for A'.

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If a and $b \in X'\backslash E'$ and $a \neq b$, then there is $G \in A$ so that $G(q^{-1}(a)) \neq G(q^{-1}(b))$ and $G|E = 0$. Then $G \circ q^{-1} \in A'$ separates a and b. This proves A' separates points on X'. It is clear that A' is uniformly closed.

If $\nu \in A'^{\perp}$, then $\nu|E' = 0$ since E' is a peak interpolation set. Since q is a homeomorphism when restricted to $X\backslash E$, we can define a Borel measure μ on X given by $\mu(K) = \nu(q(K))$ for Borel sets $K \subset X$.

Let F peak on E and set $j = 1 - F$. Then j is zero precisely on E. We now argue that $j\mu \in A^{\perp}$. Let $K = X\backslash E$. Since $\mu|E = 0$, we must show

$$\int_K fjd\mu = 0 \quad \text{for all } f \in A.$$

For each $f \in A$ there is $k \in A'$ such that $k \circ q = fj$. Therefore

$$\int_K fjd\mu = \int_{q(K)} kd\nu = \int_{X'} kd\nu = 0 \quad \text{for all } f \in A.$$

Hence, $j\mu \in A^{\perp}$.

If $\nu \neq 0$, then $\mu \neq 0$ and also $j\mu \neq 0$. Since $j\mu$ is not completely singular, there is some $\Phi \in M_A$ and $\alpha \in M_{\Phi}(A)$ such that α and μ are not singular. But $\alpha' = \alpha \circ q^{-1}$ is a representing measure for a point of $M_{A'}$, and α' and ν are not singular. We conclude that A' has no nonzero completely singular annihilating measures.

There is a homeomorphism h of E' onto Y such that $h \circ q = p$. Define $A_0 = \{f \in A' : f \circ h^{-1} \in B\}$. Then A_0 is a function algebra on X' with M_{A_0} obtained by joining M_B to $M_{A'}$ along E' by means of h (Glicksberg [**4**]). We write $M_{A_0} = M_1 \cup M_2$ where M_1 is identified with $M_{A'}\backslash E'$ and M_2 is identified with M_B.

Let μ be a nonzero completely singular measure in $B^{\perp}$. We will show that $\nu = \mu \circ h$ is a nonzero completely singular measure in $A_0{}^{\perp}$. Clearly, $0 \neq \nu \in A_0{}^{\perp}$ and supp $\nu \subset E'$. Suppose $\Phi \in M_1$ and $\alpha \in M_{\Phi}(A_0)$. Then $\alpha(E') = 0$ since there is $G \in A_0$ satisfying $G|E' = 1$ and $|G| < 1$ on $X'\backslash E'$, and

$$0 = \lim_{n\to\infty} (\Phi(G))^n = \lim_{n\to\infty} (\Phi(G^n)) = \lim_{n\to\infty} \int G^n d\alpha = \alpha(E').$$

Therefore, $\alpha \perp \nu$.

Suppose $\Phi \in M_2$ and $\alpha \in M_{\Phi}(A_0)$. Then $\alpha(X'\backslash E') = 0$ since $H = 1 - G \in A_0$ satisfies $H|M_2 = 0$ and Re $H > 0$ on M_1. We now see that $\alpha \circ h^{-1}$ is a measure representing a point of M_B. Thus $\alpha \circ h^{-1} \perp \nu \circ h^{-1}$ and consequently, $\alpha \perp \nu$. We conclude that ν is a completely singular measure in $A_0{}^{\perp}$.

Finally, we note that $J = \{f \in A' : f|E' = 0\}$ is a nonzero ideal of A' which is contained in A_0, and $\text{hull}_{A'}J$ is uncountable.

Since M_{A_0} properly contains $M_{A'}$, we also have shown that Proposition 2.1 is false without the assumption that $\text{hull}_A J$ is countable.

Let D be the closed unit disk in **C**, T the unit circle, and U the open unit disk. Let $A(D) = \{f \in C(D): f$ is analytic on $U\}$. By the maximum modulus principle we can consider $A(D)$ as a function algebra on T. In the examples that follow, we consider function algebras B on T with $B \subset A(D)$. Let m be *normalized Lebesgue measure* on T.

4.2 *Example.* Let

$$B = \left\{ f(z) \exp\left(\frac{z+1}{z-1}\right) + k : f \in A(D), f(1) = 0, \text{ and } k \in \mathbf{C} \right\}.$$

Then $J = \{f \in B : f(1) = 0\} \subset B$ and J is an ideal in $A(D)$ with $\text{hull}_{A(D)} J = \{1\}$. By Proposition 3.2 and using $A(D)^{\perp} = \{hm : h \in H_0^1(D)\}$ it follows that if $\mu \in B^{\perp}$, then there is $h \in H_0^1(D)$, $j \in J$, and $\lambda\delta_1$ where $\lambda \in \mathbf{C}$ and δ_1 is the point mass at $z = 1$ such that $\mu = (h/j)m + \lambda\delta_1$.

We show there is $\mu \in B^{\perp}$ with $\mu(\{1\}) \neq 0$. Suppose $\mu(\{1\}) = 0$ for every $\mu \in B^{\perp}$. By the Glicksberg peak set theorem [**3**, p. 58], $z = 1$ is a peak point for B. If $F \in B$ peaks at $z = 1$, then Re $(F(z) - 1) < 0$ for $z \in U$. This implies $F(z) - 1$ is an outer function [**2**], but

$$F(z) - 1 = g(z) \exp\left(\frac{z+1}{z-1}\right)$$

for some $g \in A(D)$ with $g(1) = 0$. The right side of this equation has a singular part while the left side is outer. Therefore, $z = 1$ is not a peak point for B. In connection with this example, see [**6**, Example 1.8].

4.3 *Example.* Let $\{z_k\} \subset U$ be a Blaschke sequence with the z_k's distinct and suppose the z_k's accumulate to a closed uncountable set $K \subset T$ of measure zero. Let $B = \{f \in A(D): f'(z_k) = 0$ for all k$\}$. If $b(z)$ is the Blaschke product corresponding to $\{z_k\}$ and $g(z) \in A(D)$ is zero precisely on K, then

$$B_0 = \left\{ H(z) : H(z) = \int_0^z h(w)g(w)b(w)dw \text{ for } h \in A(D) \right\} \subset B.$$

It is easy to show that $B_0|T \subset C^1(T)$ and that B_0 separates points on D [**9**]. Using these properties of B_0, an application of Theorem 2.1 of Bjork [**1**] implies $M_B = D$.

Also, $B|I$ is dense in $C(I)$ for any closed interval $I \subset T$ since $B|I$ contains a set of smooth generators [**12**]. Using this result and the fact that

$$J = \{f \in A(D): f(z_k) = f'(z_k) = 0 \text{ for all } k \text{ and } f|K = 0\} \subset B,$$

we find that if $\mu \in B^{\perp}$, then there is $h \in H_0^1(D)$, $j \in J$, and $\mu_0 \in M(K)$ such that $\mu = (h/j)m + \mu_0$. Moreover, if $h = 0$ on a set of positive measure, then $\mu = 0$.

We note $\text{hull}_{A(D)} J$ is uncountable and J is the largest ideal of $A(D)$ in B. Hence, one form of converse to Corollary 2.2 is false. We also have the same representation for elements of $B^{\perp}$ as in Proposition 3.2.

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