Can. J. Math., Vol. XXIV, No. 2, 1972, pp. 239–260

ON INFLATION AND TORSION OF AMITSUR COHOMOLOGY

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Introduction. Studies of torsion [**2**] and inflation [**11**] of Amitsur cohomology have primarily been concerned with module-finite faithful projective algebras. In this paper, our goal is to consider these topics for more general algebras. The fundamental tool, in case R is a domain with quotient field K, is the functor UK/U (defined in § 1), together with the monomorphic connecting map of Amitsur cohomology $H^1(S/R, UK/U) \to H^2(S/R, U)$ arising from a (commutative) flat R-algebra S and the unit functor U. This map was first considered in [**10**, Chapter IV, Theorem 1.6], where it was proved to be an isomorphism for certain étale faithfully flat algebras S in case R is an algebraic number ring with trivial Brauer group. In Corollary 1.5 it is shown, in the case of a module-finite faithful projective S over a regular domain R, that the connecting map is the kernel of the canonical homomorphism [**8**] from $H^2(S/R, U)$ to the split Brauer group $B(S/R)$. As the latter map is often injective [**8**, Corollary 7.7], one might expect instances of vanishing of $H^1(S/R, UK/U)$ without assuming both R Noetherian and S module-finite R-projective. Much of § 1 (viz. (1.7)–(1.9)) is devoted to such examples.

Section 2 deals with the inflation map inf: $H^2(S/R, U) \to H^2(T/R, U)$ arising from an R-algebra homomorphism $S \to T$. Using the techniques of Grothendieck topologies and spectral sequences, we obtain for arbitrary commutative R (in Theorem 2.2 and the remark following) sufficient conditions for inf to be injective. Our general approach for a domain R is to consider annihilators of elements of the kernel of inf. For flat S and T, Proposition 2.1 reduces the problem to analyzing the kernel of

$$H^1(S/R, UK/U) \to H^1(T/R, UK/U).$$

Some multiplicative functions (including the norms of Amitsur homology [**2**]) are used in (2.5)–(2.7) to show: if S and T are the integral closures of R in finite field extensions $L \subset F$ of K then, with additional hypotheses, an appropriate power of $[F:L]$ annihilates the kernel of inf. An analogous result for group cohomology is given in Theorem 2.9.

Section 3 is concerned with diverse applications to U and UK/U of a spectral sequence of Silver [**17**] that connects Amitsur cohomology and group cohomology. In particular, (3.6)–(3.8) are the only general results known to the

Received January 18, 1971 and in revised form, January 5, 1972. The work in § 2 was partially supported by an ONR Postdoctoral Research Associateship at the University of California, Los Angeles.

author concerning torsion of two-dimensional Amitsur cohomology in U for algebras not assumed to be both module-finite and projective.

1. Amitsur cohomology and UK/U**.** Throughout the paper, rings and algebras are commutative with unit elements and algebra homomorphisms are unitary. We assume familiarity with the Amitsur cohomology, Brauer group, and Pic functors (see [**2**; **7**], [**4**], and [**5**], respectively) and with R-based topologies (see [**10**, Chapter II]).

We begin by defining some coefficient functors for Amitsur cohomology. Let R be a ring and U the functor that assigns to an R-algebra S its multiplicative group of invertible elements $U(S)$. If K is an R-algebra, the functor UK is defined by $(UK)(S) = U(S \otimes_R K)$. Now assume that K is R-faithful. By restricting U and UK to the full subcategory of flat R-algebras, we obtain a monomorphism of functors $U \to UK$; let UK/U be its cokernel. As usual [**8**, (3.3)], there is a natural long exact sequence of Amitsur cohomology groups

$$\ldots \to H^n(S/R, U) \to H^n(S/R, UK) \to H^n(S/R, UK/U) \to H^{n+1}(S/R, U) \to \ldots$$

for any flat R-algebra S. Identify $H^n(S/R, UK)$ and $H^n(S \otimes_R K/K, U)$ via the standard isomorphism of the corresponding complexes [**15**, footnote, p. 224].

Consider the above connecting homomorphism for $n = 1$. If $S \otimes_R K$ is faithfully flat over K and $\mathrm{Pic}(K) = 0$ (e.g., if K is a field), then [**8**, Corollary 4.6] shows $H^1(S \otimes_R K/K, U) = 0$, whence $H^1(S/R, UK/U) \to H^2(S/R, U)$ is a monomorphism. Conditions that this map also be surjective are studied below.

First, some notation: if R is a Dedekind domain with quotient field K and $\mathfrak{J}$ is a maximal ideal of R, let $K_{\mathfrak{J}}$ be the completion of K in the $\mathfrak{J}$-adic valuation and $R_{\mathfrak{J}}$ the closure of R in $K_{\mathfrak{J}}$. In general, let $\mathrm{Int}_B\, A$ denote the integral closure of A in B.

THEOREM 1.1. *Let K be either an algebraic number field with at most one real place or a finite field extension of $k(X)$ for some finite field k. Let R be the integral closure in K of either* $\mathbf{Z}$ *or $k[X]$, as the case may be. Let S be a faithfully flat R-algebra such that $S \otimes_R K$ is finite dimensional over K and: for every maximal ideal $\mathfrak{J}$ of R, there exist a finite unramified* (*Galois*) *field extension L of $K_{\mathfrak{J}}$ and an $R_{\mathfrak{J}}$-algebra homomorphism $S \otimes_R R_{\mathfrak{J}} \to \mathrm{Int}_L(R_{\mathfrak{J}})$. Then the connecting homomorphism $H^1(S/R, UK/U) \to H^2(S/R, U)$ is an isomorphism.*

Proof. See [**10**, Chapter IV, Theorem 1.6, Theorem 3.3 and Remark (b) following, and supplement on p. 176].

Remark 1.2. Let R be as in the preceding theorem. It follows from [**10**, Chapter IV, Theorem 1.3] that the hypotheses of Theorem 1.1 hold for R-algebras which are projective, separable and faithful over $\prod R_{x_i}$ where the x_i are non-zerodivisors satisfying $(x_1, \ldots, x_n) = R$. Although such algebras are seldom module-finite over R, it is of interest to study the connecting homomorphism in the module-finite case. An appropriate tool is the theorem of Chase-Rosenberg quoted below. For motivation, we note that global class field theory and [**4**, Theorem 6.5, Propositions 7.4 and 8.2] imply that the rings R of Theorem 1.1 have trivial Brauer group.

THEOREM 1.3 (Chase-Rosenberg). *Let S be a module-finite faithful and projective R-algebra. Then there is an exact sequence natural in R and S:*

$$0 \to H^1(S/R, U) \to \mathrm{Pic}(R) \to H^0(S/R, \mathrm{Pic}) \to H^2(S/R, U) \to B(S/R) \to H^1(S/R, \mathrm{Pic}) \to H^3(S/R, U).$$

Proof. Exactness and naturality in S are proved in [**8**, Theorem 7.6]. Naturality in R follows from the explicit calculation of the maps in [**12**, Appendix].

COROLLARY 1.4. *Let R be a domain with quotient field K and S a flat R-algebra such that $S \otimes_R K$ is finite dimensional over K and $B(S \otimes_R K/K) = 0$. Then the connecting homomorphism $H^1(S/R, UK/U) \to H^2(S/R, U)$ is an isomorphism.*

Proof. As we noted earlier, $H^1(S/R, UK/U) \to H^2(S/R, U)$ is a monomorphism. By the cohomology long exact sequence arising from

$$0 \to U \to UK \to UK/U \to 0,$$

it is enough to prove $H^2(S \otimes_R K/K, U) = 0$. But $S \otimes_R K$ is artinian, hence semilocal, and so $\mathrm{Pic}(S \otimes_R K) = 0$ [**5**, Proposition 5, p. 143]. Therefore, $H^0(S \otimes_R K/K, \mathrm{Pic}) = 0$ and Theorem 1.3 applies to complete the proof.

We recall that a Noetherian domain R is called *regular* if R_M is a regular local ring for every maximal ideal M of R. Any Dedekind domain is regular. If R is regular with quotient field K, [**4**, Theorem 7.2] states that the map on Brauer groups, $B(R) \to B(K)$, is a monomorphism. In particular, if R is regular and S is an R-algebra, the induced map on split Brauer groups, $B(S/R) \to B(S \otimes_R K/K)$, is a monomorphism.

COROLLARY 1.5. *Let R be a regular domain with quotient field K and S a module-finite faithful and projective R-algebra. Then $H^1(S/R, UK/U)$, viewed as a subgroup of $H^2(S/R, U)$ via the connecting homomorphism, is the kernel of the canonical map $H^2(S/R, U) \to B(S/R)$.*

Proof. This is immediate from a chase of the exact commutative diagram

$$\begin{tikzcd} & & & 0 \arrow[d] \\ 0 \arrow[r] & H^1(S/R, UK/U) \arrow[r] & H^2(S/R, U) \arrow[r] \arrow[d] & H^2(S \otimes_R K/K, U) \arrow[d] \\ & & 0 \to B(S/R) \arrow[r] & B(S \otimes_R K/K). \end{tikzcd}$$

Remark 1.6. Let R be a Dedekind domain with quotient field K, L a finite separable field extension of K, and $S = \mathrm{Int}_L R$. Then S is a module-finite

https://doi.org/10.4153/CJM-1972-019-2 Published online by Cambridge University Press

faithful R-projective such that $S \otimes_R K \cong L$ [**18**, Chapter V, Theorem 7] and $H^2(S \otimes_R K/K, U) \cong B(L/K)$ (Theorem 1.3).

Corollary 1.5 shows that $H^1(S/R, UK/U) \to H^2(S/R, U)$ is an isomorphism if $B(S/R) = 0$. In case K is an algebraic number field, this holds if K has at most one real place or if $[L:K]$ is odd. Note that Theorem 1.1 and Remark 1.2 handle the special case in which K has at most one real place and L/K is unramified.

THEOREM 1.7. *Let R be an algebraic number ring with quotient field K and let A be the ring of all algebraic integers. Then* $H^1(A/R, UK/U) = 0$ and *there is an exact sequence*

$$0 \to H^2(A/R, U) \to B(K) \to H^2(A/R, UK/U) \to H^3(A/R, U).$$

Proof. Let F be an algebraic closure of $\mathbf{Q}$ containing K; we may take $A = \mathrm{Int}_F R$. Let S range over the inclusion-directed collection of algebraic number overrings of R contained in A, with L the quotient field of S. By working with the Hopf algebra $R[X, X^{-1}]$ in the finite topology as in [**11**, Corollary 4.4], we obtain isomorphisms

$$\varinjlim H^n(S/R, U) \xrightarrow{\cong} H^n(A/R, U)$$

and

$$\varinjlim H^n(L/K, U) \xrightarrow{\cong} H^n(F/K, U).$$

Although UK/U does not arise from a Hopf algebra (indeed UK/U is not a sheaf; see Remark 1.11 below), the same reasoning applies to it. In fact the n-th cochain group of the Amitsur complex $C(S/R, UK/U)$ is $(UK/U)(S^{n+1})$, which is isomorphic to UL^{n+1}/US^{n+1} since $S \otimes_R K \cong L$. Similarly, $A \otimes_R K \cong F$, whence

$$C^n(A/R, UK/U) \cong UF^{n+1}/UA^{n+1}.$$

As

$$\varinjlim S = A$$

and tensor products commute with direct limit, there is an isomorphism of complexes

$$\varinjlim C^n(S/R, UK/U) \xrightarrow{\cong} C^n(A/R, UK/U),$$

and hence

$$\varinjlim H^n(S/R, UK/U) \xrightarrow{\cong} H^n(A/R, UK/U)$$

[**6**, Ch. V, Proposition 9.3*, p. 100].

By the above discussion, there are isomorphisms

$$H^2(A/R, UK) \cong H^2(F/K, U) \cong \varinjlim H^2(L/K, U) \cong \varinjlim B(L/K) = B(K).$$

Applying

$$\varinjlim_{S}$$

to the cohomology sequence therefore leads to an exact sequence

$$0 \to H^1(A/R, UK/U) \to H^2(A/R, U) \xrightarrow{f} B(K)$$
$$\to H^2(A/R, UK/U) \to H^3(A/R, U).$$

To complete the proof, we need only to prove f is a monomorphism. However, f is factored into the product of a monomorphism $g: H^2(A/R, U) \to B(R)$ [**11**, Corollary 4.4] and the canonical monomorphism $B(R) \to B(K)$. Indeed, since g is also obtained via direct limit, the verification reduces to checking commutativity of diagrams of the form

$$\begin{array}{ccc} H^2(S/R, U) & \to & B(S/R) \\ \downarrow & & \downarrow \\ H^2(L/K, U) & \to & B(L/K), \end{array}$$

and this is handled by the naturality assertion of Theorem 1.3.

We now provide two more examples of vanishing one-dimensional UK/U-cohomology. In view of Corollary 1.5 (and the fact that $B(\mathbf{Z}) = 0$), Propositions 1.8 and 1.9 may each be regarded as generalizations of the result that $H^2(\mathbf{Z}[i]/\mathbf{Z}, U) = 0$ [**14**, Theorem 6.3.2].

PROPOSITION 1.8. *Let R be an integrally closed domain with ordered quotient field K. Let $L = K(\sqrt{m})$ for some negative nonsquare m in R; assume $S = \text{Int}_L R$ is R-flat. If $U(R) = \{1, -1\}$, then $H^1(S/R, UK/U) = 0$.*

Note. The assumption that S is R-flat is used to guarantee that

$$H^1(S/R, UK/U)$$

is defined.

Proof. By means of the usual identifications

$$C^n(S/R, UK/U) \cong U(L^{n+1})/U(S^{n+1}),$$

the problem becomes: given $\xi \in UL^2$ such that the Amitsur coboundary $d^1\xi \in US^3 \subset UL^3$, find $l \in UL$ such that $\xi \equiv l^{-1} \otimes l \bmod (US^2)$.

Let $\xi = \sum \alpha_i \otimes \beta_i$ and $G = \text{gal}(L/K) = \{1, g\}$. Under the isomorphism $U(L^2) \to \prod_G U(L) = L \times L$, let ξ be sent to (a, b); i.e.

$$a = \sum \alpha_i \beta_i \text{ and } b = \sum \alpha_i g(\beta_i).$$

Now under the algebra isomorphism $L^3 \to \prod_{G^2} L = L \times L \times L \times L$,

$$d^1(\xi) = (\sum 1 \otimes \alpha_i \otimes \beta_i)(\sum \alpha_i \otimes 1 \otimes \beta_i)^{-1}(\sum \alpha_i \otimes \beta_i \otimes 1)$$

corresponds to $(a, b, g(a), g(b))(a, b, b, a)^{-1}(a, a, b, b)$. Since G maps S into itself, it follows that $(a, a, g(a), a^{-1}bg(b)) \in \prod U(S)$.

Let $N = N_{L/K}$. Then $N(b) = bg(b) \in aU(S) \cap K = U(S) \cap K = \{\pm 1\}$; similarly, $N(a) \in N(U(S)) = \{\pm 1\}$.

Case 1: $N(a) = N(b)$. Then Hilbert's Theorem 90 proves $l \in L$ with $ab^{-1} = lg(l^{-1})$. Since $a \otimes 1 \in U(S^2)$ corresponds to $(a, a) = (a, blg(l^{-1})) = (a, b)(1, lg(l^{-1})) \in L \times L$, we conclude $\xi(l \otimes l^{-1}) = a \otimes 1$, to finish the proof of this case.

Case 2: $N(a) = -N(b)$. We shall prove that this case cannot actually arise. Express ξ as

$$\xi = k_{11}1 \otimes 1 + k_{12}1 \otimes \sqrt{m} + k_{21}\sqrt{m} \otimes 1 + k_{22}\sqrt{m} \otimes \sqrt{m},$$

for elements $k_{ij} \in K$. Since $g(\sqrt{m}) = -\sqrt{m}$, the hypothesis of Case 2 implies (after a simple computation) that $(k_{11})^2 = m[(k_{12})^2 + (k_{21})^2 - (k_{22})^2 m]$. As m is negative, we conclude $(k_{11})^2 = 0 = (k_{12})^2 + (k_{21})^2 - (k_{22})^2 m$. Then $k_{11} = k_{12} = k_{21} = k_{22} = 0$ and $0 = \xi \in U(L^2)$, the desired contradiction.

PROPOSITION 1.9. *Let R be an integrally closed domain with quotient field K of characteristic $\neq 2, 3$. Let L be a quadratic field extension of K; then $L = K(\sqrt{m})$ for some $m \in R$. Let S be the free R-subalgebra of L with basis $\{1, \sqrt{m}\}$. If $U(R) = \{1, -1\}$, then $H^1(S/R, UK/U) = 0$.*

Proof. With the notation and argument of Proposition 1.8, we are reduced to showing that $N(a) = -N(b)$ leads to a contradiction. There exist elements $q_j \in R$ such that

$$\begin{aligned} d^1(\xi) = {} & q_1 1 \otimes 1 \otimes 1 + q_2 1 \otimes 1 \otimes \sqrt{m} + q_3 1 \otimes \sqrt{m} \otimes 1 \\ & + q_4 1 \otimes \sqrt{m} \otimes \sqrt{m} + q_5 \sqrt{m} \otimes 1 \otimes 1 + q_6 \sqrt{m} \otimes 1 \otimes \sqrt{m} \\ & + q_7 \sqrt{m} \otimes \sqrt{m} \otimes 1 + q_8 \sqrt{m} \otimes \sqrt{m} \otimes \sqrt{m}. \end{aligned}$$

Viewed in $S \times S \times S \times S$, this gives rise to the following equations (recall $g(\sqrt{m}) = -\sqrt{m}$):

$$\begin{aligned} &(\mathrm{I})\ a = (q_1 + q_4 m + q_6 m + q_7 m) + \sqrt{m}(q_2 + q_3 + q_5 + q_8 m), \\ &(\mathrm{II})\ a = (q_1 - q_4 m - q_6 m + q_7 m) + \sqrt{m}(-q_2 + q_3 + q_5 - q_8 m), \\ &(\mathrm{III})\ g(a) = (q_1 + q_4 m - q_6 m - q_7 m) + \sqrt{m}(-q_2 - q_3 + q_5 + q_8 m), \\ &(\mathrm{IV})\ a^{-1}bg(b) = (q_1 - q_4 m + q_6 m - q_7 m) + \sqrt{m}(q_2 - q_3 + q_5 - q_8 m). \end{aligned}$$

Since $g(g(a)) = a$ and $a^{-1}bg(b) = -g(a)$, we may rewrite (III) and (IV) as:

$$\begin{aligned} &(\mathrm{III}')\ a = q_1 + q_4 m - q_6 m - q_7 m) + \sqrt{m}(q_2 + q_3 - q_5 - q_8 m), \\ &(\mathrm{IV}')\ a = (-q_1 + q_4 m - q_6 m + q_7 m) + \sqrt{m}(q_2 - q_3 + q_5 - q_8 m). \end{aligned}$$

Since S is R-free on $\{1, \sqrt{m}\}$, a comparison of (I) with (II) shows $q_4 = -q_6$ and $q_2 = -q_8 m$. A similar comparison of (III′) with (IV′) implies $q_1 = q_7 m$ and $q_3 = q_5$. Hence $a = 2mq_4 - 2mq_8\sqrt{m}$ and $\pm 1 = N(a) = ag(a) = 4m^2(q_4)^2 - 4m^2(q_8)^2 m$; then $2 \in U(R) = \{1, -1\}$, contradicting the hypothesis that $\mathrm{char}(K) \neq 3$. This completes the proof.

Remark 1.10. (a) It is perhaps interesting to note that the hypotheses of Propositions 1.8 and 1.9 do not imply that R is Prüfer. For example, $R = \mathbf{Z}[X]$ is not Prüfer; its quotient field $K = \mathbf{Q}(X)$ is ordered by defining a nonzero fraction f/g in lowest terms to be positive if and only if the leading coefficients of f and g have the same sign. If $L = K(\sqrt{-1})$, then $S = \mathrm{Int}_L R$ is R-free on $\{1, \sqrt{-1}\}$ and the preceding propositions each imply $H^1(S/R, UK/U) = 0$.

(b) If S is any quadratic algebraic number ring, it is known that $H^2(S/\mathbf{Z}, U) = 0$ [**14**, Theorems 6.4.2 and 6.4.3]. Another proof could proceed as follows. The quotient field of S is $\mathbf{Q}(\sqrt{m})$, for some nonzero squarefree rational integer $m \neq 1$. If $m \equiv 2$ or $3 \pmod 4$, Proposition 1.9 shows $H^1(S/\mathbf{Z}, U\mathbf{Q}/U) = 0$ and, by Corollary 1.5, $H^2(S/\mathbf{Z}, U) = 0$ since $B(S/\mathbf{Z}) = 0$. In case $m \equiv 1 \pmod 4$, then $\{1, (1 + \sqrt{m})/2\}$ is a $\mathbf{Z}$-basis of S. With the notation of Proposition 1.9, we are reduced to deriving a contradiction if $N(a) = -N(b)$. If $d^1(\xi)$ is expressed in terms of the induced $\mathbf{Z}$-basis of S^3, several different descriptions of a are obtained, and a basis argument similar to that of Proposition 1.9 shows $4 \in m\mathbf{Z}$, the desired contradiction.

Remark 1.11. Let R be an algebraic number ring with quotient field K. Let $T(R)$ be the R-based topology whose underlying category is that of all module-finite flat R-algebras and whose covers are singleton sets consisting of faithfully flat R-algebra maps. The purpose of this remark is to show that UK/U is not a $T(R)$-sheaf in case $\mathrm{Pic}(R) \neq 0$ (e.g., $R = \mathbf{Z}[\sqrt{-5}]$).

As in [**9**, Theorem 20.14], there exists a finite field extension L of K such that the canonical map $\mathrm{Pic}(R) \to \mathrm{Pic}(S)$ is zero, where $S = \mathrm{Int}_L R$. Now S is a module-finite faithful R-projective, hence faithfully flat [**8**, p. 67]; i.e., $\{R \to S\}$ is a $T(R)$-cover. If UK/U were a sheaf, then the map $UK/UR \to UL/US$ would induce an isomorphism

$$UK/UR \xrightarrow{\cong} (UK/U)(R) \cong H^0(S/R, UK/U).$$

However, [**8** Proposition 3.9(a)] shows

$$H^0(S/R, UK) \cong UK \text{ and } H^0(S/R, U) \cong UR;$$

the cohomology sequence and [**8**, Corollary 4.6] then provide an exact sequence

$$0 \to UR \to UK \to H^0(S/R, UK/U) \to \mathrm{Pic}(R) \to 0,$$

thus proving UK/U is not a sheaf.

By adapting the argument of [**10**, supplement on p. 176] to $T(R)$, one may construct $T(R)$-sheaves which are not $T(R)$-additive. In the case just considered, UK/U is $T(R)$-additive but is not a $T(R)$-sheaf; additivity may be established by the five lemma since U and UK are each additive.

2. Inflation and norms. Let $f: S \to T$ be an R-algebra homomorphism and J an Ab-valued functor defined on a full subcategory of R-algebras containing all the tensor products S^n and T^n. The homomorphisms

$$J(f \otimes \ldots \otimes f): J(S^n) \to J(T^n)$$

induce a map of Amitsur complexes $C(S/R, J) \to C(T/R, J)$ which yields *inflation* homomorphisms

$$\inf = \inf_n(S, T, R, J) \colon H^n(S/R, J) \to H^n(T/R, J).$$

In this section, we study the kernel of inf for the case $n = 2$ and $J = U$. Our work is motivated by the role played in Theorem 1.7 by direct limits of systems of Amitsur cohomology groups, where the maps of the directed sets are given by inflation. In addition to the background material for §1, we assume familiarity with the rudiments of group cohomology.

PROPOSITION 2.1. *Let R be a domain with quotient field K, F a finite extension field of K, and S and T flat R-subalgebras of F such that there exists an R-algebra homomorphism $S \to T$. Then* $\ker(\inf_1(S, T, R, UK/U)) = \ker(\inf_2(S, T, R, U))$.

Proof. $S \otimes_R K$ (respectively, $T \otimes_R K$) is a K-subspace of $F \otimes_R K = F$ and, hence, is finite dimensional. Theorem 1.3 then supplies a commutative diagram

$$\begin{array}{ccc} H^2(S \otimes_R K/K, U) & \to & B(S \otimes_R K/K) \\ \downarrow & & \downarrow \\ H^2(T \otimes_R K/K, U) & \to & B(T \otimes_R K/K) \end{array}$$

in which the horizontal maps are isomorphisms, the left vertical map is inf and the right vertical map is inclusion of subgroups of $B(K)$. We then have an exact commutative diagram

$$\begin{array}{ccccccc} & & & & & & 0 \\ & & & & & & \downarrow \\ 0 \to & H^1(S/R, UK/U) & \to & H^2(S/R, U) & \to & B(S \otimes_R K/K) \\ & \downarrow & & \downarrow & & \downarrow \\ 0 \to & H^1(T/R, UK/U) & \to & H^2(T/R, U) & \to & B(T \otimes_R K/K) \end{array}$$

in which the left and middle vertical maps are the infs in question. A diagram chase completes the proof.

THEOREM 2.2 (cf. Morris [**14**, Theorem 3.2.1]). *Let S be an R-algebra and T a faithfully flat S-algebra such that T^2 is S^2-faithfully flat. If* $\operatorname{Pic}(T^2) = 0 = \operatorname{Pic}(S^2)$, *then* $\inf_2(S, T, R, U)$ *is a monomorphism.*

Proof. Consider the R-based topology X, the objects of whose underlying category are all R-algebras A whose cardinality satisfies

$$\operatorname{card}(A) \leqq \max(\operatorname{card}(S), \operatorname{card}(T), \aleph_0),$$

with $\operatorname{Cov}(X)$ consisting of all singleton sets containing a faithfully flat morphism in $\operatorname{Cat}(X)$. As X is dual to a Grothendieck topology, [**3**, Chapter II, 1.8 (i)] shows that the category of X-sheaves has enough injectives.

https://doi.org/10.4153/CJM-1972-019-2 Published online by Cambridge University Press

Now U is an X-sheaf [**8**, Proposition 3.9 (a)]; let

$$0 \to U \to U^* \to U' \to 0$$

be an exact sequence in the category of X-sheaves with U^* injective.

There is a commutative diagram

$$\begin{array}{ccccccccc}
0 \to & U(S) & \to & U^*(S) & \to & U'(S) & \to & H_X{}^1(S, U) \\
& \downarrow & & \downarrow & & \downarrow & & \\
0 \to & U(S^2) & \to & U^*(S^2) & \to & U'(S^2) & \to & H_X{}^1(S^2, U) \\
& \downarrow & & \downarrow & & \downarrow & & \\
0 \to & U(S^3) & \to & U^*(S^3) & \to & U'(S^3) & & \\
& \downarrow & & \downarrow & & \downarrow & & \\
0 \to & U(S^4) & \to & U^*(S^4) & \to & U'(S^4) & &
\end{array}$$

whose rows are exact and whose columns are the initial segments of $C(S/R, U)$, $C(S/R, U^*)$ and $C(S/R, U')$, respectively. The Grothendieck cohomology group $H_X{}^1(S^2, U)$ is isomorphic to the Cech cohomology group $\check{H}_X{}^1(S^2, U)$ [**3**, Chapter II, Corollary 3.6] which, [**8**, Corollary 4.6] shows, can be embedded in $\mathrm{Pic}(S^2) = 0$; hence $\check{H}_X{}^1(S^2, U) = 0$. Similarly, $H_X{}^1(S, U)$ embeds in $\mathrm{Pic}(S)$, which is easily seen to vanish (cf. [**11**, Corollary 4.2]). A standard chase of the above diagram (cf. [**6**, p. 40]) then yields an exact sequence

$$H^1(S/R, U^*) \to H^1(S/R, U') \to H^2(S/R, U) \to H^2(S/R, U^*).$$

However, $H^1(S/R, U^*) = 0 = H^2(S/R, U^*)$ since U^* is injective [**3**, Chapter I, Corollary 3.1 and Chapter II, 1.8 (ii)], and the connecting map is therefore an isomorphism

$$H^1(S/R, U') \xrightarrow{\cong} H^2(S/R, U).$$

We may deal with T similarly, to obtain an isomorphism

$$H^1(T/R, U') \xrightarrow{\cong} H^2(T/R, U).$$

It follows from the commutativity of the diagram

$$\begin{array}{ccc}
H^1(S/R, U') & \to & H^2(S/R, U) \\
\downarrow & & \downarrow \\
H^1(T/R, U') & \to & H^2(T/R, U)
\end{array}$$

that we need only to establish that $\inf_1(S, T, R, U')$ is a monomorphism. Since $\{S \to T\}$ and $\{S^2 \to T^2\}$ are in $\mathrm{Cov}(X)$ and U' is an X-sheaf, the argument of [**14**, Theorem 3.1.3] may be adapted to show that $\inf_1(S, T, R, U')$ is indeed a monomorphism, thus completing the proof.

Remark. It is interesting to compare Theorem 2.2 with the following consequence of [**11**, Corollary 3.2]. Let S be an R-algebra and T an S-algebra which is faithfully flat over R. If the canonical map $\mathrm{Pic}(S) \to \mathrm{Pic}(T \otimes_R S)$ is a monomorphism, then $\mathrm{inf}_2(S, T, R, U)$ is also a monomorphism.

We pause to recall a connection between Amitsur cohomology and group cohomology [7]. If G is a finite group of automorphisms of a ring T with fixed ring $R = T^G$ and J is an additive Ab-valued functor defined on a full subcategory of R-algebras containing all T^n, then there exist canonical homomorphisms $H^n(T/R, J) \to H^n(G, JT)$. (The precise definition of these maps, apart from their naturality, will not be needed in what follows.) These maps are isomorphisms if T/R is Galois with respect to G [**7**, Theorem 5.4].

PROPOSITION 2.3. *Let G be a finite group of automorphisms of a ring T, H a normal subgroup of G, $R = T^G$ and $S = T^H$. Assume S/R is Galois with respect to the canonical G/H-structure (this holds if T/R is Galois with respect to G [**7**, Theorem 2.2]). If $H^1(H, UT) = 0$, then $\mathrm{inf}_2(S, T, R, U)$ is a monomorphism.*

Proof. [**10**, Chapter I, Theorem 2.5] provides a commutative diagram

$$\begin{array}{ccc} H^2(S/R, U) & \to & H^2(T/R, U) \\ \downarrow & & \downarrow \\ H^2(G/H, US) & \to & H^2(G, UT) \end{array}$$

in which the left vertical map is an isomorphism (since U is additive and S/R is Galois). As $(UT)^H = US$, the inflation-restriction theorem of group cohomology [**16**, Chapter VII, Proposition 5] shows the lower horizontal map is a monomorphism. By commutativity of the diagram, so is the upper horizontal map, namely $\mathrm{inf}_2(S, T, R, U)$.

Remark 2.4. The preceding theorem, remark, and proposition generally fail to apply in case $R \subset S \subset T$ is a tower of domains. Indeed, for algebraic number rings, the hypotheses of Theorem 2.2 imply that S and T are principal ideal domains, while Proposition 2.3 requires that the quotient field of S is unramified over the quotient field of R [**7**, Remark 1.5(d)]. A more useful result for algebraic number rings may be deduced from [**11**, Remark 3.3]. With the goal of an eventual generalization of this result (cf. Corollary 3.9 below), we devote the remainder of this section to exploiting Proposition 2.1. As a first application, we re-prove Proposition 2.3 in the special case that R is a Prüfer domain with quotient field K, $K \subset L \subset F$ a tower of finite Galois field extensions, $G = \mathrm{gal}(F/K)$, $H = \mathrm{gal}(F/L)$, $S = \mathrm{Int}_L R$ is Galois over R with respect to G/H, $T = \mathrm{Int}_F R$ and $H^1(H, UT) = 0$.

Since R is Prüfer, S and T are R-flat and it therefore suffices (by Proposition 2.1) to prove $\mathrm{inf}_1(S, T, R, UK/U)$ is a monomorphism. This will follow (cf. [**14**, Theorem 3.1.3]) in case:

(a) $(UK/U)(S^2) \to (UK/U)(T^2)$ is a monomorphism;

(b) $(UK/U)(S)$ maps onto the difference kernel of

$$(UK/U)(T) \rightrightarrows (UK/U)(T \otimes_S T).$$

https://doi.org/10.4153/CJM-1972-019-2 Published online by Cambridge University Press

Since R is integrally closed, multiplication induces isomorphisms

$$S \otimes_R K \xrightarrow{\cong} L \quad \text{and} \quad T \otimes_R K \xrightarrow{\cong} F.$$

Therefore, (a) reduces to proving $(UL^2)/(US^2) \to (UF^2)/(UT^2)$ is a monomorphism. As S/R is Galois, the canonical map $S^2 \to \prod_{G/H} S$ is an isomorphism [**7**, Theorem 1.3]. Since $U(T) \cap U(L) = U(S)$, the conclusion then follows easily from the commutative diagrams

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \to & S^2 & \to & \prod_{G/H} S & \to & 0, \\ & & \downarrow & & \downarrow & & \\ 0 & \to & L^2 & \to & \prod_{G/H} L & \to & 0, \end{array} \qquad \begin{array}{ccccccc} & & L^2 & & \longrightarrow & & F^2 \\ & & \downarrow & & & & \downarrow \\ 0 & \to & \prod_{G/H} L & \to & \prod_G L & \to & \prod_G F, \end{array}$$

and

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & T^2 & \to & \prod_G T & & \\ & & \downarrow & & \downarrow & & \\ 0 & \to & F^2 & \to & \prod_G F & \to & 0. \end{array}$$

As for (b), it is enough to show that (UL)/(US) maps onto the difference kernel of

$$(UF)/(UT) \rightrightarrows [(UK/U)(T \otimes_S T) \to \prod_H (UK/U)(T) \xrightarrow{\cong} \prod_H (UF)/(UT)];$$

i.e., that $(UF/UT)^H = (UL)/(US)$. We have an exact commutative diagram

$$\begin{array}{ccccccccc} 0 \to & (UT)^H & \to & (UF)^H & \to & (UF/UT)^H & \to & H^1(H, UT) & = 0 \\ & \| & & \| & & \uparrow & & \| & \\ 0 \to & US & \to & UL & \to & UL/US & \longrightarrow & 0 & \end{array}$$

from which the conclusion follows by the five lemma, thus completing the proof.

THEOREM 2.5. *Let R be a Prüfer domain with quotient field K, $K \subset L \subset F$ a tower of finite Galois field extensions, $G = \mathrm{gal}(F/K)$ and $H = \mathrm{gal}(F/L)$ such that $S = \mathrm{Int}_L R$ is Galois over R with respect to G/H. Let T be an S-subalgebra of $\mathrm{Int}_F R$ such that multiplication induces an isomorphism*

$$T \otimes_R K \xrightarrow{\cong} F.$$

Then $\ker(\inf_2(S, T, R, U))$ *is $[F:L]$-torsion.*

Proof. We shall construct a group homomorphism N: $U(F^2) \to U(L^2)$ such that the following three conditions hold:

(a) $N(\xi) = \xi^{[F:L]}$ for all $\xi \in U(L^2)$;
(b) $N(f \otimes 1) = N_{F/L}(f) \otimes 1$ and $N(1 \otimes f) = 1 \otimes N_{F/L}(f)$ for all $f \in U(F)$;
(c) $N(UT^2) \subset U(S^2)$;
where we are viewing the standard monomorphisms $L^2 \to F^2$, $T^2 \to F^2$ and $S^2 \to L^2$ as inclusions.

Explicitly, since L/K and F/K are Galois, there are canonical R-algebra isomorphisms

$$L^2 \xrightarrow{\cong} \prod_{G/H} L \quad \text{and} \quad F^2 \xrightarrow{\cong} \prod_{G} F,$$

which give rise to group isomorphisms

$$U(L^2) \xrightarrow{\cong} \prod_{G/H} U(L) \quad \text{and} \quad U(F^2) \xrightarrow{\cong} \prod_{G} U(F).$$

To construct N, it therefore suffices to define a group homomorphism $\prod_G U(F) \to \prod_{G/H} U(L)$. If $\{g_1, \ldots, g_i, \ldots\}$ is a fixed set of coset representatives for H in G, we need only define group homomorphisms N_i: $\prod_G U(F) \to U(L)$; take N_i to be the composition of the g_i-th projection map $\prod_G U(F) \to U(F)$ with the usual field norm $N_{F/L}$: $U(F) \to U(L)$.

To establish (a), let $\xi = \sum a_j \otimes b_j \in U(L^2)$. Fix $1 \leqq i \leqq [L:K]$. The Hg_i-th component of $N(\xi)$ (viewed in $\prod_{G/H} U(L)$) is $N_{F/L}(\sum_j a_j g_i(b_j))$, which we must show equal to $(\sum_j a_j g_i(b_j))^{[F:L]}$. For $h \in H$, let $h^* = g_i^{-1} h g_i (\in H)$. Since H fixes each a_j and b_j, we compute

$$\begin{aligned} N_{F/L}(\textstyle\sum a_j g_i(b_j)) &= \prod_{h \in H} (\textstyle\sum a_j g_i h^*(b_j)) \\ &= \prod_{h \in H} (\textstyle\sum a_j g_i(b_j)) \\ &= (\textstyle\sum a_j g_i(b_j))^{|H|} \\ &= (\textstyle\sum a_j g_i(b_j))^{[F:L]}, \end{aligned}$$

as required.

The first assertion of (b) is trivial. As for the second, we must prove (for fixed i) that $N_{F/L}(g_i(f)) = g_i N_{F/L}(f)$; i.e. that

$$\prod_{h \in H} h g_i(f) = \prod_{h \in H} g_i h(f).$$

This is immediate since the function $h \to h^*$ defined above is clearly a bijection.

As for (c), the isomorphism

$$U(S^2) \xrightarrow{\cong} \prod_{G/H} U(S)$$

reduces the problem to showing $N_{F/L}(UT) \subset US$. Now

$$N_{F/L}(UT) \subset U(N_{F/L}(T)) \subset U(N_{F/L}(\operatorname{Int}_F R)) \subset U(L \cap \operatorname{Int}_F R) = US,$$

and so N has the stated properties.

Since R is Prüfer, S and T are R-flat and Proposition 2.1 reduces the theorem to proving $\ker(\inf_1(S, T, R, UK/U))$ is $[F:L]$-torsion. As R is integrally closed,

$$S \otimes_R K \xrightarrow{\cong} L$$

and there are identifications of the cochains $C^n(S/R, UK/U) = (UL^{n+1})/(US^{n+1})$; similarly $C^n(T/R, UK/U) = (UF^{n+1})/(UT^{n+1})$. We must therefore prove that, if $\xi \in U(L^2)$ and $f \in UF$ satisfy $\xi(f \otimes f^{-1}) \in U(T^2)$, then there exists $l \in UL$ with $\xi^{[F:L]}(l \otimes l^{-1}) \in U(S^2)$.

The homomorphism N and properties (a)-(c) were made to order for this formulation of the problem. If $l = N_{F/L}(f)$ then

$$N(f \otimes f^{-1}) = N(f \otimes 1) \cdot N(1 \otimes f)^{-1} = l \otimes l^{-1}.$$

Hence $\xi^{[F:L]}(l \otimes l^{-1}) = N(\xi)N(f \otimes f^{-1}) = N(\xi(f \otimes f^{-1})) \in N(UT^2) \subset U(S^2)$ and the proof is complete.

Remark. The hypothesis that R is Prüfer was used in the preceding proof only to guarantee that S and T are R-flat and that

$$S \otimes_R K \xrightarrow{\cong} L.$$

COROLLARY 2.6. *With the hypotheses of Theorem 2.5, we assume also that K is perfect of characteristic $p > 0$ and that $[F:L]$ is a power of p. Then $\inf_2(S, T, R, U)$ is a monomorphism.*

Proof. As above, we must prove that, if $\xi \in U(L^2)$ and $f \in UF$ satisfy $\xi(f \otimes f^{-1}) \in U(T^2)$, then there exists $l \in UL$ with $\xi(l \otimes l^{-1}) \in U(S^2)$. Let $q = [F:L]$ and $m = N_{L/K}(f)$; then applying N yields $\xi^q(m \otimes m^{-1}) \in U(S^2)$. Choose $l \in UL$ with $l^q = m$. Then $[\xi(l \otimes l^{-1})]^q \in U(S^2)$, and the result will follow once we prove that $U(S^2)$ is closed under q-th roots in $U(L^2)$. This, in turn, follows from the exact commutative diagram

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ 0 & \to & U(S^2) & \to & \prod_{G/H} U(S) & \to & 0 & & \\ & & & & \downarrow & & & & \\ 0 & \to & U(L^2) & \to & \prod_{G/H} U(L) & \to & 0 & & \end{array}$$

since US is closed under q-th roots in UL.

THEOREM 2.7. *Let R be a domain with quotient field K, $K \subset L \subset F$ a tower of finite field extensions, and S and T flat R-subalgebras of F such that multiplication induces isomorphisms*

$$S \otimes_R K \xrightarrow{\cong} L \quad \text{and} \quad T \otimes_R K \xrightarrow{\cong} F.$$

Assume also that $S \subset T$ *and that* T^2 *is a module-finite* S^2*-projective. Then* $\ker(\inf_2(S, T, R, U))$ *is* $[F{:}L]^2$*-torsion.*

Proof. As in the proof of Theorem 2.5, it suffices to construct a group homomorphism N: $U(F^2) \to U(L^2)$ such that the following three conditions hold:

(a) $N(\xi) = \xi^{[F:L]^2}$ for all $\xi \in U(L^2)$;

(b) $N(f \otimes 1) = N_{F/L}(f)^{[F:L]} \otimes 1$ and $N(1 \otimes f) = 1 \otimes N_{F/L}(f)^{[F:L]}$ for all $f \in U(F)$;

(c) $N(UT^2) \subset U(S^2)$.

Since F^2 is L^2-free of rank $[F{:}L]^2$, we may define N by

$$N(\eta) = \mathrm{Norm}(F^2/L^2; \eta),$$

the determinant of the L^2-linear endomorphism of F^2 effected by multiplication by η. Then it is standard that N is a homomorphism. If $\xi \in U(L^2)$ then $N(\xi)$ is the determinant of the scalar matrix ξI, and (a) is immediate.

As for (b), we need only prove the assertion about $N(f \otimes 1)$ (apply [**2**, Proposition 2(a)] to the "switch" map $F^2 \to F^2$). Assume that multiplication by f on F is represented by the matrix $A = (a_{ij})$ with respect to an L-basis $\{x_1, x_2, \ldots\}$; i.e., $fx_i = \sum a_{ij}x_j$ for all i. Then multiplication by $f \otimes 1$ in F^2 is represented, with respect to the L^2-basis

$$\{x_1 \otimes x_1, \ldots, x_1 \otimes x_{[F:L]}, x_2 \otimes x_1, \ldots, x_2 \otimes x_{[F:L]}, \ldots, x_{[F:L]} \otimes x_1, \ldots, x_{[F:L]} \otimes x_{[F:L]}\}$$

by the matrix B whose entry at the (i, j)-th row and (k, l)-th column is $(a_{ik} \otimes 1)\delta_{j,l}$. (For example, if $[F{:}L] = 3$, then

$$B = \left.\begin{bmatrix} a_{11}\otimes 1 & 0 & 0 & a_{12}\otimes 1 & 0 & 0 & a_{13}\otimes 1 & 0 & 0 \\ 0 & a_{11}\otimes 1 & 0 & 0 & a_{12}\otimes 1 & 0 & 0 & a_{13}\otimes 1 & 0 \\ & & & & \vdots & & & & \\ 0 & 0 & a_{31}\otimes 1 & 0 & 0 & a_{32}\otimes 1 & 0 & 0 & a_{33}\otimes 1 \end{bmatrix}\right).$$

It is easy to see that, if we regard B as representing a transformation t with respect to an ordered basis $\{v_1, \ldots, v_{[F:L]^2}\}$, then t is represented with respect to the basis

$$\{v_1, v_{[F:L]+1}, \ldots, v_{[F:L]([F:L]-1)+1}, v_2, v_{[F:L]+2}, \ldots, v_{[F:L]^2}\}$$

by the $[F{:}L]$-square block matrix

$$C = \begin{bmatrix} A\otimes 1 & & & \\ & A\otimes 1 & & \\ & & \ddots & \\ & & & A\otimes 1 \end{bmatrix}.$$

Hence

$$N(f \otimes 1) = \det(B) = \det(t) = \det(C) = \det(A)^{[F:L]} \otimes 1 = N_{L/K}(f)^{[F:L]} \otimes 1,$$

thus proving (b).

The flatness assumptions permit us to view S^2 as a subalgebra of T^2 and thus to define $N_1 = \mathrm{Norm}(T^2/S^2)$ in the sense of Goldman-Amitsur [**13**; **2**]. Explicitly, let $\lambda \in T^2$. If

$$h: \oplus(S^2) \xrightarrow{\cong} Y \oplus T^2$$

is an isomorphism exhibiting T^2 as an S^2-summand of a direct sum of finitely many copies of S^2 and if g = multiplication by λ on T^2, then

$$N_1(\lambda) = \det(h^{-1}(1_Y \oplus g)h).$$

Since

$$S \otimes_R K \xrightarrow{\cong} L \quad \textit{and} \quad T \otimes_R K \xrightarrow{\cong} F,$$

we may view $h \otimes 1_K$ as an L^2-isomorphism

$$\oplus L^2 \xrightarrow{\cong} (Y \otimes K) \oplus F^2.$$

Now $g \otimes 1_K$ is just multiplication by λ on F^2 and so [**13**, Proposition 1.2] implies

$$N(\lambda) = \det([h \otimes 1_K]^{-1}[1_{Y \otimes K} \oplus (g \otimes 1_K)][h \otimes 1_K])$$

which, by [**13**, Proposition 1.4], equals $\det(h^{-1}(1_Y \oplus g)h) = N_1(\lambda)$. Hence $N(\lambda) \in \mathrm{image}(N_1) \subset S^2$. As $N(1) = 1 \in S^2$, it is clear that $N(UT^2) \subset U(S^2)$, to establish (c) and complete the proof.

Remark. It is perhaps worthwhile to note that the preceding proof actually yielded a commutative diagram

$$\begin{array}{ccc} T^2 & \to & S^2 \\ \downarrow & & \downarrow \\ F^2 & \to & L^2 \end{array}$$

where the horizontal maps are the norms, N_1 and N, and the vertical maps are the canonical inclusions.

Remark 2.8. We should note that the conclusion of Theorem 2.7 may be strengthened in the following case. Let R be a Dedekind domain with quotient field K, $K \subset L \subset F$ a tower of finite separable field extensions, $S = \mathrm{Int}_L R$ and $T = \mathrm{Int}_F R$. Then (by [**2**, Theorem 6] and [**11**, Remark 3.3]) we may conclude $\ker(\inf_2(S, T, R, U))$ is $([F:L], [L:K])$-torsion. This is the result alluded to in Remark 2.4, and generalizations of it will appear in Proposition 3.7 and Corollary 3.9. For the present, it is interesting to observe the following group cohomological analogue.

THEOREM 2.9. *Let R be a domain with quotient field K, $K \subset L \subset F$ a tower of finite Galois field extensions with groups $G = \mathrm{gal}(F/K)$ and $H = \mathrm{gal}(F/L)$, $S = \mathrm{Int}_L R$ and $T = \mathrm{Int}_F R$. Then $([L:K], [F:L])$ annihilates the kernel of the group cohomology homomorphism $H^2(G/H, US) \to H^2(G, UT)$ induced by the maps $G \to G/H$ and $US \to UT$.*

Note. If T/R is Galois with respect to G, then the homomorphism in question may be identified with $\mathrm{inf}_2(S, T, R, U)$.

Proof of Theorem. [**7**, Theorem 5.4] implies $H^1(G, UF)$ is isomorphic to $H^1(F/K, U)$ which vanishes by [**8**, Corollary 4.6]; similarly, $H^1(G/H, UL) = 0$. With the aid of the classical natural isomorphisms $H^2(G, UF) \cong B(F/K)$ and $H^2(G/H, UL) \cong B(L/K)$, the long cohomology sequences provide an exact commutative diagram

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & \alpha & \downarrow \\
H^1(G/H, UL/US) & \to & H^1(G, UF/UT) \\
\downarrow & \beta & \downarrow \\
H^2(G/H, US) & \to & H^2(G, UT) \\
\downarrow & & \downarrow \\
0 \to B(L/K) & \to & B(F/K)
\end{array}$$

A diagram chase shows $\ker(\alpha) = \ker(\beta)$. Moreover, α factors through group cohomological inflation $H^1(G/H, (UF/UT)^H) \to H^1(G, UF/UT)$, which is a monomorphism by the usual inflation-restriction result for group cohomology [**16**, Chapter VII, Proposition 5]. Hence we need only to consider $\ker(\gamma)$, where γ: $H^1(G/H, UL/US) \to H^1(G/H, (UF/UT)^H)$. As $[G:H] = [L:K]$ annihilates $H^1(G/H, UL/US)$, it remains only to prove that $\ker(\gamma)$ is $|H| = [F:L]$-torsion.

We use the standard nonhomogeneous complex (the "bar resolution") for group cohomology. If $\xi \in \prod_{G/H} UL/US$ is a 1-cocycle which becomes a 1-coboundary in $\prod_{G/H} (UF/UT)^H$ (i.e., lies in $\ker(\gamma)$), then there exists $\eta \in UF$ such that $\xi(Hg) = (g\eta)\eta^{-1}(UT)$ for all $g \in G$. In other words, if $a_g \in UL$ satisfies $\xi(Hg) = a_g(US)$, then

$$a_g \equiv (g\eta)\eta^{-1} \bmod (UT).$$

Let $l = N_{F/L}(\eta)$. Then (writing $gh = h^*g$ as before) we compute

$$(gl)l^{-1} = \left[\prod_{h \in H} h^*g(\eta)\right] \Big/ \left[\prod_{h \in H} h(\eta)\right] = \prod_{h \in H} h((g\eta)\eta^{-1})$$

$$\equiv \prod_{h \in H} h(a_g) = (a_g)^{|H|} \bmod (UT).$$

Hence

$$(gl)l^{-1} \equiv (a_g)^{|H|} \bmod (UL \cap UT);\ \text{i.e., } \bmod (US).$$

The 1-coboundary of $l(US)$ is therefore $\xi^{|H|}$, completing the proof.

https://doi.org/10.4153/CJM-1972-019-2 Published online by Cambridge University Press

Remark 2.10. Let R be a domain with quotient field K, $K \subset L \subset F$ a tower of finite field extensions with F/K Galois, and T a flat R-subalgebra of $\mathrm{Int}_F R$. In Theorems 2.5 and 2.7, we had reason to consider elements $\xi \in U(L^2)$ for which there exist $f \in UF$ such that $\xi(f \otimes f^{-1}) \in U(T^2)$. We conclude this section by proving that such elements always satisfy

$$\mathrm{Norm}(L^2, K; \xi) \in U(\mathrm{Int}_K R).$$

Let $N = \mathrm{Norm}(F^2, K)$, $N_1 = \mathrm{Norm}(L^2, K)$, $N_2 = \mathrm{Norm}(F^2, L^2)$, $k = N_{F/K}(f)$ and $n = [F{:}L]$. As in the proof of Theorem 2.7, we have $N(f \otimes f^{-1}) = N(f \otimes 1)N(1 \otimes f)^{-1} = (k^n)(k^{-n}) = 1$ and $N_2(\xi) = \xi^{n^2}$. By [**2**, Corollary 3], $N(\xi) = N_1(N_2(\xi)) = (N_1\xi)^{n^2}$. Since $U(\mathrm{Int}_K R)$ is closed under roots in UK, and N is a homomorphism, it therefore suffices to show $N(U(T^2)) \subset U(\mathrm{Int}_K R)$.

Let $G = \mathrm{gal}(F/K)$, $I = \mathrm{Int}_F R$ and $\lambda \in U(T^2)$. By means of the standard isomorphism

$$F^2 \xrightarrow{\cong} \prod_G F,$$

we may identify λ with $(\ldots, \lambda_g, \ldots) \in \prod_G U(I)$. Then

$$N(\lambda) = \textstyle\prod_g N_{F/K}(\lambda_g) \in U(N_{F/K}(I)) \subset U(\mathrm{Int}_K R),$$

completing the proof.

3. Relations with group cohomology. In this final section we study some connections between Amitsur and group cohomology. The following result will prove to be basic.

THEOREM 3.1 (Silver). *Let G be a finite group of automorphisms of an (respectively, a flat; respectively, a module-free; respectively, a projective) R-algebra S. Let J be an additive Ab-valued functor defined on the category of all (respectively, flat; respectively, module-free; respectively, projective) R-algebras. For $q \geq 0$, define the functor J^q by $J^q(A) = H^q(G, J(A \otimes_R S))$. Then there exists a first quadrant spectral sequence $H^p(S/R, J^q) \Rightarrow H^{p+q}(G, J(S))$.*

Proof. One may adapt the arguments leading to [**17**, Theorem 2.3, p. 31] to the category of all (respectively, flat; respectively, module-free; respectively, projective) R-algebras and thus derive a double complex for which the above is an associated spectral sequence.

We pause to record the exact sequence of low terms of the above spectral sequence. (Note that an apparent error in [**17**] identifies J^0 with J in stating this result.)

COROLLARY 3.2. *Under the conditions of the theorem, there is an exact sequence*

$$\begin{aligned} 0 &\to H^1(S/R, J^0) \to H^1(G, J(S)) \to H^0(S/R, J^1) \\ &\to H^2(S/R, J^0) \to \ker[H^2(G, J(S)) \to H^2(G, J(S^2))] \\ &\to H^1(S/R, J^1) \to H^3(S/R, J^0). \end{aligned}$$

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COROLLARY 3.3. *If S is the ring of algebraic integers of a quadratic number field L and $G = \mathrm{gal}(L/\mathbf{Q})$, then $H^1(G, US)$ is nonzero and is the difference kernel of the two homomorphisms from $H^1(G, US^2)$ to $H^1(G, US^3)$ induced by the face maps $\epsilon_0, \epsilon_1: S^2 \to S^3$.*

Proof. Since $S^G = \mathbf{Z}$, it is easy to see that the canonical map $U(A) \to U(A \otimes_R S)^G$ is an isomorphism for any module-free R-algebra A. Then the natural transformation $U \to U^0$ is an equivalence, and the preceding corollary yields an exact sequence

$$H^1(S/\mathbf{Z}, U) \to H^1(G, US) \to H^0(S/\mathbf{Z}, U^1) \to H^2(S/\mathbf{Z}, U).$$

However, Theorem 1.3 shows $H^1(S/\mathbf{Z}, U) = 0$ (since $\mathrm{Pic}(\mathbf{Z}) = 0$) and, as noted in Remark 1.10 (b), $H^2(S/\mathbf{Z}, U) = 0$; hence, $H^1(G, US) \cong H^0(S/\mathbf{Z}, U^1)$, the difference kernel in question. It remains only to prove that $H^1(G, US) \neq 0$. If N is the restriction of $N_{L/Q}$ viewed as an endomorphism of US and if g is the nontrivial element of G, then $H^1(G, US) \cong [\ker(N)]/\{s^{-1}g(s): s \in US\}$.

Case 1: *S is real.* By the Dirichlet unit theorem, US is the direct product of $\{1, -1\}$ with the free multiplicative abelian group generated by a "fundamental unit", u. Since $g^2 = 1$, freeness implies $g(u) = \pm u^{\pm 1}$; of course $g(u) \neq u$ because $u \notin \mathbf{Z}$.

Subcase (*i*): $g(u) = -u$. Then, for all $s \in US$, one checks easily that $s^{-1}g(s) \in \{1, -1\}$. Since $N(u^2) = N(u)^2 = (\pm 1)^2 = 1$ and $u^2 \neq \pm 1$, the proof of this subcase is complete.

Subcase (*ii*): $g(u) = tu^{-1}$, *where* $t = \pm 1$. If $t' = \pm 1$ and $j \in \mathbf{Z}$ then $(t'u^j)^{-1}g(t'u^j) = t^j u^{-2j} \neq -1 \in \ker(N)$.

Case 2: *S is complex.* Express $L = \mathbf{Q}(\sqrt{m})$ for some negative squarefree rational integer m. If $m = -1$, then $US = \{1, -1, i, -i\}$ where $i^2 = -1$; note $i \notin \{s^{-1}g(s): s \in US\}$ although $N(i) = 1$. If $m = -3$, then

$$US = \{1, -1, (-1 + \sqrt{-3})/2, (-1 - \sqrt{-3})/2, (1 - \sqrt{-3})/2, (1 + \sqrt{-3})/2\};$$

note $-1 \notin \{s^{-1}g(s): s \in US\}$ although $N(-1) = 1$. For other values of m, $US = \{1, -1\}$ and the assertion is clear.

PROPOSITION 3.4. *Let R be a domain with ordered quotient field K. Let $L = K(\sqrt{m})$ for some negative nonsquare $m \in K$. Let $G = \mathrm{gal}(L/K)$ and $S = \mathrm{Int}_L R$. If $U(R) = \{1, -1\}$, then $H^1(G, UL/US) = 0$.*

Proof. As in the preceding argument, the usual formula for the cohomology of a cyclic group shows $H^1(G, UL/US) \cong \ker(N)/I_G(UL/US)$ where I_G is the augmentation ideal of G and N is the endomorphism of UL/US induced by the field norm $N_{L/K}$. Note that $lUS \in \ker(N)$ if and only if $N_{L/K}(l) = \pm 1$; if $G = \{1, g\}$, then $I_G(UL/US)$ is just the collection of cosets of elements of the form $g(l)l^{-1}$. By Hilbert's Theorem 90, it is enough to show that no $l \in L$ can satisfy $N_{L/K}(l) = -1$. If $l = a + b\sqrt{m}$ (with $a, b \in K$) and $N_{L/K}(l) = -1$ then (since $g(\sqrt{m}) = -\sqrt{m}$) we obtain $a^2 - mb^2 = -1$, contradicting negativity of m.

Remark 3.5. (a) Corollary 3.3 provides an example in which corresponding one-dimensional Amitsur and group cohomology groups (in U) differ. For another example, let $G = \{1, g\}$ act on $S = \mathbf{Z}[X]/(X^2) = \mathbf{Z}[x]$ by $g(m + nx) = m - nx$; then S is $\mathbf{Z}$-free on $\{1, x\}$ and $S^G = \mathbf{Z}$. The existence of ring maps, $\mathbf{Z} \to S$ and $S \to \mathbf{Z}$, implies $H^n(S/\mathbf{Z}, U) = H^n(\mathbf{Z}/\mathbf{Z}, U) = 0$ for all $n \geq 1$, by the fundamental homotopy property of Amitsur cohomology [**1**, Lemma 2.7]. However, $H^{2n+1}(G, US) \cong \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$ and $H^{2n}(G, US) \cong \mathbf{Z}/2\mathbf{Z}$ for all $n \geq 0$. Of course, $S/\mathbf{Z}$ is not Galois since $(1, -1)$ is not in the image of the canonical map $S^2 \to S \times S$.

(b) Remark 1.10 (b) and Proposition 3.4 show that the corresponding connecting homomorphisms, $H^1(G, UL/US) \to H^2(G, US)$ and

$$H^1(S/\mathbf{Z}, U\mathbf{Q}/U) \to H^2(S/\mathbf{Z}, U),$$

are both zero maps if S is the ring of algebraic integers of a complex quadratic number field L. However, the former map is not epimorphic (it is $0 \to \mathbf{Z}/2\mathbf{Z}$), while the latter map is the trivial isomorphism. In particular, the connecting homomorphism of group cohomology is not the kernel of a map into the split Brauer group (cf. Corollary 1.5).

We next show that certain (not necessarily module-finite) R-algebras have torsion Amitsur cohomology groups.

PROPOSITION 3.6. *Let S be a faithful projective R-algebra and G a finite group of R-algebra automorphisms of S such that $S^G = R$. If G has exactly n elements, then $H^2(S/R, U)$ is n^2-torsion.*

Proof. For each R-module, A, let f_A: $A \to (A \otimes_R S)^G$ be the canonical homomorphism. By a standard argument, $f_{A \oplus B}$ is an isomorphism if and only if f_A and f_B are each isomorphisms. However, f_F is an isomorphism for any free R-module F, and hence also for any R-projective. It is then easy to show that the natural transformation $U \to U^0$ is an equivalence on the category of projective R-algebras, and an analysis of the spectral sequence of Theorem 3.1 leads to the exact sequence $H^0(S/R, U^1) \to H^2(S/R, U) \to H^2(G, US)$. Since $H^0(S/R, U^1)$ is a subgroup of $H^1(G, U(S^2))$ and group cohomology in G is annihilated by $|G| = n$, the conclusion is immediate.

PROPOSITION 3.7. *Let R be a Prüfer domain with quotient field K, L a finite field extension of K, and S an R-subalgebra of* $\mathrm{Int}_L R$. *Then* $[L{:}K]$ *annihilates* $H^n(S/R, U)$ *for all* $n \geq 0$.

Proof. Let M range over the inclusion-directed collection of finitely generated R-subalgebras of S. Each M is a module-finite R-flat, hence R-projective of rank dividing $[L{:}K]$. (One uses [**5**, Théorème 1, p. 138], noting that R is con-

https://doi.org/10.4153/CJM-1972-019-2 Published online by Cambridge University Press

nected, to see the rank of M exists and is the K-dimension of the subspace $M \otimes_R K$ of L). Hence [**2**, Theorem 6] shows $[L:K]$ annihilates each $H^n(M/R, U)$. Since

$$\varinjlim M = S,$$

we may argue as in Theorem 1.7 to show

$$\varinjlim H^n(M/R, U) \xrightarrow{\cong} H^n(S/R, U),$$

which readily yields the desired result.

THEOREM 3.8. *Let R be a domain with quotient field K, L a finite Galois field extension of K, and $S = \operatorname{Int}_L R$. Assume that S is R-flat and that multiplication induces an isomorphism*

$$S \otimes_R K \xrightarrow{\cong} L.$$

If $n = [L:K]$, then $H^1(S/R, UK/U)$ is n^2-torsion and hence $H^2(S/R, U)$ is n^3-torsion.

Proof. The last statement follows from the exact sequence

$$H^1(S/R, UK/U) \to H^2(S/R, U) \to H^2(L/K, U)$$

since $n \cdot H^2(L/K, U) = 0$ [**2**, Theorem 6].

For the first statement, let $J = UK/U$. Since Theorem 3.1 supplies a monomorphism $H^1(S/R, J^0) \to H^1(G, J(S))$, it follows that $H^1(S/R, J^0)$ is annihilated by $|G| = n$. If X is the kernel of the homomorphism $H^1(S/R, J) \to H^1(S/R, J^0)$ induced by the natural transformation $J \to J^0$, it suffices to prove $n \cdot X = 0$.

Consider the commutative diagram

$$\begin{array}{ccccc} J(S) & \rightrightarrows & J(S^2) & \overrightarrow{\rightrightarrows} & J(S^3) \\ \alpha\downarrow & & \beta\downarrow & & \downarrow \\ J^0(S) & \rightrightarrows & J^0(S^2) & \overrightarrow{\rightrightarrows} & J^0(S^3). \end{array}$$

Under the usual identifications, β may be regarded as the face map $J(\epsilon_2): J(S^2) = UL^2/US^2 \to J(S^3) = UL^3/US^3$, with image restricted to lie in $J(S^3)^G$. Hence β is a monomorphism and it therefore suffices to prove that $n \cdot J^0(S) \subset \operatorname{im}(\alpha)$; i.e., that the n-th power of any element $\bar{\xi}$ of $(UL^2/US^2)^G$ is in the image of UL/US.

Let $\xi = \sum a_i \otimes b_i$ be a coset representative of $\bar{\xi}$. Since G acts on L^2 via the second factor, we have

$$\sum a_i \otimes b_i \equiv \sum a_i \otimes g(b_i) \bmod (US^2)$$

for all $g \in G$. Multiplying the congruences yields

$$\xi^n \equiv \lambda = \prod_g \left(\sum a_i \otimes g(b_i)\right) \bmod (US^2).$$

Evidently, $\lambda \in (L^2)^G = L \otimes 1$. Viewing $\lambda \in UL$, we see $\bar{\xi}^n = \alpha(\lambda \cdot US)$, and the proof is complete.

Combining Theorems 2.7 and 3.8 leads to the following result.

COROLLARY 3.9. *Let R be a domain with quotient field K, $K \subset L \subset F$ a tower of finite field extensions with L/K Galois, $S = \mathrm{Int}_L R$, and T a flat R-subalgebra of F containing S such that multiplication induces isomorphisms*

$$S \otimes_R K \xrightarrow{\cong} L \quad \text{and} \quad T \otimes_R K \xrightarrow{\cong} F.$$

Assume also that S is R-flat and that T^2 is a module-finite S^2-projective. Then $([F:L]^2, [L:K]^3)$ *annihilates* $\ker(\inf_2(S, T, R, U))$.

Remark. Although we have dealt exclusively with UK/U, we close by stating a result that indicates the usefulness of a new coefficient functor (proof via standard techniques). If R is a domain with quotient field K and S is a flat R-subalgebra of a field extension L of K, then the canonical map

$$H^1(S/R, UL/U) \to H^2(S/R, U)$$

is an isomorphism.

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