Compactifications of moduli of del Pezzo surfaces via line arrangement and K-stability

Junyan Zhao

Abstract. In this paper, we study compactifications of the moduli of smooth del Pezzo surfaces using K-stability and the line arrangement. We construct K-moduli of log del Pezzo pairs with sum of lines as boundary divisors, and prove that for $d = 2, 3, 4$, these K-moduli of pairs are isomorphic to the K-moduli spaces of del Pezzo surfaces. For $d = 1$, we prove that they are different by exhibiting some walls.

1 Introduction

The moduli spaces of del Pezzo surfaces have been extensively studied thanks to the development of different approaches to construct moduli spaces.

A weighted marked del Pezzo surface is a pair (S, B) with S a del Pezzo surface and $B = \sum b_i B_i$ a boundary divisor, where $b_i \in [0, 1]$ and B_i's are all the finitely many lines on S. The Kollár, Shepherd-Barron, and Alexeev (KSBA) moduli spaces parameterize all such pairs such that (S, B) is semi-log-canonical and $K_S + B$ is ample. For example, when $b_i = 1$, for each i, the moduli of such quartic del Pezzo pairs are isomorphic to the moduli $\overline{M}_{0,5}$ of rational curves with five marked points (cf. [HKT09, HKT13]). Another interesting case is when $b_i = b$, where b is a positive number such that (S, B) is close to a Calabi–Yau pair (cf. [GKS21, HKT09, HKT13]).

The development of K-stability provides a moduli theory for log Fano pairs, called K-moduli spaces. The general K-moduli theory was established by a group of people in the past decade (cf. [ABHLX20, BHLLX21, BLX22, BX19, CP21, LWX21, LXZ22, XZ20]). Roughly speaking, for fixed numerical invariants, we have a projective scheme parameterizing the K-polystable log Fano pairs with these invariants. Moreover, in the surface case, there is a wall crossing structure when we vary the coefficients of the boundary divisors (cf. [ADL19, ADL21]). See Section 2 for definitions and more details. We can ask the similar questions in the K-moduli spaces setting:

Question *Fix a degree $d \in \{1, 2, 3, 4\}$ and a rational number $0 \le c < 1$.*

(1) *Can we describe the K-moduli space $\overline{M}_{d,c}^K$ of K-polystable log Fano pairs (X, cD) admitting a $\mathbb{Q}$-Gorenstein smoothing to a pair (X_0, cD_0), where X_0 is a smooth*

del Pezzo surface of anticanonical degree d and $D_0 = \sum L_i$ is the sum of lines (i.e., $-K_{X_0}$-degree 1 rational curves)?

(2) *When varying c in the range such that (X_0, cD_0) is log Fano, do the K-moduli spaces of pairs in (1) change? If yes, can we determine all the values c_i's at which the moduli spaces get changed?*

In particular, when $c = 0$, the K-moduli space parameterizes all the del Pezzo surfaces of a fixed degree, which admit a Kähler–Einstein metric (cf. [MM93, OSS16]).

For $c > 0$, if a pair (X, cD) is a K-polystable degeneration of pairs $(X_0, c \sum L_i)$, then D is unique by the separatedness of the K-moduli spaces. In particular, when X_0 has du Val singularities, the boundary divisor D can also be written as the sum $\sum L_i'$ of lines, where L_i''s do not have to be distinct.

We prove in this paper that for $d = 2, 3, 4$, there are no walls for the K-moduli spaces $\overline{M}_{d,c}^K$, and they are all isomorphic to K-moduli of del Pezzo surfaces. For $d = 1$, we exhibit examples to see that there is indeed some walls.

Theorem 1.1 (Corollary 3.8, Proposition 3.10, and Theorem 4.12) *For any fixed degree $d \in \{2, 3, 4\}$, there are no walls for the K-moduli $\overline{M}_{d,c}^K$ when c varies from 0 to the log Calabi–Yau threshold, and they are isomorphic to the K-moduli space of del Pezzo surfaces of degree d.*

Theorem 1.2 (Proposition 5.2) *Let $0 < c < \frac{1}{240}$ be a number, and let $\overline{M}_{1,c}^K$ be the K-moduli space defined above. Then there is a wall $c = c' < \frac{1}{288}$, which is given by the destabilization of the surface pair, where the surface acquires an A_7-singularity.*

2 Preliminaries

Convention   In this paper, we work over the field of complex numbers $\mathbb{C}$. By a surface, we mean a complex connected projective algebraic surface. We keep the notions on singularities of surface pairs the same as [KM98, Chapters 2 and 4].

For the reader's convenience, we state the definitions and results in simple versions, especially in dimension 2. Most of them can be generalized to higher dimension.

Definition 2.1 Let X be a normal projective variety, and let D be an effective $\mathbb{Q}$-divisor. Then (X, D) is called a log Fano pair if $K_X + D$ is $\mathbb{Q}$-Cartier and $-(K_X + D)$ is ample. A normal projective variety X is called a $\mathbb{Q}$-Fano variety if $(X, 0)$ is a klt log Fano pair.

Definition 2.2 Let (X, D) be an n-dimensional log Fano pair, and let E be a prime divisor on a normal projective variety Y, where $\pi : Y \to X$ is a birational morphism. Then the log discrepancy of (X, E) with respect to E is

$$A_{(X,D)}(E) := 1 + \text{coeff}_E(K_Y - \pi^*(K_X + D)).$$

We define the S-invariant of (X, D) with respect to E to be

$$S_{(X,D)}(E) := \frac{1}{(-K_X - D)^n} \int_0^\infty \text{vol}_Y(\pi^*(-K_X - D) - tE)dt,$$

and the β-invariant of (X, D) with respect to E to be

$$\beta_{(X,D)}(E) := A_{(X,D)}(E) - S_{(X,D)}(E).$$

The first definition of K-(poly/semi)stability of log Fano pairs used test configurations. We refer the reader to [ADL19, Xu21]. There is an equivalent definition using valuations, which is called *valuative criterion* for K-stability. The advantage of this definition is that it is easier to check.

Theorem 2.3 (cf. [BX19, Fuj19, Li17, BX19])   *A log Fano pair (X, D) is:*

(1) *K-semistable if and only if $\beta_{(X,D)}(E) \geq 0$ for any prime divisor E over X;*
(2) *K-stable if and only if $\beta_{(X,D)}(E) > 0$ for any prime divisor E over X.*

The following powerful result is called *interpolation* of K-stability. We only state a version that we will use later. For a more general statement, see, for example, [ADL19, Proposition 2.13] or [Der16, Lemma 2.6].

Theorem 2.4   *Let X be a K-semistable $\mathbb{Q}$-Fano variety, and let $D \sim_{\mathbb{Q}} -rK_X$ be an effective divisor.*

(1) *If $\left(X, \frac{1}{r}D\right)$ is klt, then (X, cD) is K-stable for any $c \in \left(0, \frac{1}{r}\right)$.*
(2) *If $\left(X, \frac{1}{r}D\right)$ is log canonical, then (X, cD) is K-semistable for any $c \in \left(0, \frac{1}{r}\right)$.*

Theorem 2.5   *Let (X, D) be a klt log Fano pair which isotrivially degenerates to a K-semistable log Fano pair (X_0, D_0). Then (X, D) is also K-semistable.*

This follows immediately from the openness of K-(semi)stability (cf. [BLX22, Xu20]), which is useful in our analysis of K-moduli spaces.

Recall that the volume of a divisor D on an n-dimensional normal projective variety Y is

$$\mathrm{vol}_Y(D) := \lim_{m \to \infty} \frac{\dim H^0(Y, mD)}{m^n/n!}.$$

The divisor D is big by definition if and only if $\mathrm{vol}_Y(D) > 0$.

Definition 2.6   Let $x \in (X, D)$ be an n-dimensional klt singularity. Let $\pi : Y \to X$ be a birational morphism such that $E \subseteq Y$ is an exceptional divisor whose center on X is $\{x\}$. Then the volume of $(x \in X)$ with respect to E is

$$\mathrm{vol}_{x,X,D}(E) := \lim_{m \to \infty} \frac{\dim \mathcal{O}_{X,x}/\{f \in \mathcal{O}_{X,x} : \mathrm{ord}_E(f) \geq m\}}{m^n/n!},$$

and the normalized volume of $(x \in X)$ with respect to E is

$$\widehat{\mathrm{vol}}_{x,X,D}(E) := A_{(X,D)}(D) \cdot \mathrm{vol}_{x,X,D}(E).$$

We define the local volume of $x \in (X, D)$ to be

$$\widehat{\mathrm{vol}}(x, X, D) := \inf_E \widehat{\mathrm{vol}}_{x,X,D}(E),$$

where E runs through all the prime divisor over X whose center on X is $\{x\}$.

Theorem 2.7 (cf. [Fuj18, Liu18, LL19])   *Let (X, D) be an n-dimensional K-semistable log Fano pair. Then, for any $x \in X$, we have*

$$(-K_X - D)^n \le \left(1 + \frac{1}{n}\right)^n \widehat{\mathrm{vol}}(x, X, D).$$

Now, let us briefly review some results on the K-moduli spaces of log Fano pairs. We mainly state the results in our setting. For more general statements, see [ADL19, Theorem 3.1 and Remark 3.25] or [ADL21, Theorem 2.21].

Definition 2.8   Let $f : (\mathcal{X}, \mathcal{D}) \to B$ be a proper flat morphism to a reduced scheme with normal, geometrically connected fibers of pure dimension n, where $\mathcal{D}$ is an effective relative Mumford $\mathbb{Q}$-divisor (cf. [Kol19, Definition 1]) on $\mathcal{X}$ which does not contain any fiber of f. Then f is called a $\mathbb{Q}$-Gorenstein flat family of log Fano pairs if $-(K_{\mathcal{X}/B} + \mathcal{D})$ is $\mathbb{Q}$-Cartier and ample over B.

Definition 2.9   Let $0 < c < 1/r$ be a rational number, and let (X, cD) be a log Fano pair such that $D \sim -rK_X$. A $\mathbb{Q}$-Gorenstein flat family of log Fano pairs $f : (\mathcal{X}, c\mathcal{D}) \to C$ over a pointed smooth curve $(0 \in C)$ is called a $\mathbb{Q}$-Gorenstein smoothing of (X, D) if:

(1) the divisors $\mathcal{D}$ and $K_{\mathcal{X}/C}$ are both $\mathbb{Q}$-Cartier, f-ample, and $\mathcal{D} \sim_{\mathbb{Q}, f} -rK_{\mathcal{X}/B}$;
(2) both f and $f|_{\mathcal{D}}$ are smooth over $C\backslash\{0\}$; and
(3) $(\mathcal{X}_0, c\mathcal{D}_0) \simeq (X, cD)$.

Theorem 2.10 (cf. [ADL19])   *Let χ be the Hilbert polynomial of an anti-canonically polarized smooth del Pezzo surface X of degree d. Let r be the positive integer, and let $c \in (0, 1/r)$ be a rational number. Consider the moduli pseudo-functor sending a reduced base S to*

$$\mathcal{M}_{d,c}^K(S) = \left\{ (\mathcal{X}, \mathcal{D})/S \, \middle| \, \begin{matrix} (\mathcal{X}, c\mathcal{D})/S \text{ is a } \mathbb{Q}\text{-Gorenstein smoothable log Fano family,} \\ \mathcal{D} \sim_{S, \mathbb{Q}} -rK_{\mathcal{X}/S}, \text{ each fiber } (\mathcal{X}_s, c\mathcal{D}_s) \text{ is K-semistable, and} \\ \chi(\mathcal{X}_s, \mathcal{O}_{\mathcal{X}_s}(-mK_{\mathcal{X}_s})) = \chi(m) \text{ for } m \text{ sufficiently divisible.} \end{matrix} \right\}.$$

Then there is a smooth quotient stack $\mathcal{M}_d^K(c)$ of a smooth scheme by a projective general linear group which represents this pseudo-functor. The $\mathbb{C}$-points of $\mathcal{M}_d^K(c)$ parameterize K-semistable $\mathbb{Q}$-Gorenstein smoothable log Fano pairs (X, cD) with Hilbert polynomial $\chi(X, \mathcal{O}_X(-mK_X)) = \chi(m)$ for sufficiently divisible $m \gg 0$ and $D \sim_{\mathbb{Q}} -rK_X$. Moreover, the stack $\mathcal{M}_d^K(c)$ admits a good moduli space $\overline{M}_d^K(c)$, which is a normal projective reduced scheme of finite type over $\mathbb{C}$, whose $\mathbb{C}$-points parameterize K-polystable log Fano pairs.

Let X be a smooth del Pezzo surface of degree $d \in \{1, 2, 3, 4\}$, let L_i's be the lines on X, and let r be the integer such that $\sum L_i \sim -rK_X$.

Definition 2.11   Let $\mathcal{M}_{d,c}^K$ be the stack-theoretic closure of the locally closed substack of $\mathcal{M}_d^K(c)$ parameterizing pairs (X, cD), where X is a smooth del Pezzo surface and $D = \sum L_i$, and let $\overline{M}_{d,c}^K$ be its good moduli space.

Remark 2.12   The good moduli space $\overline{M}_{d,c}^K$ is exactly the closed subscheme of $\overline{M}_d^K(c)$, which is the scheme-theoretic closure of the locus parameterizing smooth del Pezzo surfaces with the sum of lines.

Theorem 2.13 (cf. [ADL19], Theorem 1.2])   *Keep the notation as in the last theorem. There are rational numbers*

$$0 = c_0 < c_1 < c_2 < \cdots < c_n = \frac{1}{r}$$

such that for every $0 \le j < n$, the K-moduli stacks $\mathcal{M}_{d,c}^K$ are independent of the choice of $c \in (c_j, c_{j+1})$. Moreover, for every $0 \le j < n$ and $0 < \varepsilon \ll 1$, one has open immersions

$$\mathcal{M}_{d,c_j-\varepsilon}^K \hookrightarrow \mathcal{M}_{d,c_j}^K \hookleftarrow \mathcal{M}_{d,c_j+\varepsilon}^K,$$

which descend to projective birational morphisms

$$\overline{M}_{d,c_j-\varepsilon}^K \to \overline{M}_{d,c_j}^K \leftarrow \overline{M}_{d,c_j+\varepsilon}^K.$$

3 Cubic and quartic del Pezzo pairs

In this section, we focus on the cases when $d = 3, 4$. The K-moduli of cubic (resp. quartic) del Pezzo surfaces are isomorphic to the GIT moduli spaces of cubics (resp. quartics) (cf. [MM93, OSS16]). In particular, the K-polystable limits of the smooth ones are still embedded in $\mathbb{P}^3$ (resp. $\mathbb{P}^4$) as a cubic surface (resp. complete intersection of quadric hypersurfaces), and they have at worst A_1 and A_2-singularities (resp. A_1-singularities). Thus, it makes sense to discuss lines on cubic and quartic del Pezzo surfaces with these mild singularities. The lines on singular cubic surfaces were first studied in [Cay69]: they are the degeneration of the 27 lines on smooth cubic surfaces with *multiplicities*, which are nothing but the number of lines which reduce to a given one on a singular cubic surface.

We will prove that for all c in the Fano region, the K-moduli spaces $\overline{M}_{d,c}^K$ of cubic (and quartic) del Pezzo pairs do not have wall crossings, and the moduli spaces are all isomorphic to the K-moduli space of cubic (resp. quartic) del Pezzo surfaces. The proof will proceed by first showing that for $0 < c = \varepsilon \ll 1$, we have the desired isomorphism induced by the natural forgetful map, and then showing that $\overline{M}_{d,c}^K \simeq \overline{M}_{d,\varepsilon}^K$.

3.1 The cubic case

In [OSS16], the authors essentially proved that for cubic del Pezzo surfaces, the GIT-(semi/poly)stability is equivalent to the K-(semi/poly)stability. As a consequence, the two moduli spaces are isomorphic. However, the K-moduli space was constructed only in recent years. Although this is well known to experts, for the reader's convenience, here we state a more recent proof using local–global volumes comparison.

Theorem 3.1 (cf. [OSS16], Section 4.2])   *A cubic surface is K-semistable if and only if it is GIT-semistable. In particular, the K-moduli space of degree 3 del Pezzo surfaces is isomorphic to the GIT moduli space of cubic surfaces.*

Proof   We know that the Fermat cubic surface is K-stable (cf. [Tia87]). By the openness of the K-stability (cf. Theorem 2.5), a general cubic surface is K-stable. Denote by $\overline{M}_3^K$ the K-moduli space of cubic del Pezzo surfaces, which parameterizes

K-stable smooth cubic surfaces and their K-polystable limits. Let $X \in \overline{M}_3^K$ be the limit of a one-parameter family of $\{X_t\}_{t \in T \setminus \{0\}}$ K-semistable smooth cubics.

As the K-semistable surfaces have klt singularities (cf. [Oda13]), in particular have quotient singularities, we let $(x \in X) \simeq (0 \in \mathbb{A}^2/G_x)$ be a singular point (if it exists). Then, by Theorem 2.7, we have that

$$3 = (-K_X)^2 \leq \frac{9}{4}\widehat{\mathrm{vol}}(x, X) = \frac{9}{4} \cdot \frac{4}{|G_x|} = \frac{9}{|G_x|},$$

which implies that $|G_x| \leq 3$. If $|G_x| = 2$, then x is an A_1-singularity. If $|G_x| = 3$, then x is either an A_2-singularity or a $\frac{1}{3}(1,1)$-singularity. The latter case is ruled out by [KSB88, Proposition 3.10]. We thus conclude that X has at worst A_1- or A_2-singularities. It follows from [Fuj90, Section 2] that X can be embedded by $|-K_X|$ into $\mathbb{P}^3$ as a cubic surface.

Since for hypersurfaces the K-stability implies GIT-stability (cf. [OSS16, Theorem 3.4] or [PT09, Theorem 2]), then we get an open immersion $\mathcal{M}_3^K \hookrightarrow \mathcal{M}_3^{\mathrm{GIT}}$ of moduli stacks, which descends to a birational and injective morphism $\Phi : \overline{M}_3^K \to \overline{M}_3^{\mathrm{GIT}}$ between good moduli spaces. Notice that the $\overline{M}_3^{\mathrm{GIT}}$ is normal by the properties of GIT quotients. As both of these two moduli spaces are proper, then Φ is a finite map, and thus Φ is an isomorphism by Zariski Main Theorem. ∎

Remark 3.2   We will frequently call $\overline{M}_3^K$ the K-moduli spaces of cubic surfaces, by which we mean the K-moduli compactification of smooth K-polystable cubic surfaces.

In the proof of the theorem, we deduce that a K-polystable cubic del Pezzo surface has at worst A_1- or A_2-singularities. This partially recovers the following classical result of Hilbert.

Theorem 3.3 (cf. [Hil70])   *A cubic surface $X \subseteq \mathbb{P}^3$ is:*

(i) *GIT-stable if and only if it has at worst A_1-singularities;*
(ii) *GIT-strictly polystable if and only if it is isomorphic to the cubic X_0 defined by $x_0^3 = x_1 x_2 x_3$;*
(iii) *GIT-semistable if and only if it has at worst A_1- or A_2-singularities.*

For the semistable cubic surfaces, we know how the lines degenerate. In other words, we know the multiplicities of the lines (cf. [Tu05, Table 1]).

Proposition 3.4   *Let X be a semistable cubic surface, and let L_i's be the 27 lines on it. Then the pair $(X, c \sum L_i)$ is log canonical for $0 < c < \frac{1}{9}$.*

Proof   It is proven in [GKS21, Proposition 5.8] that the log canonical threshold of X is larger than $\frac{1}{9}$ if X has at worst A_1-singularities.

Now, let us deal with the surfaces with A_2-singularities. We first give an explicit construction of the surface $X_0 = \{x_0^3 = x_1 x_2 x_3\}$. Let $(x : y : z)$ be the homogeneous coordinates of $\mathbb{P}^2$, then blow up the tangent vector given by $\{y = 0\}$ at $(1 : 0 : 0)$, the tangent vector given by $\{z = 0\}$ at $(0 : 1 : 0)$, and the tangent vector given by $\{x = 0\}$ at $(0 : 0 : 1)$. Denote by E_i and F_i the exceptional divisors on the blowup with self-intersection -2 and -1, respectively, and H_i the (-2)-curves in the class

$$H - E_1 - 2F_1 - E_2 - F_2, \quad H - E_2 - 2F_2 - E_3 - F_3, \quad H - E_3 - 2F_3 - E_1 - F_1,$$

respectively, where $i = 1, 2, 3$. Denote this surface by $\widetilde{X}_0$. Finally, contract the three pairs of (-2)-curves to get X_0. Notice that there are four sections of $\omega_{\widetilde{X}_0}^*$ coming from $xyz, y^2z, z^2x, x^2y \in H^0(\mathbb{P}^2, \omega_{\mathbb{P}^2}^*)$. They give rise to a morphism from $\widetilde{X}_0$ to $\mathbb{P}^3$ contracting (-2)-curves, with image $X_0 \subseteq \mathbb{P}^3$ given by $x_0^3 = x_1 x_2 x_3$.

Let $\pi : \widetilde{X}_0 \to X_0$ be the contraction map. This surface has three A_2-singularities, and the 27 lines degenerate to the images of F_1, F_2, F_3, denoted by G_1, G_2, G_3, each of which is of multiplicity 9 (cf. [Tu05, Proposition 4.1(ii)]). Then, by computing intersection numbers, we deduce that

$$\pi^* G_1 = F_1 + \frac{1}{3}E_1 + \frac{2}{3}H_1 + \frac{1}{3}H_2 + \frac{2}{3}E_2,$$

and the other two relations for $\pi^* G_2$ and $\pi^* G_3$. As a result, we obtain that

$$A_{(X_0, c \sum L_j)}(F_i) = A_{(X_0, c \sum L_j)}(E_i) = A_{(X_0, c \sum L_j)}(H_i) = 1 - 9c > 0$$

for $0 < c < \frac{1}{9}$. Thus, the pair $(X_0, c \sum L_j)$ is log canonical for $0 < c < \frac{1}{9}$. The other pairs whose surfaces have A_2-singularities specially degenerate to this case, so we get the result we desire. ∎

Proposition 3.5 *Let $0 < \varepsilon \ll 1$ be a rational number. Then the forgetful map $\varphi : \mathcal{M}_3^K(\varepsilon) \to \mathcal{M}_3^K$ is proper.*

Proof Let $\pi^\circ : (\mathscr{X}^\circ, c\mathscr{D}^\circ) \to T \backslash \{0\}$ be a family of K-semistable pairs, where $(\mathscr{X}^\circ, c\mathscr{D}^\circ)_t$ is isomorphic to a smooth del Pezzo surface with the sum of lines, and let $\mathscr{X} \to T$ be an extension of $\mathscr{X}^\circ \to T \backslash \{0\}$. It suffices to show that there exists a unique extension $\pi : (\mathscr{X}, c\mathscr{D}) \to T$ of π°. The uniqueness is apparent to us: as $\mathscr{X}_0$ is normal, the filling $\mathscr{D}_0$ must be obtained in the following way if it exists. Let $\mathscr{X}^{sm}$ be the open locus of $\mathscr{X}$ such that $\pi : \mathscr{X} \to T$ is smooth on $\mathscr{X}^{sm}$. Taking the scheme-theoretic closure of $\mathscr{D}^\circ$, and restricting it to $\mathscr{X}_0^{sm}$, the $\mathscr{D}_0$ is the extension (by taking closure) of this restricted divisor on $\mathscr{X}_0^{sm}$ to $\mathscr{X}_0$ as a Weil divisor.

Now, we only need to display such a filling. Recall that the central fiber $\mathscr{X}_0$ has at worst A_2-singularities. It was displayed in [Cay69] that the lines on general fibers degenerate to lines on $\mathscr{X}_0$ with multiplicities, denoted by $\sum L_{i,0}$. It follows from Proposition 3.4 that the pair $(\mathscr{X}_0, c \sum L_{i,0})$ is log canonical when $c = \frac{1}{9}$. It follows from interpolation that the pair $(\mathscr{X}_0, c \sum L_{i,0})$ is K-semistable for $0 < c \ll 1$, and this gives a desired filling. ∎

Proposition 3.6 *Let $0 < \varepsilon \ll 1$ be a rational number. Then there is an isomorphism $\overline{M}_{3,\varepsilon}^K \simeq \overline{M}_3^K$.*

Proof We claim that the forgetful map $\varphi : \mathcal{M}_{3,\varepsilon}^K \to \mathcal{M}_3^K$ is finite. Since φ is representable, by Proposition 3.5, it suffices to show that it is quasi-finite. In the proof of Proposition 3.5, we in fact show that for each K-semistable pair (X, cD) in the stack $\mathcal{M}_{3,\varepsilon}^K$, the divisor D is the sum of the lines on X counted with multiplicities. For every cubic surface X with at worst A_2-singularities, there are only finitely many lines on X; hence, φ is quasi-finite. The finite forgetful map φ descends to a finite morphism between good moduli spaces $\psi : \overline{M}_{3,\varepsilon}^K \to \overline{M}_3^K$. As the K-moduli $\overline{M}_3^K$ is normal, and the morphism ψ is birational and finite, it follows from Zariski's Main Theorem that ψ is an isomorphism. ∎

Remark 3.7 In fact, the forgetful map φ between stacks is an isomorphism: for the universal family $\mathscr{X}$ over $\mathcal{M}_3^K$, as we can still define lines with multiplicities on mildly singular cubic surfaces, there is a divisor $\mathscr{D}$ on $\mathscr{X}$ such that each fiber $(\mathscr{X}, \mathscr{D})_t$ is a cubic with sum of lines. This gives the inverse morphism of ϕ.

Corollary 3.8 *Let X be a K-semistable cubic surface, and let L_i's be the 27 lines on it. Then the pair $(X, c \sum L_i)$ is K-semistable for $0 < c < \frac{1}{9}$. In other words, there are no walls for $\overline{M}_{3,c}^K$ when c varies in $(0, \frac{1}{9})$. We have a natural isomorphism*

$$\overline{M}_{3,c}^K \xrightarrow{\sim} \overline{M}_3^K$$

induced by the forgetful map.

Proof This follows immediately from Propositions 3.4 and 3.6 that there are no walls for the K-moduli spaces $\overline{M}_{3,c}^K$, and they are all isomorphic to the K-moduli of cubic del Pezzo surfaces, the isomorphism being induced by the forgetful map φ. ∎

3.2 The quartic case

The existence of Kähler–Einstein metric on quartic surfaces was studied in [MM93, OSS16].

Theorem 3.9 (cf. [OSS16, Theorems 4.1 and 4.2]) *A K-semistable quartic del Pezzo surface has at worst A_1-singularities.*

By [HW81, Theorem 3.4], we know that every ADE del Pezzo surface is the contraction of the (-2)-curves of a blowup of $\mathbb{P}^2$ at points in *almost general position* (cf. [HW81, Definition 3.2]), that is, in the position such that no $(-k)$-curves will be created under the blowup for any $k \geq 3$. Also, it follows from [Fuj90, Section 2] that an ADE del Pezzo surface of degree 4 can always be anti-canonically embedded into $\mathbb{P}^4$ as an complete intersection of two quadric 3-folds. Therefore, similar as in the cubic case, we have a canonical choice of the degeneration of the 16 lines on those ADE del Pezzo quartic surfaces.

Proposition 3.10 *Let X be a K-semistable quartic del Pezzo surface, and let L_i's be the 16 lines on it. Then the pair $(X, c \sum L_i)$ is K-semistable for $0 < c < \frac{1}{4}$. Moreover, there is a natural isomorphism $\overline{M}_{4,c}^K \simeq \overline{M}_4^K$ of the K-moduli spaces, induced by the forgetful map $\mathcal{M}_{4,\varepsilon}^K \to \mathcal{M}_4^K$.*

Proof As in the proof of Corollary 3.8, we have a family $(\mathcal{X}, \mathcal{D})$ over $\mathcal{M}_4^K$ such that a general fiber is a smooth quartic with the 16 lines on it. By interpolation of K-stability (cf. Theorem 2.4), if we prove that these pairs are log canonical for $0 < c \leq \frac{1}{4}$, then $(\mathcal{X}, c\mathcal{D})$ has K-semistable fibers for any $0 \leq c < 1/4$. In particular, the same argument of the second statement of Corollary 3.8 shows that $\overline{M}_{4,c}^K \simeq \overline{M}_4^K$ for any $c \in (0, \frac{1}{4})$.

Now, we prove that each fiber in the family $(\mathcal{X}, c\mathcal{D})$ is log canonical for $0 \leq c \leq 1/4$. Let $(X, cD = c \sum L_i)$ be an arbitrary fiber.

(i) If X is smooth, then the 16 lines are distinct and $\sum L_i$ is normal crossing. Thus, the pair is log canonical automatically.

(ii) If X has one A_1-singularity p, then we claim that there exist exactly four lines of multiplicity 2 and eight lines of multiplicity 1, and that these double lines pass through p, while the remaining eight lines avoid it.

First, notice that a quintic del Pezzo surface X with exactly one A_1-singularity is obtained by contracting the (-2)-curve on the blowup of $\mathbb{P}^2$ along three distinct points p_1, p_2, p_3 and a tangent vector supported at p_4. In fact, let $p \in X$ be the singularity, and let $\widetilde{X} \to X$ be the blowup of X at p with an exceptional divisor, which is a (-2)-curve. By [HW81, Theorem 3.4], the surface $\widetilde{X}$ is obtained from blowing up $\mathbb{P}^2$; thus, there must be a (-1)-curve C intersecting E. Contracting first C, then the proper transform of E, one get a smooth del Pezzo surface of degree 6, which is a blowup of $\mathbb{P}^2$ at three distinct points.

Consider a one-parameter family of projective planes $\mathbb{P}^2 \times \mathbb{A}^1 \to \mathbb{A}^1$, and let $l_1, \ldots, l_5$ be five sections of $\mathbb{A}^1$ such that over $0 \neq t \in \mathbb{A}^1$, $l_1, \ldots, l_5$ does not intersect, while over 0, only l_4 and l_5 intersect transversely. Blowing up $\mathbb{P}^2 \times \mathbb{A}^1$ along $l_1 \cup \cdots \cup l_5$ with reduced scheme structure, one gets a degeneration $\mathfrak{X} \to \mathbb{A}^1$ of smooth quartic del Pezzo surfaces to a singular one with exactly one A_1-singularity. Denote $p_i(t)$ the intersection of l_i with the fiber $\mathbb{P}^2_t$ over $t \in \mathbb{A}^1$. Then, under the family $\mathfrak{X} \to \mathbb{A}^1$, the lines $L(p_i(t), p_4(t))$ and $L(p_i(t), p_5(t))$ (for i=1,2,3) on general fibers $\mathfrak{X}_t$ degenerate to the same line $L(p_i(0), p_4(0)) = L(p_i(0), p_5(0))$, and the exceptional divisors over $p_4(t)$ and $p_5(t)$ also degenerate to the same line, which is the exceptional divisor over $p_4(0) = p_5(0)$. The other eight lines over general fibers degenerate to distinct lines on $\mathfrak{X}_0$. Finally, observe that $-K_{\mathfrak{X}/\mathbb{A}^1}$ gives rise to an embedding of $\mathfrak{X}$ into $\mathbb{P}^4 \times \mathbb{A}^1$ over $\mathbb{A}^1$. Thus, this degeneration indeed occurs in the Hilbert scheme.

Let E be the exceptional divisor of the blowup $\pi : \widetilde{X} \to X$. Then $E + \pi^* \sum L_i$ has simple normal crossing support. Moreover, we have that

$$A_{(X, c \sum L_i)}(E) = 1 - 4c > 0$$

for any $0 < c < \frac{1}{4}$. As the multiplicity of the proper transform of L_i is at most two, then the pair is log canonical.

(iii) If X has two A_1-singularities p and q, then there exist one line of multiplicity 4, four lines of multiplicity 2, and four lines of multiplicity 1. This follows from the same argument as in (i), and in fact the line with multiplicity 4 passes through both p and q, the line with multiplicity 2 passes through either p or q, and the line with multiplicity 1 avoids both p and q. For each singularity, there are exactly eight lines (counted with multiplicities) passing through it. Let E be the exceptional divisor of the blowup at p. Then we have that

$$A_{(X, c \sum L_i)}(E) = 1 - 4c > 0$$

for any $0 < c < \frac{1}{4}$. As the multiplicity of the proper transform of L_i is at most four, then the pair is log canonical.

The proofs of cases (iii) and (iv) are completely the same; we will omit part of the details.

(iv) Suppose that X has three A_1-singularities. Then X is obtained as follows: blow up $\mathbb{P}^2$ at a general tangent vector and at three curvilinear points on a general

line, where two of them collide, then take the ample model. There exist two lines
of multiplicity 4, and four lines of multiplicity 2. For each singularity, there are
exactly eight lines (counted with multiplicities) passing through it. For the same
reason as in (ii), the pair is log canonical.

(v) If X has four A_1-singularities, then it is obtained as follows: blow up $\mathbb{P}^2$ at a general
point P and at two tangent vectors whose supporting lines pass through P, then
take the ample model. There are four lines of multiplicity 4. For each singularity,
there are exactly eight lines (counted with multiplicities) passing through it. For
the same reason as in (ii), the pair is log canonical. ∎

In the proof of Proposition 3.10, we can deduce the following result, which is proved
in [Tu05] for cubic surface case. The explicit equations of the lines on any A_1 cubic del
Pezzo surface can be carried out from the normal form as a cubic hypersurface in $\mathbb{P}^3$.
When reducing from the smooth surface to a singular one, the 27 lines on a smooth
surface reduce to the lines on the corresponding singular surface. The *multiplicity* (cf.
[Cay69]) of a line l of a singular surface is nothing but the number of lines which
reduce to l.

Corollary 3.11   *Let $X \subseteq \mathbb{P}^4$ be a quartic del Pezzo surface with at worst A_1-singularities,
and let l be a line on it. Then:*

(1) *If l does not contain any singular point, then l is of multiplicity 1.*

(2) *If l contains exactly one singularity, then l is of multiplicity 2.*

(3) *If l contains two singularities, then l is of multiplicity 4.*

Remark 3.12   Using the same argument as in Theorem 3.1, one can prove that the
K-moduli space of quartic del Pezzo surfaces is isomorphic to the GIT-moduli space

$$\mathrm{Gr}\big(2, H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2))\big)^{ss} /\!\!/ \mathrm{PGL}(5).$$

See [OSS16, SS17] for details. As a consequence, each K-moduli space $\overline{M}_{4,c}^{K}$ is isomor-
phic to this GIT-moduli space.

4 Degree 2 case

In this section, we prove that there are no walls for the K-moduli spaces $\overline{M}_{2,c}^{K}$ when
we vary the coefficient c from 0 to $\frac{1}{28}$.

Recall that a smooth del Pezzo surface of degree 2 is a double cover of $\mathbb{P}^2$ branched
along a quartic curve, and the 56 lines are sent pairwise to the 28 bitangent lines of
the quartic. Thus, the K-stability of a degree 2 del Pezzo surface X is equivalent to the
K-stability of a del Pezzo pair $(\mathbb{P}^2, \frac{1}{2}C_4)$ (cf. [ADL19, Remark 6.2]), where C_4 is the
quartic plane curve along which the double cover $X \to \mathbb{P}^2$ is branched.

In [OSS16], the authors give a description of the K-moduli space of del Pezzo sur-
faces of degree 2. It is diffeomorphic to the blowup of $\mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(4))^{ss} /\!\!/ \mathrm{PGL}(3)$ at
the point parameterizing the double conic. Moreover, each point $[s]$ on the exceptional
divisor E represents a surface which is a double cover of $\mathbb{P}(1,1,4)$ branched along
a hyperelliptic curve $z^2 = f_8(x, y)$, where f_8 is a GIT-polystable octic binary form.
In [ADL19], the authors study the wall crossing of the K-moduli of pairs $(\mathbb{P}^2, cC_4)$
when c varies from 0 to $\frac{3}{4}$. They proved that there is a unique wall $c = \frac{3}{8}$. As a result,

the K-moduli of degree 2 del Pezzo surfaces is isomorphic to a weighted blowup of the GIT moduli space $\mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(4))^{ss} /\!\!/ \mathrm{PGL}(3)$ at the point parameterizing the double conic. For a rigorous proof, we refer the reader to [ADL21, Proposition 6.12], where the authors prove that the K-moduli of the quartic double solids are isomorphic to the K-moduli space of K-polystable pairs $(X, \frac{1}{2}D)$ which admit a $\mathbb{Q}$-Gorenstein smoothing to $(\mathbb{P}^3, \frac{1}{2}S)$ with $S \in |-K_{\mathbb{P}^3}|$ a smooth quartic K3 surface.

Observe that the degeneration of $\mathbb{P}^2$ to $\mathbb{P}(1,1,4)$ can occur in $\mathbb{P}(1,1,1,2)$ (cf. [ADL19, Theorem 5.14]); thus, there is a canonical choice of the degeneration of the curves and the bitangent lines, and the multiplicity is well defined. Therefore, we can apply the same approach as in the degree $d = 3, 4$ cases, to show that there are no walls. By interpolation of K-semistability, it suffices to check that the $(X, \frac{1}{28}\sum E_i)$ is log canonical, where X is a K-polystable degree quadric del Pezzo surface and E_i's are the lines on it. It further reduces to checking that $(\mathbb{P}^2, \frac{1}{2}C + \frac{1}{28}\sum L_i)$ or $(\mathbb{P}(1,1,4), \frac{1}{2}C + \frac{1}{28}\sum L_i)$ is log canonical (cf. [KM98, Proposition 5.20]), where C is the branched curve and L_i's are the bitangent lines of it.

The classification of the semistable plane quartics is well known to the experts (cf. [MFK94]). For the reader's convenience, we state the result here.

Lemma 4.1 (cf. [HL10, Theorem 2]) *Let $G = \mathrm{PGL}(3)$ act on the space of plane quartic. A plane quartic curve $C_4 \subseteq \mathbb{P}^2$ is:*

(i) *stable if and only if it has at worst A_1 or A_2 singularities;*

(ii) *strictly semistable if and only it is a double conic or has a tacnode. Moreover, C_4 belongs to a minimal orbit if and only if it is either a double conic or the union of two tangent conics, where at least one is smooth.*

Remark 4.2 In the case (ii), if both of the (distinct) conics are smooth, then we call it *cateye*; if there is a singular one, we call it *ox*. These are the only two types of polystable quartics with infinite stabilizers.

4.1 Singularities of plane curves with infinite stabilizers

Let us first compute the log canonical property for the special cases where the quartics have infinite stabilizers.

Lemma 4.3 *The pair $(\mathbb{P}^2, \frac{1}{2}(C_2 + C_2') + c\sum L_i)$ is log canonical for $0 < c < \frac{1}{28}$, where C_2 and C_2' are two smooth conics tangential at two points p and q.*

Proof First, observe that the $(\mathbb{P}^2, C_2 + C_2')$ is the degeneration (in the Hilbert scheme) of pairs $(\mathbb{P}^2, C_2(t) + C_2'(t))_{t \in T}$ where the boundary divisors consist of two smooth conics with a tacnode and two nodes. It follows from [CS03, Section 3.4 case 2] that the arrangement of the 28 bitangent lines of $C_2 + C_2'$ is $\sum L_i = 6L_p + 6L_q + 16L_{pq}$, where L_p and L_q are tangent lines of the conics at p and q, respectively, and L_{pq} is the line connecting p and q. It follows immediately that the coefficients of $\frac{1}{2}(C_2 + C_2') + c\sum L_i$ are all smaller than 1.

Take the minimal log resolution of $(\mathbb{P}^2, \frac{1}{2}(C_2 + C_2') + c\sum L_i)$. By symmetry, we only need to look at the log resolution at q. Let E and F be the exceptional divisors over q with self-intersection -2 and -1, respectively. Then one has $A(E) = 1 - 22c$ and $A(F) = 1 - 28c$, which are positive when $0 < c < \frac{1}{28}$. ∎

The ox case is similar.

Lemma 4.4   *The pair* $(\mathbb{P}^2, \frac{1}{2}(C_2 + M_1 + M_1') + c \sum L_i)$ *is log canonical for* $0 < c < \frac{1}{28}$, *where C_2 is a smooth conic, and M_1, M_1' are two distinct lines tangential to C_2 at two points p and q, respectively.*

Proof   First, observe that the $(\mathbb{P}^2, C_2 + M_1 + M_1')$ is the degeneration (in the Hilbert scheme) of pairs $(\mathbb{P}^2, C_2(t) + M_1(t) + M_1'(t))_{t \in T}$ where the boundary divisors consist of two distinct lines $M_1(t), M_1'(t)$ and a smooth conic $C_2(t)$ tangent to $M_1(t)$ at a point $p(t)$ and meeting $M_1'(t)$ transversely. It follows from [CS03, Section 3.4 case 10] that the arrangement of the 28 bitangent lines of $C_2 + M_1 + M_1'$ is $\sum L_i = 6M_1 + 6M_1' + 16L_{pq}$, where L_{pq} is the line connecting p and q. It follows immediately that the coefficients of $\frac{1}{2}(C_2 + C_2') + c \sum L_i$ are all smaller than 1.

Taking the minimal log resolution of $(\mathbb{P}^2, \frac{1}{2}(C_2 + M_1 + M_1') + c \sum L_i)$. By symmetry, we only need to look at the log resolution at q. Let E and F be the exceptional divisors over q with self-intersection -2 and -1, respectively. Then one has $A(E) = 1 - 22c$ and $A(F) = 1 - 28c$, which are positive when $0 < c < \frac{1}{28}$.   ∎

4.2   Singularities of reducible plane curves with finite stabilizers

Now, we consider the reducible quartics with finite stabilizers.

Proposition 4.5   *Let C_4 be a reducible quartic curve in $\mathbb{P}^2$ with finite stabilizers under the $\mathrm{PGL}(3)$-action, and let L_i's be the degeneration on C_4 of the 28 bitangent lines of smooth quartics. Then $(\mathbb{P}^2, \frac{1}{2}C_4 + c \sum L_i)$ is log canonical for $0 \le c \le \frac{1}{28}$.*

Proof   The classification of C_4 is listed in [CS03, Section 3.4]. For each class, we can run the same argument. For simplicity, we only prove the statement for one class where $C_4 = M_1 + M_2 + M_3 + M_4$ is the union of four general lines (forming three pairs of nodes). In this case, we have that $\sum L_i = 4(M_1 + \cdots + M_7)$, where M_5, M_6, M_7 are the lines joining the three pairs of nodes (cf. [CS03, Section 3.4 case 11]). Then the blowup of the pair $(\mathbb{P}^2, \frac{1}{2}C_4 + c \sum L_i)$ at the six nodes (with exceptional divisors $E_1, \ldots, E_6$) is log smooth. As the boundary divisors $\frac{1}{2}C_4 + c \sum L_i$ have coefficients less than 1 and $A_{(\mathbb{P}^2, \frac{1}{2}C_4 + c \sum L_i)}(E_i) = 1 - 12c > 0$, then the pair we consider is log canonical.   ∎

4.3   Singularities of irreducible plane curves with finite stabilizers

Now, we deal with the general case: an irreducible plane quartic C_4 with at worst A_1 or A_2 singularities. We first assume that C_4 is irreducible. Then [CS03, Table 3.2 and Lemma 3.3.1] describe how these bitangent lines degenerate. We claim that all these curves give log canonical pairs $(\mathbb{P}^2, \frac{1}{2}C_4 + c \sum L_i)$.

Lemma 4.6   *The pair $(\mathbb{P}^2, \frac{1}{2}C_4 + c \sum L_i)$ is log canonical for $0 < c < \frac{1}{28}$, where C_4 has three nodes.*

Proof   We know from [CS03, Lemma 3.3.1] that the 28 lines degenerate to 4 bitangent lines of multiplicity 1 not passing through the nodes, 6 tangent lines of multiplicity 2 passing through exactly one node, and 3 lines of multiplicity 4 containing two nodes. Thus, for each node p, there are 12 lines passing through it, and at most 4

lines tangential to it. One can blow up at p twice to get a log resolution. We have $A(E) = 1 - 16c > 0$ and $A(F) \geq \frac{3}{2} - 20c > 0$ for $0 < c < 1/28$, where E and F are the two exceptional divisors over p with self-intersection -2 and -1, respectively. The singularities at other points are milder; thus, $(\mathbb{P}^2, \frac{1}{2}C_4 + c \sum L_i)$ is log canonical as desired. ∎

Lemma 4.7 *The pair* $(\mathbb{P}^2, \frac{1}{2}C_4 + c \sum L_i)$ *is log canonical for* $0 < c < \frac{1}{28}$*, where* C_4 *has three cusps.*

Proof We know from [CS03, Lemma 3.3.1] that the 28 lines degenerate to 4 bitangent lines of multiplicity 1 not passing through the cusps, and 3 lines of multiplicity 9 containing two cusps. Thus, for each cusp p, there are 18 lines passing through it, and none of them are tangential to it. One can blow up at p three times to get a log resolution. Let E, F, G be the exceptional divisors with self-intersection $-3, -2, -1$, respectively. We have $A(E) = 1 - 18c > 0$, $A(F) = \frac{3}{2} - 18c > 0$, and $A(G) = 2 - 36c > 0$ for $0 < c < 1/28$. The singularities at other points are milder; thus, $(\mathbb{P}^2, \frac{1}{2}C_4 + c \sum L_i)$ is log canonical as desired. ∎

Lemma 4.8 *The pair* $(\mathbb{P}^2, \frac{1}{2}C_4 + c \sum L_i)$ *is log canonical for* $0 < c < \frac{1}{28}$*, where* C_4 *has one node and two cusps.*

Proof By [CS03, Lemma 3.3.1], we see that the 28 lines degenerate to 1 bitangent line of multiplicity 1 not passing through the singularities, 2 tangent lines of multiplicity 3 passing through exactly one cusp, 1 line of multiplicity 9 containing two cusps, and 2 lines of multiplicity 6 passing through one node and one cusp.

Thus, for the node p, there are 12 lines passing through it, and none of them are tangential to it. One can blow up at p to get a log resolution and the log discrepancy with respect to the exceptional divisor is $1 - 12c > 0$.

For each cusp q, there are 18 lines passing through it, and at most 3 lines tangential to it. Resolving q by blowing up three times, one gets three exceptional divisors E, F, G with self-intersection $-3, -2, -1$, respectively. We have $A(E) = 1 - 18c > 0$, $A(F) \geq \frac{3}{2} - 21c > 0$, and $A(G) \geq 2 - 39c > 0$ for $0 < c < 1/28$. The singularities at other points are milder; thus, $(\mathbb{P}^2, \frac{1}{2}C_4 + c \sum L_i)$ is log canonical as desired. ∎

Lemma 4.9 *The pair* $(\mathbb{P}^2, \frac{1}{2}C_4 + c \sum L_i)$ *is log canonical for* $0 < c < \frac{1}{28}$*, where* C_4 *has one cusp and two nodes.*

Proof By [CS03, Lemma 3.3.1], we see that the 28 lines degenerate to 2 bitangent lines of multiplicity 1 not passing through the singularities, 2 tangent lines of multiplicity 3 passing through exactly the cusp, 2 tangent lines of multiplicity 2 passing through exactly one node, 1 line of multiplicity 4 containing two nodes, and 2 lines of multiplicity 6 passing through one node and one cusp.

Thus, for the cusp p, there are 18 lines passing through it, and at most 6 of them are tangential to it. Resolving p by blowing up three times, one gets three exceptional divisors E, F, G with self-intersection $-3, -2, -1$, respectively. We have $A(E) = 1 - 18c > 0$, $A(F) \geq \frac{3}{2} - 24c > 0$, and $A(G) \geq 2 - 42c > 0$ for $0 < c < 1/28$.

For each node q, there are 12 lines passing through it, and at most 2 lines tangential to it. One can blow up at p twice to get a log resolution. We have $A(E') = 1 - 12c > 0$ and $A(F') \geq \frac{3}{2} - 14c > 0$ for $0 < c < 1/28$, where E' and F' are the two exceptional

divisors over p with self-intersection -2 and -1, respectively. The singularities at other points are milder; thus, $(\mathbb{P}^2, \frac{1}{2}C_4 + c\sum L_i)$ is log canonical as desired. ■

The other pairs with C_4 irreducible have milder singularity than the four types in the above four lemmas. Applying the same argument to all these pairs, we conclude the following.

Proposition 4.10 *Let C_4 be a GIT-semistable plane quartic which is not a double conic. Then the pair $(\mathbb{P}^2, \frac{1}{2}C_4 + c\sum L_i)$ is log canonical for $0 < c < \frac{1}{28}$. In particular, these pairs are K-semistable for $0 < c < 1/28$.*

4.4 Singularities of curves in $\mathbb{P}(1,1,4)$

We first point out an explicit degeneration of $\mathbb{P}^2$ to $\mathbb{P}(1,1,4)$ in $\mathbb{P}(1,1,1,2)$. Let $(x_0 : x_1 : x_2 : x_3)$ be the homogeneous coordinate of $\mathbb{P}(1,1,1,2)$ and then consider the hypersurface

$$\mathfrak{X} := \left\{ (t, (x_0 : x_1 : x_2 : x_3)) : t \cdot x_3 = x_0 x_2 - x_1^2 \right\} \subseteq \mathbb{A}^1 \times \mathbb{P}(1,1,1,2).$$

The natural projection $\pi : \mathfrak{X} \to \mathbb{A}^1$ gives a special degeneration of $\mathbb{P}^2$ to $\mathbb{P}(1,1,4)$.

Recall that an octic binary form $f_8(x,y)$ is GIT-semistable if and only if each of its zeros has multiplicity at most 4 (cf. [MFK94] Section 4.1). Also, notice that if the point $(x : y : z)$ is a singularity of the curve $C = \{z^2 = f_8(x,y)\}$, then $z = 0$ and $(x : y)$ is a multiple root of $f_8(x,y)$.

Let p_i be the intersection points of C and $\{z = 0\}$, where $i = 1, \ldots, 8$. Another important observation is that a bitangent line of a plane quartic degenerates under π to two rulings of $\mathbb{P}(1,1,4)$, each of which passes through some p_i, and there are $\binom{8}{2} = 28$ pairs in total. Thus, we only need to focus on the vertex p of $\mathbb{P}(1,1,4)$ and the singularities of C.

Proposition 4.11 *The pair $(\mathbb{P}(1,1,4), \frac{1}{2}C + c\sum L_i)$ is log canonical for $0 < c < \frac{1}{28}$.*

Proof Notice that C does not pass through the vertex p. Let E be the exceptional divisor of $\mathrm{Bl}_p \mathbb{P}(1,1,4) \to \mathbb{P}(1,1,4)$. Then $A(E) = \frac{1}{2} - 14c > 0$ for $0 < c < \frac{1}{28}$.

Suppose that $p_i = (x : y : 0)$ is a singular point of C. We may assume that $x = 0$ and $y = 1$, and locally the equation of C is $z^2 = x^t$, where $t \in \{2,3,4\}$. When $t = 4$, p_i is a tacnode, and there are 28 lines passing through it. Let F and G be the exceptional divisors of the minimal resolution of p_i with self-intersection -2 and -1, respectively. Then $A(F) = 1 - 28c > 0$ and $A(G) = 1 - 28c > 0$ for $0 < c < \frac{1}{28}$. This is also true for the cases $t = 2$ and $t = 3$. ■

To sum up, we conclude the following by the same argument as in Corollary 3.8.

Theorem 4.12 *Let $\overline{M}_2^K$ be the K-moduli space of degree 2 del Pezzo surfaces. Then there are no walls for the K-moduli stacks $\mathcal{M}_{2,c}^K$, and there is an isomorphism*

$$\overline{M}_{2,c}^K \simeq \overline{M}_2^K$$

for any $0 < c < \frac{1}{28}$.

5 Discussion on degree 1 case

In this section, we display some examples to show that there exist some walls for $M_{1,c}^K$ when c varies from 0 to $\frac{1}{240}$. The following result describes the K-polystable del Pezzo surfaces with at worst ADE singularities. For the analytic description of the K-moduli space M_1^K, see [OSS16, Section 5].

Proposition 5.1 (cf. [OSS16, Section 6.1.2]) *A nodal del Pezzo surface X of degree 1 is K-polystable if and only if it has either only A_k-singularities with $k \leq 7$, or exactly two D_4-singularities, and X is not isomorphic to one of the surfaces parameterized by $(a_1 : a_2) \in \mathbb{P}^1$, which are hypersurfaces in $\mathbb{P}(1, 1, 2, 3)$ defined by the equations*

$$w^2 = a_1 z^3 + z^2 x^2 + zy^4 + a_2 z^2 y^2.$$

5.1 Degree 1 del Pezzo with an A_7-singularity

We aim to prove the following statement.

Proposition 5.2 *Let Y be a del Pezzo surface of degree 1 with an A_7-singularity, and let L_i's be the degeneration of the 240 lines. Then the pair $\left(Y, c\sum L_i\right)$ is K-unstable for $c \geq \frac{1}{288}$.*

Proof Let Y be a degree 1 del Pezzo surface with an A_7-singularity. Then Y is obtained by blowing up $\mathbb{P}^2$ at eight curvilinear points on a smooth cubic curve supported at a single point and taking the ample model. Let $E_1, \ldots, E_8$ be the eight exceptional divisor on the blowup $\widetilde{Y}$ in the order of blowup, and let H be the class of the pullback of $\mathcal{O}_{\mathbb{P}^2}(1)$. Then the proper transform to $\widetilde{Y}$ of the 240 lines on Y are of the following types:

(a) E_8, intersecting E_7, of multiplicity 8;
(b) $H - E_1 - 2E_2 - 2E_3 - \cdots - 2E_8$, intersecting E_2, of multiplicity 28;
(c) $2H - E_1 - 2E_2 - 3E_3 - 4E_4 - 5\sum_{i \geq 5} E_i$, intersecting E_5, of multiplicity 56;
(d) $3H - 2E_1 - 3E_2 - \cdots - 8E_7 - 8E_8$, intersecting E_1 and E_7, of multiplicity 56;
(e) $4H - 2E_1 - 4E_2 - 6E_3 - \sum_{i \geq 4}(i + 3)E_i$, intersecting E_3 and E_8, of multiplicity 56;
(f) $5H - \sum_{i=1}^6 2iE_i - 13E_7 - 14E_8$, intersecting E_6 and E_8, of multiplicity 28;
(g) $6H - 3E_1 - \sum_{i \geq 2}(2i + 1)E_i$, intersecting E_1 and intersecting E_8 twice, of multiplicity 8;

Denote by L_i and l_i the line and its image on Y of one of the above types, and a_i be the multiplicity of the line L_i, where $i = 0, 1, \ldots, 6$ is the coefficient of H in the class. Let $\pi : \widetilde{Y} \to Y$ be the blow-down morphism. Then we have

$$\pi^* l_0 = E_8 + \frac{7}{8}E_7 + \frac{6}{8}E_6 + \frac{5}{8}E_5 + \frac{4}{8}E_4 + \frac{3}{8}E_3 + \frac{2}{8}E_2 + \frac{1}{8}E_1,$$

$$\pi^* l_1 = L_1 + \frac{1}{4}E_7 + \frac{2}{4}E_6 + \frac{3}{4}E_5 + \frac{4}{4}E_4 + \frac{5}{4}E_3 + \frac{6}{4}E_2 + \frac{3}{4}E_1,$$

$$\pi^* l_2 = L_2 + \frac{5}{8}E_7 + \frac{10}{8}E_6 + \frac{15}{8}E_5 + \frac{12}{8}E_4 + \frac{9}{8}E_3 + \frac{6}{8}E_2 + \frac{3}{8}E_1,$$

$$\pi^* l_3 = L_3 + E_7 + E_6 + E_5 + E_4 + E_3 + E_2 + E_1,$$

$$\pi^* l_4 = L_4 + \frac{3}{8}E_7 + \frac{6}{8}E_6 + \frac{9}{8}E_5 + \frac{12}{8}E_4 + \frac{15}{8}E_3 + \frac{10}{8}E_2 + \frac{5}{8}E_1,$$

$$\pi^* l_5 = L_5 + \frac{3}{4} E_7 + \frac{6}{4} E_6 + \frac{5}{4} E_5 + \frac{4}{4} E_4 + \frac{3}{4} E_3 + \frac{2}{4} E_2 + \frac{1}{4} E_1,$$

$$\pi^* l_6 = L_6 + \frac{1}{8} E_7 + \frac{2}{8} E_6 + \frac{3}{8} E_5 + \frac{4}{8} E_4 + \frac{5}{8} E_3 + \frac{6}{8} E_2 + \frac{7}{8} E_1.$$

In particular, one gets that the log discrepancy

$$A_{(Y, c \sum a_i L_i)}(E_3) = A_{(Y, c \sum a_i L_i)}(E_4) = A_{(Y, c \sum a_i L_i)}(E_5) = 1 - 288c < 0$$

when $\frac{1}{288} < c < \frac{1}{240}$. Thus, there is a wall $0 < c < \frac{1}{288}$ given by the degeneration of such pairs. ∎

5.2 Degree 1 del Pezzo with two D_4-singularities

Proposition 5.3 *Let Z be a del Pezzo surface of degree 1 with exactly two D_4-singularities, and let L_i's be the degeneration of the 240 lines. Then the pair $(Z, c \sum L_i)$ is K-stable for any $0 < c < \frac{1}{240}$.*

Proof Let Z be a degree 1 del Pezzo surface with exactly D_4-singularities. Such surfaces are weighted hypersurfaces in $\mathbb{P}(1, 1, 2, 3)$, and there is a one-parameter family parameterizing them (cf. [OSS16, Example 5.19]). For us, a description by blowing up projective plane is more useful, since we need to figure out the degeneration of the 240 lines.

We see that Z can be obtained by blowing up $\mathbb{P}^2$ in the following way. Fix four distinct points p_1, p_2, p_3, p_4 on a line L in $\mathbb{P}^2$, and another point $p \in \mathbb{P}^2 \backslash L$. Blow up $\mathbb{P}^2$ at p_1 along the tangent direction to p with exceptional divisors E_1, F_1; at p_2 along the tangent direction to p with exceptional divisors E_2, F_2; and at the point p_3 with exceptional divisors E. Finally, blow up the length 3 zero-dimensional curvilinear subscheme supported on the line $\overline{pp_4}$ and concentrated at p with exceptional divisors E_3, F_3, G_3 (see Figure 1 for the blowup procedure and Figure 2 for the configuration of the (-1)-curves and (-2)-curves on $\widetilde{Z}$). Denote this surface by $\widetilde{Z}$, and the ample model by Z.

Let H be the class of the pullback of $\mathcal{O}_{\mathbb{P}^2}(1)$. Then the proper transform to $\widetilde{Z}$ of the 240 lines on Z are of the following types:

(a) E, of multiplicity 24;
(b) $H - E - E_3 - F_3 - G_3$, of multiplicity 24;
(c) F_1, of multiplicity 64;
(d) F_2, of multiplicity 64;
(e) G_3, of multiplicity 64.

Notice that the configurations of the curves in Figure 2 are symmetric: the black curves denote the (-2)-curves and the red lines denote the (-1)-curves. Using the same computation as in Section 5.1, one sees that the minimal log discrepancy of the pair $(Z, c \sum a_i L_i)$ is $1 - 240c > 0$, when $0 < c < \frac{1}{240}$. Thus, the surfaces with exactly two D_4-singularities do not contribute any wall. ∎

Remark 5.4 Notice that the four points $p_1, \ldots, p_4$ on $\mathbb{P}^1$ have a cross-ratio. This also explains why we have a one-dimensional family of surfaces with exactly two D_4-singularities.
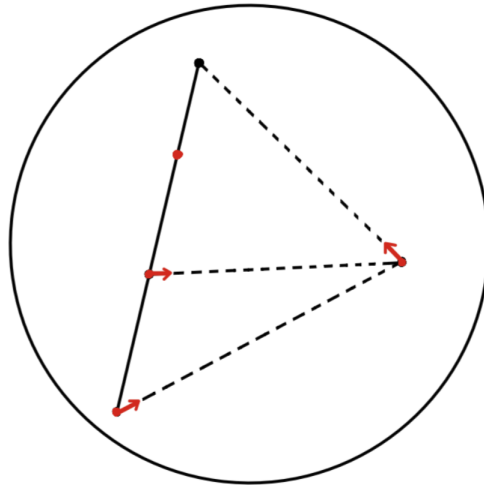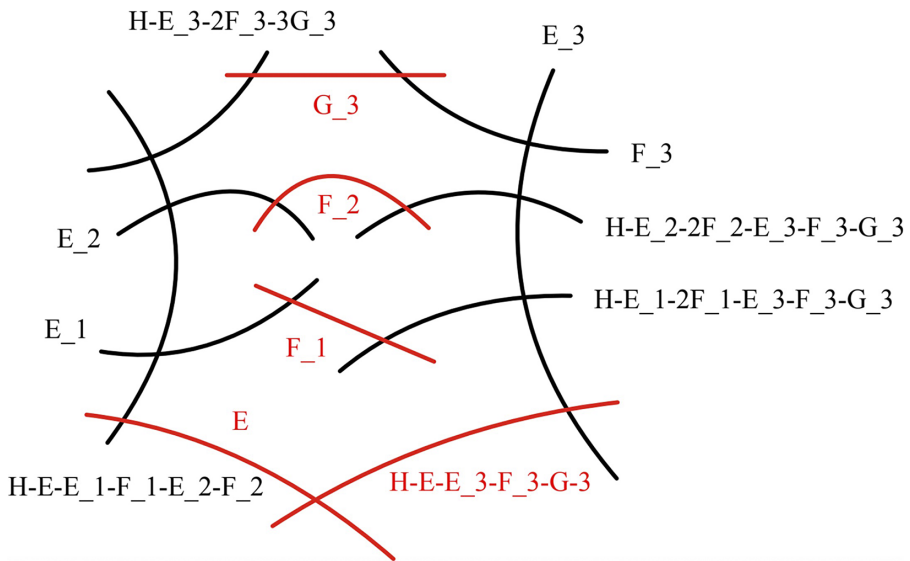
Figure 1: Blowup of $\mathbb{P}^2$ to get $\widetilde{Z}$.

Figure 2: Configuration of the lines on $\widetilde{Z}$.

5.3 Degree 1 del Pezzo with an A_8-singularity and two $\frac{1}{9}(1,2)$-singularities

Among all K-polystable degenerations of smooth degree 1 del Pezzo surface, there is a special one X_∞ with an A_8-singularity and two $\frac{1}{9}(1,2)$-singularities. The surface X_∞ can be viewed as a degree 18 hypersurface in $\mathbb{P}(1,2,9,9)$ given by the equation $z_3 z_4 = z_2^9$, where $(z_1 : z_2 : z_3 : z_4)$ is the coordinates of weights $1, 2, 9, 9$, respectively.

Moreover, projecting to $\mathbb{P}(1, 2, 9)$ by

$$(z_1 : z_2 : z_3 : z_4) \mapsto (z_1 : z_2 : z_3 + z_4)$$

realizes X_∞ as a double cover of $\mathbb{P}(1, 2, 9)$ branched along the curve $C_\infty = \{v^9 = w^2\}$, where $(u : v : w)$ is the coordinate of $\mathbb{P}(1, 2, 9)$. In fact, this double cover map is given by the linear system $|-2K_{X_\infty}|$.

Recall that for a smooth del Pezzo surface X_t of degree 1, the map given by the linear system $|-2K_{X_t}|$ is a double cover to $\mathbb{P}(1, 1, 2) \subseteq \mathbb{P}^3$ branched along a sextic curve C_t. The 240 lines on X_t are sent pairwise to the 120 conics on $\mathbb{P}(1, 1, 2)$ obtained by intersecting the $\mathbb{P}(1, 1, 2)$ with the 120 tritangent planes of C_t in $\mathbb{P}^3$. See [KRSNS18] for details.

The degeneration of $\mathbb{P}(1, 1, 2)$ to $\mathbb{P}(1, 2, 9)$ can be observed in $\mathbb{P}^{15}$: they are embedded into $\mathbb{P}^{15}$ by the complete linear series $|\mathcal{O}(6)|$ and $|\mathcal{O}(18)|$, respectively. In particular, the 120 conics degenerate to curves in $\mathbb{P}(1, 2, 9)$ of degree 6. Notice that the pair $(\mathbb{P}(1, 2, 9), \frac{1}{2}C_\infty)$ is a $\mathbb{T}$-variety of complexity one: there is a $\mathbb{G}_m$-action given by

$$\lambda \cdot (u : v : w) = (u : \lambda^2 v : \lambda^9 w).$$

In particular, each of the 120 sextic curves is $\mathbb{G}_m$-invariant, and hence is defined by one of the following four equations:

$$u^6 = 0, \quad u^4 v = 0, \quad u^2 v^2 = 0, \quad v^3 = 0.$$

However, the multiplicity of them is not clear to us. We have the following partial result.

Proposition 5.5 *Let X_∞ be the degree 1 del Pezzo surface as above, and let L_i's be the 240 lines on it counted with multiplicities. Then the pair $(X_\infty, c \sum L_i)$ is either K-polystable for any $0 < c < \frac{1}{240}$ or K-unstable for any $0 < c < \frac{1}{240}$. Moreover, it is K-polystable if and only if*

$$\mathrm{ord}_{u=0}\left(\sum l_j\right) = 240 \quad and \quad \mathrm{ord}_{u=0}\left(\sum l_j\right) = 240,$$

where l_i's are the 120 sextic curves in $\mathbb{P}(1, 2, 9)$ given by the images of L_i's.

Proof The main tool here we use is equivariant K-stability (cf. [Zhu21]). Assume that we are in the case when $\mathrm{ord}_{u=0}\left(\sum l_j\right) = 240$ and $\mathrm{ord}_{u=0}\left(\sum l_j\right) = 240$. By [LZ22, Theorem 1.2], it suffices to check the K-stability of the pair $(\mathbb{P}(1, 2, 9), \frac{1}{2}C_\infty + c \sum l_i)$, which is a $\mathbb{T}$-pair of complexity one. By [ACC+21, Theorem 1.3.9], we only need to compute the β-invariant of the pair with respect to all $\mathbb{G}_m$-invariant divisors on $\mathbb{P}(1, 2, 9)$ (see also [Zha22, Theorem 2.9] for the statement for pairs). The divisor $\{u = 0\}$ is the unique horizontal divisor on X_∞. We have that

$$A_{(\mathbb{P}(1,2,9), \frac{1}{2}C_\infty + c \sum l_i)}(\{u = 0\}) = 1 - 240c$$

and that

$$S_{(\mathbb{P}(1,2,9), \frac{1}{2}C_\infty + c \sum l_i)}(\{u = 0\}) = \frac{(3 - 720c)}{\mathcal{O}(1)^2} \int_0^1 \mathcal{O}(1 - t)^2 dt = 1 - 240c,$$

and thus $\beta_{(\mathbb{P}(1,2,9),\frac{1}{2}C_\infty+c\sum l_i)}(\{u=0\})=0$. For the vertical divisor $\{v=0\}$, we have that

$$A_{(\mathbb{P}(1,2,9),\frac{1}{2}C_\infty+c\sum l_i)}(\{v=0\}) = 1-240c$$

and that

$$S_{(\mathbb{P}(1,2,9),\frac{1}{2}C_\infty+c\sum l_i)}(\{v=0\}) = \frac{(3-720c)}{\mathcal{O}(1)^2}\int_0^{\frac{1}{2}}\mathcal{O}(1-2t)^2dt = \frac{1}{2}(1-240c).$$

For other vertical divisors $D_t = \{v^9 = tw^2\}$ with $\lambda \in \mathbb{C}^*$, similarly, we have that $\beta_{(\mathbb{P}(1,2,9),\frac{1}{2}C_\infty+c\sum l_i)}(D_t) > 0$. This concludes for the case $\mathrm{ord}_{u=0}\left(\sum l_j\right) = \mathrm{ord}_{u=0}\left(\sum l_j\right) = 240$. For the other case, the β-invariant for the horizontal divisor $\{u=0\}$ is nonzero when $0 < c < \frac{1}{240}$, so the pair is K-unstable. ∎

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Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, 851 South Morgan Street, Chicago, IL 60607, United States
e-mail: jzhao81@uic.edu