TIME REVERSIONS OF MARKOV PROCESSES

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§ 0. Introduction

A time reversion of a Markov process was discussed by Kolmogoroff for Markov chains in 1936 [6] and for a diffusion in 1937 [7]. He described it as a process having an adjoint transition probability. Although his treatment is purely analytical, in his case if the process x_t has an invariant distribution, the reversed process $z_t = x_{-t}$ is the process with the adjoint transition probability. In this discussion, however, it is very restrictive that the initial distribution of the process must be an invariant measure.

On the other hand, the adjoint (or dual) process of a Markov process can be defined with respect to any sub-invariant (or excessive) measure, and this was done by Nelson [15] and Hunt [3]. Ever since the notion of the adjoint processes has an important rôle in the theory of Markov processes (cf. e.g. [3], [10], [12], [14], [15]). The relation between the adjoints and the time reversions of a Markov process was, however, not fairly clear. For example, an adjoint of a temporally homogeneous Markov process is always temporally homogeneous by the definition, but the time reversion $z_t = x_{-t}$ or $z_t = x_{T-t}$ (where T is a positive constant) is, in general, temporally inhomogeneous.

Recently Hunt [2] proved for Markov chains that if the time reversion is performed from the last exit time from a subset, then the reversed process has temporally homogeneous Markov property. Ikeda, Nagasawa and Sato [5] also proved that for Markov processes obtained by killing, the reversed processes from the killing times have temporal homogeneity.

The purpose of this paper is to prove that the reversed processes from appropriate random times (*L-times* cf. § 2) preserve temporally homogeneous Markov property (cf. § 3). Time reversions of approximate Markov processes introduced by Hunt in [2] will be treated in § 4.

The works of this paper grew up from the joint works with Ikeda and

Received December 9, 1963.

Sato [5]. In the course of the study K. Sato took verious discussions with me and details of some proofs were improved by his suggestions, and N. Ikeda and M. Motoo gave me several advices. I wish to express my hearty thanks to them.

§ 1. Notations and definitions

In this section we shall recall some of definitions and notations on Markov processes, which follow mostly Dynkin's book [1].

Let E be a locally compact Housdorff space with a countable base and $\mathscr{B}$ denote the topological Borel field of E. Let $\{\partial\}$ be adjoined to E as an extra point and denote $E^* = E \cup \{\partial\}$.

W is the space of mappings w from $[0, \infty]$ to E^* satisfying the followings; (w_1) There exists the ***killing time*** $\zeta(w)$ of w with values in $[0, \infty]$ such as $w(t) \in E$ for $t < \zeta(w)$ and $w(t) = \partial$ for $t \geqq \zeta(w)$; (w_2) $w(t)$ is right-continuous and with left hand limits in $[0, \zeta(w))$. Let x_t denote the coordinate mapping i.e. $x_t(w) = w(t)$. Shifted path $w_t (t \geqq 0)$ of w is defined by $x_s(w_t) = x_{t+s}(w)$ for any $s \geqq 0$. Let $\mathscr{N}$ be the σ-field of W generated by $\{x_s \in A\}$ ($s \geqq 0$, and $A \in \mathscr{B}$). Put $W_t = \{w : w \in W, \zeta(w) > t\}$ and let $\mathscr{N}_t$ be the σ-field of W_t generated by $\{x_s \in A, \zeta > t\}$ ($s \in [0, t]$ and $A \in \mathscr{B}$).

Let $\{\boldsymbol{P}_a;\ a \in E\}$ be a system of probability measures on $(W, \mathscr{N})$ satisfying; (p_1) For every $t \geqq 0$ and $A \in \mathscr{B}$, $\boldsymbol{P}_a[x_t \in A]$ is $\mathscr{B}$-measurable in $a \in E$; (p_2) $\boldsymbol{P}_a[x_0 = a] = 1$ for each $a \in E$; and (p_3) $\boldsymbol{P}_a[x_{t+s} \in A \mid \mathscr{N}_t] = \boldsymbol{P}_{x_t}[x_s \in A]$, $\boldsymbol{P}_a - a.e.$ on W_t ($t, s \geqq 0$, $a \in E$ and $A \in \mathscr{B}$). A system $X = (x_t, \zeta, \mathscr{N}_t, \boldsymbol{P}_a)$ is said to be a (temporally homogeneous) *Markov process*.

For a measure ν on $(E, \mathscr{B})$, we put

$$\boldsymbol{P}_\nu[B] = \int \nu(da) \boldsymbol{P}_a[B], \qquad (B \in \mathscr{N}). \tag{1.1}$$

Let $\overline{\mathscr{N}} = \bigcap_\nu \mathscr{N}(\boldsymbol{P}_\nu)$ and $\overline{\mathscr{N}}_t = \bigcap_\nu \mathscr{N}_t(\boldsymbol{P}_\nu)$, where $\mathscr{N}(\boldsymbol{P}_\nu)$ (resp. $\mathscr{N}_t(\boldsymbol{P}_\nu)$) is the completion of $\mathscr{N}$ (resp. $\mathscr{N}_t$) by $\boldsymbol{P}_\nu$ (ν, here, varies over all probability measures on $(E, \mathscr{B})$).

A random time $\sigma(w)$ is a *Markov time* if $0 \leqq \sigma(w) \leqq +\infty$ ($w \in W$) and $\{\sigma < t < \zeta\} \in \overline{\mathscr{N}}_t$, ($t \geqq 0$). Given a Markov time σ, we denote $W_\sigma = \{w : \sigma(w) < \zeta(w)\}$ and $\overline{\mathscr{N}}_\sigma$ the σ-field of W_σ consisted of all $B \subset W_\sigma$ such as $B \cap \{\sigma < t < \zeta\} \in \overline{\mathscr{N}}_t$ ($t \geqq 0$).

A Markov process X is said to have the *strong Markov property* if for each Markov time σ, $\boldsymbol{P}_a[x_{\sigma+s}\in A\mid\mathcal{N}_\sigma]=\boldsymbol{P}_{x_\sigma}[x_s\in A]$, $\boldsymbol{P}_a$-*a.e.* on W_σ, $(s\geqq 0$, $a\in E$ and $A\in\mathcal{B})$, and is said to have the *quasi-left continuity*, if for every sequence of Markov times $\{\sigma_n\}$, $\lim_{n\to\infty} x_{\sigma_n}=x_{\lim\sigma_n}$, $\boldsymbol{P}_a$-*a.e.* on $\{\sigma_n\uparrow,\ \lim_{n\to\infty}\sigma_n<\zeta\}$. A Markov process X is called a *standard process* if it has the strong Markov property and quasi-left continuity.

A mapping $b(t, w)$ from $[0, \infty]\times W$ to $[0, \infty]$ is called a (non-negative continuous) *additive functional of* X, if; (a_1) $b(t, w)$ is $\mathcal{N}$-measurable and $\{w : b(t, w)<r,\ \zeta(w)>t\}\in\mathcal{N}_t$ for all $r\geqq 0$, $(t\geqq 0)$; (a_2) $0=b(0, w)\leqq b(t, w)<\infty\ (0\leqq t<\infty)$; (a_3) $b(t+s, w)=b(t, w)+b(s, w_t)$, $(w\in W)$; (a_4) $b(t, w)=b(\zeta, w)$ for $t\geqq\zeta(w)$; and (a_5) $b(t, w)$ is continuous in t, $(w=W)$ (cf. e.g. [17], [11]).

Let $\zeta'(w)$ be an $\mathcal{N}$-measurable function on W with values in $[0, \infty]$, and $z_t(w)$ be defined for $w\in W_0\subset W(W_0\in\mathcal{N})$ and $t\in[0, \zeta'(w))$ (resp. $(0, \zeta'(w))$) with values in E, and put $W'_t=\{\zeta'>t\}\cap W_0$ for $t\geqq 0$ (resp. $t>0$). Let $\mathcal{M}_t$ be the σ-field of W'_t generated by $\{z_s\in A,\ \zeta'>t\}$, $(A\in\mathcal{B},\ s\in[0, t]$ (resp. $(0, t]$)), and $\mathcal{M}$ be a σ-field on W_0 containing all $\mathcal{M}_t$ $(t\geqq 0)$.

Let $\boldsymbol{P}$ be a measure on $(W_0, \mathcal{M})$, which is σ-finite on $(W'_t, \mathcal{M}_t)$ for every $t\geqq 0$ (resp. $t>0$). A system $(z_t, \zeta', \mathcal{M}_t, \boldsymbol{P})$ (for brevity, $(z_t, \boldsymbol{P})$) is said to have (temporally homogeneous) *Markov property with a transition probability* $P(t, a, A)$, if, for every compact set A,

$$
\begin{aligned}
(1.2)\qquad \boldsymbol{P}[z_t\in A\mid\mathcal{M}_s]&=\boldsymbol{P}[z_t\in A\mid z_s]\\
&=P(t-s, z_s, A),\ \boldsymbol{P}\text{-}a.e.\ \text{on } W'_s,\ \ (0\underset{(=)}{<}s<t).^{0)}
\end{aligned}
$$

Further, if z_0 is defined and

$$(1.3)\qquad \boldsymbol{P}[z_0\in A]=\mu(A),$$

then $(z_t, \boldsymbol{P})$ is said *to have the initial measure* μ.

Let $X=(x_t, \zeta, \mathcal{N}_t, \boldsymbol{P}_a)$ be a Markov process and ν be a σ-finite measure on $(E, \mathcal{B})$ satisfying that

$$(1.4)\qquad \boldsymbol{P}_\nu[x_t\in A]<\infty,\ \text{for every } t\geqq 0 \text{ and every compact } A,$$

[0] Because of σ-finiteness of $\boldsymbol{P}$ on W_t, we may define $\boldsymbol{P}[z_t\in A\mid\mathcal{M}_s]$, etc. by the same way as defining conditional probabilities using Radon-Nikodym theorem.

https://doi.org/10.1017/S0027763000011405 Published online by Cambridge University Press

then $(x_t, \zeta, \mathcal{N}_t, \boldsymbol{P}_\nu)$ (for brevity, $(x_t, \boldsymbol{P}_\nu)$) has temporally homogeneous Markov property with a transition probability $P(t, a, A)=\boldsymbol{P}_a[x_t \in A]$.

§ 2. Random times of type L

In this section we shall investigate a new class of random times (L-time), which is a generalization of the last exit time from a subset of E.

DEFINITIN 2.1. *A function $\tau(w)$ from W to $\{-\infty\} \cup [0, \infty]$ is called a random time of type L (briefly, L-time) if it has the properties;*

(L₁) $\tau(w)$ *is $\mathcal{N}$-measurable and* $\tau(w) \leqq \zeta(w)$, *and*

(L₂) $\{s<\tau(w)-t<\infty\}=\{s<\tau(w_t)<\infty\}, \quad (t, s \geqq 0).$

Also, τ is said to be an almost L-time if it satisfies (L₁) *and, instead of* (L₂),

(L′₂) $\{s<\tau(w)-t<\infty\}=\{s<\tau(w_t)<\infty\}, \ \boldsymbol{P}_a-a.e. \ (t, s \geqq 0).$

An (almost) L-time is an appropriate time at which the time scale is reversed as to preserve the temporally homogeneous Markov property of the reversed process $z_t=x_{\tau-t-0}$. This fact will be proved in the next section. In the following, we shall prove some properties of (almost) L-times and give several examples of them.

LEMMA 2.2. *Let τ be an L-time, then*

$$\{0<\tau-t<\infty\} \subset \{\tau(w)=t+\tau(w_t)\}, \text{ for any } t \geqq 0. \tag{2.1}$$

If τ is an almost L-time, we have, for any $t \geqq 0$,

$$\boldsymbol{P}_a[\tau(w)=t+\tau(w_t) \mid 0<\tau-t<\infty]=1, \quad (a \in E). \tag{2.1′}$$

Proof. Let τ be an L-time, then

$$\begin{aligned}&\{\tau(w)>\tau(w_t)+t,\ 0<\tau(w)-t<\infty\}\\&=\bigcup_r \{\tau(w)>r>\tau(w_t)+t,\ 0<\tau(w)-t<\infty\}\end{aligned}$$

where r varies over the rationals larger than t. Hence, by making use of (L₂), we have

$$=\bigcup_r \{\infty>\tau(w_t)>r-t>\tau(w_t)\}=\phi.$$

We have also

$$\{\tau(w)<\tau(w_t)+t,\ 0<\tau(w)-t<\infty\}=\phi.$$

https://doi.org/10.1017/S0027763000011405 Published online by Cambridge University Press

Consequently, we have verified that

$$\{0<\tau(w)-t<\infty\}=\{\tau(w)=\tau(w_t)+t,\ 0<\tau(w)-t<\infty\},$$

proving (2.1). The proof for almost L-times is similar.

LEMMA 2.3. *Let τ be an almost L-time, then*

$$L_\alpha(a)=M_a[e^{-\alpha\tau};\ 0<\tau],\ (\alpha>0),^{1)} \\ =P_a[0<\tau(w)<\infty],\ (\alpha=0), \tag{2.2}$$

is an α-excessive function.[1]

Proof. Let $\alpha>0$, then we have, by Lemma 2.2,

$$M_a[L_\alpha(x_t)e^{-\alpha t}]=M_a[e^{-\alpha t-\alpha\tau(w_t)};\ 0<\tau(w_t)] \\ =M_a[e^{-\alpha\tau(w)};\ t<\tau(w)]\uparrow M_a[e^{-\alpha\tau(w)};\ 0<\tau(w)],\ (t\downarrow 0).$$

The proof for $\alpha=0$ is similar.

PROPOSITION 2.4. *The killing time $\zeta(w)$ is an L-time.*

Proof is obvious.

DEFINITION 2.5. *For a subset $D\subset E$, the last exit time from D is defined by*

$$\xi_D(w)=\sup\{t\geqq 0;\ x_t(w)\in D\},\ (\sup\phi=-\infty). \tag{2.3}$$

PROPOSITION 2.6. *Let D be an open set, then the last exit time ξ_D is an L-time.*[2]

Proof. Since paths are right continuous,

$$\{\xi_D>t\}=\{\exists\ \text{rational}\ r>t,\ x_r(w)\in D\}\in\mathcal{N}.$$

Let $w\in\{s<\xi_D-t<\infty\}$, then

$$\xi_D(w)=\sup\{t+r;\ \exists r\geqq 0,\ x_{t+r}(w)\in D\} \\ =\sup\{t+r;\ \exists r\geqq 0,\ x_r(w_t)\in D\} \\ =t+\xi_D(w_t).$$

Therefore, $w\in\{s<\xi_D(w_t)<\infty\}$. Converse is obvious.

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1) $M_a[f(w);\ B]=\int_B f(w)P_a[dw]$.
u is α-excessive if $M_a[u(x_t)e^{-\alpha t}]\uparrow u(a)\geqq 0,\ (t\downarrow 0)$.

2) If X is standard, ξ_D is an L-time for nearly analytic set D. For, $\{\xi_D\leqq t\}=\bigcap_n\{\sigma_D(w_{t+(1/n)})=\infty\}\in\mathcal{N}$, which is communicated from K. Sato.

Remark. Let ξ_E denote the last exit time from E, then

$$\xi_E(w)=\zeta(w),\ \text{for}\ w\in\{\zeta(w)>0\}. \tag{2.4}$$

Consequently, ξ_E and ζ can be identified P_a-*a.e.*

We shall prove that there are many kinds of L-times connected with a given L-time. For the purpose, we need to define the reversed path w^* of w and reversed Markov times from an L-time. We fix an L-time τ in the following.

Put

$$W_0=\{w\ :\ 0<\tau(w)<\infty\}. \tag{2.5}$$

DEFINITION 2.7. *For $w\in W_0$, we define a new path w^* by*

$$w^*(t)=w(\tau(w)-t-0),\ \textit{for}\ 0<t<\tau(w), \tag{2.6}$$

(*if* $w(\tau(w)-0)$ *exists, we add* $t=0$ *in* (2.6)) *and call* w^* *the reversed path of* w *from* $\tau(w)$.

Put

$$W^*=\{w^*\ :\ w\in W_0\}, \tag{2.7}$$

and let $\mathscr{N}^*$ be a σ-field of W^* induced by the mapping $w\to w^*$, i.e. $B^*\in\mathscr{N}^*$ if and only if $\{w\ :\ w^*\in B^*\}\in\bar{\mathscr{N}}$.

DEFINITION 2.8. *A function $\sigma^*(w^*)$ from W^* to $[0,\ \infty]$ is said to be a reversed Markov time (corresponding to the fixed L-time τ), if it has the properties:*

(m_1) $\sigma^*(w^*)$ *is* $\mathscr{N}^*$-*measnrable; and*

(m_2) *For any* $s\geqq 0$, *take any* $w\in W_0$ *such as* $w_s\in W_0$. *If* $\sigma^*((w_s)^*)<\tau(w_s)$, *then we have* $\sigma^*((w_s)^*)=\sigma^*(w^*)$.[3)]

LEMMA 2.9.

$$(w_s)^*(t)=w^*(t)\ \textit{for any}\ 0<t<\tau(w_s). \tag{2.8}$$

Proof. For $0<t<\tau(w_s)$, we have, making use of Lemma 2.2, $(w_s)^*(t)=w_s(\tau(w_s)-t-0)=w(\tau(w_s)+s-t-0)=w(\tau(w)-t-0)=w^*(t)$.

[3)] This was suggested by K. Sato. Our terminology is reasonable, if we note Lemma 2.9 and refer to Galmarino's test (cf. e g. [9]).

LEMMA 2.10. *For any random time* $\sigma(w)\geqq 0$, *we have*

$$\{\sigma<\tau-s<\infty\}=\{\sigma<\tau(w_s)<\infty\}. \tag{2.9}$$

Proof.

$$\{\sigma<\tau-s<\infty\}=\bigcup_r\{\sigma<r<\tau-s<\infty\}=\bigcup_r\{\sigma<r<\tau(\omega_s)<\infty\}$$
$$=\{\sigma<\tau(w_s)<\infty\}.$$

DEFINITION 2.11. *Taking the fixed L-time* τ, *and a reversed Markov time* σ^* *(corresponding to* τ*), we define* τ' *by*

$$\begin{aligned}\tau'(w)&=\tau(w)-\sigma^*(w^*),\ (w\in W_0),\\ &=\tau(w),\ (w\in W\setminus W_0).\end{aligned} \tag{2.10}$$

Then we obtain the following

PROPOSITION 2.12.[4] $\tau'(w)$ *defined in* (2.10) *is an L-time.*

Proof. For any s, $t\geqq 0$, it follows from (m_2) and Lemma 2.10 that

$$\{t<\tau'(w_s)<\infty\}=\{t<\tau(w_s)-\sigma^*((w_s)^*)<\infty\}$$
$$=\{t<\tau(w_s)-\sigma^*(w^*)<\infty\}=\{t<\tau(w)-s-\sigma^*(w^*)<\infty\}$$
$$=\{t<\tau'(w)-s<\infty\},$$

completing the proof.

Remark. If τ is an almost L-time, then τ' defined in (2.10) is an almost L-time.

We denote the coordinate mapping of w^* also by x_t, i.e. $x_t(w^*)=w^*(t)$.

DEFINITION 2.13. *Put, for a subset* $D\subset E$,

$$\sigma_D^*(w^*)=\inf\{t>0;\ x_t(w^*)\in D\},\ \textit{for}\ w^*\in W^*, \tag{2.11}$$
$$(\inf\phi=\infty),$$

and $\sigma_D^*(w^*)$ *is said to be the reversed first passage time to D.*

PROPOSITION 2.14. *If D is an open set, then the reversed first passage time* $\sigma_D^*(w^*)$ *is a reversed Markov time.*

[4] This proposition combined with Theorems in §3 shows that the reversed processes of a process from L-times have a common transition probability independent of L-times.

Proof. (m_1) is verified by

$$\{\sigma_D^*(w^*) < s\} = \{\exists \text{ rational } r,\ 0 < r < s;\ x_r(w^*) \in D\}.$$

If $w_s \in W_0$ and $\sigma_D^*((w_s)^*) < \tau(w_s)$, then we have evidently

$$\sigma_D^*((w_s)^*) = \sigma_D^*(w^*),$$

by Lemma 2.9.

COROLLARY 2.15. *Let D be an open set, then*

$$\begin{aligned} \tau'(w) &= \tau(w) - \sigma_D^*(w^*),\ (w \in W_0), \\ &= \tau(w),\ (w \in W \setminus W_0), \end{aligned} \tag{2.12}$$

is an L-time.

The following property of the last exit time ξ_D is used in the next section to verify the conditions for a time reversion.

PROPOSITION 2.16. *If a Markav process X is standard and D is an open set with regular boundary,*[5] *then there exists an additive functional $b(t, w)$ satisfying*

$$L_0(a) \equiv \boldsymbol{P}_a[0 < \xi_D < \infty] = \boldsymbol{M}_a[(b(\infty, w)]. \tag{2.13}$$

Proof. Let $\sigma_n \uparrow \sigma$ be any sequence of Markov times, then it suffices to prove, according to Meyer [10] and Sur [16],

$$\lim_{n\to\infty} \boldsymbol{M}_a[L_0(x_{\sigma_n})] = \boldsymbol{M}_a[L_0(x_\sigma)],\ (a \in E). \tag{2.14}$$

Now, we have

$$\begin{aligned} &\lim_{n\to\infty} \boldsymbol{M}_a[L_0(x_{\sigma_n})] = \lim_{n\to\infty} \boldsymbol{M}_a[\boldsymbol{P}_{x_{\sigma_n}}[0 < \xi_D < \infty]] \\ &= \lim_{n\to\infty} \boldsymbol{P}_a[0 < \xi_D(w_{\sigma_n}) < \infty] \\ &= \lim_{n\to\infty} \boldsymbol{P}_a[\exists s > 0,\ x_{\sigma_n+s} \in D,\ \xi_D(w_{\sigma_n}) < \infty] \\ &= \boldsymbol{P}_a[\bigcap_n \{\exists s > 0,\ x_{\sigma_n+s} \in D,\ \xi_D(w_{\sigma_n}) < \infty\}] \\ &= \boldsymbol{P}_a[\bigcap_n A_n,\ \sigma > \xi_D] + \boldsymbol{P}_a[\bigcap_n A_n,\ \sigma < \xi_D] + \boldsymbol{P}_a[\bigcap_n A_n,\ \sigma = \xi_D] \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III}, \text{ say,} \end{aligned}$$

where $A_n = \{\exists s > 0,\ x_{\sigma_n+s} \in D,\ \xi_D(w_{\sigma_n}) < \infty\}$.

$$\mathrm{I} = \boldsymbol{P}_a[\bigcap_n A_n,\ \sigma > \xi_D] = \boldsymbol{P}_a[\bigcap_n A_n,\ \sigma \geqq \exists \sigma_{n_0} > \xi_D] = 0,$$

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[5] An open set D is said to have regular boundary, if $\boldsymbol{P}_a[\sigma_D = 0] = 1$ for each $a \in \overline{D}$, where $\sigma_D = \inf\{t > 0;\ x_t \in D\}$, $(\inf \phi = \infty)$.

https://doi.org/10.1017/S0027763000011405 Published online by Cambridge University Press

since A_{n_0} and $\{\sigma_{n_0} > \xi_D\}$ are disjoint.

$$\mathrm{II} = \boldsymbol{P}_a[\bigcap_n A_n,\ \sigma < \xi_D] = \boldsymbol{P}_a[\exists s > 0,\ x_{\sigma+s} \in D,\ \xi_D(w_\sigma) < \infty].$$

$$\begin{aligned}\mathrm{III} &= \boldsymbol{P}_a[\bigcap_n A_n,\ \sigma = \xi_D,\ x_\sigma \in \overline{D}] + \boldsymbol{P}_a[\bigcap_n A_n,\ \sigma = \xi_D,\ x_\sigma \notin \overline{D}] \\ &= \mathrm{III}_1 + \mathrm{III}_2, \text{ say.}\end{aligned}$$

Because of $\{\sigma = \xi_D\} = \{\xi_D(w_\sigma) = 0 \text{ or } -\infty,\ \sigma = \xi_D\}$, we have

$$\begin{aligned}\mathrm{III}_1 &\leqq \boldsymbol{P}_a[\sigma = \xi_D,\ x_\sigma \in \overline{D}] \leqq \boldsymbol{P}_a[x_\sigma \in \overline{D},\ \xi_D(w_\sigma) = 0 \text{ or } -\infty] \\ &= \boldsymbol{M}_a[\boldsymbol{P}_{x_\sigma}[\xi_D = 0 \text{ or } -\infty];\ x_\sigma \in \overline{D}] = 0.\end{aligned}$$

For, $\boldsymbol{P}_{x_\sigma}[\xi_D = 0 \text{ or } -\infty] = 0$, $\boldsymbol{P}_a - a.e.$ on $\{x_\sigma \in \overline{D}\}$, since any point of $\overline{D}$ is regular for D by the assumption. Because of

$$\boldsymbol{P}_a[\bigcap_n A_n,\ \sigma = \xi_D,\ \exists \sigma_n = \sigma] = 0,$$

putting

$$B = \{\bigcap_n A_n,\ \sigma = \xi_D,\ x_\sigma \notin \overline{D},\ \bigcap_n \{\rho_n < \xi_D\}\},$$

we have

$$\mathrm{III}_2 = \boldsymbol{P}_a[B].$$

Put $\sigma_n'(w) = (\sigma_n(w) + \sigma_D(w_{\sigma_n})) \wedge \sigma$, then σ_n' is a Markov time with the property $\sigma_n \leqq \sigma_n' \leqq \xi_D = \sigma < \zeta$ on B, from which it follows that $\sigma_n'(w) \uparrow \sigma(w) < \zeta(w)$, and $x_{\sigma_n'}(w) \in \overline{D}$ for $w \in B$. The quasi-left continuity, therefore, implies that

$$\boldsymbol{P}_a[B] = \boldsymbol{P}_a[\lim_{n\to\infty} x_{\sigma_n'} = x_\sigma;\ B] = \boldsymbol{P}_a[x_\sigma = \lim_{n\to\infty} x_{\sigma_n'} \in \overline{D}\ ;\ B] = 0.$$

Hence, we have

$$\begin{aligned}&\lim_{n\to\infty} \boldsymbol{M}_a[L_0(x_{\sigma_n})] = \boldsymbol{P}_a[\exists s > 0,\ x_{\sigma+s} \in D,\ \xi_D(w_\sigma) < \infty] \\ &= \boldsymbol{M}_a[\boldsymbol{P}_{x_\sigma}[\exists s > 0,\ x_s \in D,\ \xi_D < \infty]] \\ &= \boldsymbol{M}_a[L_0(x_\sigma)],\end{aligned}$$

completing the proof.

Remark 1. Since E is open and closed, and $\boldsymbol{P}_a[\sigma_E = 0] = 1$ for $a \in E$ by definition, Proposition 2.16 is valid for ξ_E (or ζ).

Remark 2. Let X be a (not necessarily standard) Markov process and φ_t be an additive functional of X. Denote $\dot{X}$ the process with killing by φ_t, and $\dot{\zeta}$ the killing time of $\dot{X}$. There exists an additive functional $b(t,\ \dot{w})$ satisfying

$\dot{P}_a[0<\dot{\zeta}<\infty]=\dot{M}_a[b(\infty,\dot{w})]$, if

$$v(a)=M_a\left[\int_0^\infty e^{-\alpha t}d\varphi_t\right]$$

is uniformly α-excessive for some $\alpha \geqq 0$. In particular, this is applied for $e^{-\alpha t}$-subprocess X^α.

Proof.[6] Put $u(a)=\dot{P}_a[0<\dot{\zeta}<\infty]$, then

$$\begin{aligned} u(a)-\dot{M}_a[u(\dot{x}_t)] &= \dot{P}_a[\dot{\zeta}<0]-\dot{P}_a[\dot{\zeta}>t] \\ &= 1-M_a[e^{-\varphi_t}] \leqq M_a[\varphi_t] \\ &\leqq e^{\alpha t}M_a\left[\int_0^t e^{-\alpha s}d\varphi_s\right] \\ &= e^{\alpha t}\{v(a)-M_a[e^{-\alpha t}v(x_t)]\} \to 0, \text{ uniformly } (t\downarrow 0), \end{aligned}$$

Therefore, there exists an additive functional $b^\prime t, \dot{w})$ satisfying $u(a)=\dot{M}_a[b(\infty,\dot{w})]$. (cf. [4], [17]).

§ 3. Time reversions of Markov processes from almost *L*-times

DEFINITION 3.1. *Let τ be an almost L-time and $W_0=\{w;\ 0<\tau(w)<\infty\}$. Put, for $w\in W_0$,*

$$z_t(w)=x_{\tau(w)-t-0}(w),\ (0<t<\tau(w)), \\ =\partial,\ (t\geqq\tau(w)), \tag{3.1}$$

(if there exists $x_{\tau-0}$, we permit $t=0$ in (3.1)). The process $(z_t, \boldsymbol{P}_\nu)$ defined on the space $(W_0, \mathscr{N}|_{W_0})$ is said to be the reversed process of $(x_t, \boldsymbol{P}_\nu)$ from an almost L-time τ, where ν is a σ-finite measure on $(E, \mathscr{B})$.

In this section we shall prove the temporally homogenous Markov property of the reversed process $(z_t, \boldsymbol{P}_\nu)$. For this purpose, it suffices to prove for any $n\geqq 0$, $0=t_0<t_1<\cdots<t_n$, and $f_0, f_1, f_2, \ldots, f_n\in C_0(E)$[7] that

$$\begin{aligned} &M_\nu\left[\prod_{j=0}^n f_j(z_{t_j});\ t_n<\tau<\infty\right] \\ &= M_\nu\left[\prod_{j=0}^{n-1} f_j(z_{t_j})\int\hat{P}(t_n-t_{n-1}, z_{t_{n-1}}, db)f_n(b);\ t_{n-1}<\tau<\infty\right], \end{aligned} \tag{3.2}$$

---

6) This was given by K. Sato.

7) $B(E)$, $C(E)$ and $C_0(E)$ are the spaces of bounded $\mathscr{B}$-measurable functions, bounded continuous functions, and continuous functions with compacts upports, respectively. $C_\infty(E)$ is uniform closure of $C_0(E)$.

where $\hat{P}(t, a, db)$ is a (temporally homogeneous) transition probability, ($j=0$ may be omitted when z_0 is not consulted with).

In the present paper we proceed as follows: At first we prove (3.2) under the form of the Laplace transforms, in the second place we check the conditions under which the Laplace transform can be stripped off.

The coming lemmas are fundamental in the first stage, further they show the temporally homogeneous Markov property of the reversed process in essence.

LEMMA 3.2. *Let τ be an almost L-time. For any $n \geqq 1$, $0=t_0<t_1<t_2<\cdots<t_n$, and $f_0, f_1, \ldots, f_n \in B(E)$ ($f_0 \equiv 1$, when z_0 does not exist), we have, for $\alpha, \beta > 0$,*

$$\int_{t_{n-1}}^{\infty} e^{-\beta t_n} dt_n M_a\Big[e^{-\alpha\tau}\prod_{j=0}^{n} f_j(z_{t_j});\ t_n<\tau<\infty\Big] \tag{3.3}$$
$$=\int G_\alpha(a, db) f_n(b) M_b\Big[e^{-(\alpha+\beta)\tau}\prod_{j=0}^{n-1} f_j(z_{t_j});\ t_{n-1}<\tau<\infty\Big],$$

where

$$G_\alpha(a, A)=M_a\Big[\int_0^{\infty} e^{-\alpha t}\chi_A(x_t)\,dt\Big]. \tag{3.4}$$

Further, if $G_0(a, .)$ is σ-finite, and $f_n \in B_0(E)$, then (3.3) *is valid for $\alpha=0$.*

Proof. It suffices to prove (3.3) for $f_0, f_1, \ldots, f_n \in C(E)$.

$$\int_{t_{n-1}}^{\infty} e^{-\beta t_n}\, dt_n M_a\Big[e^{-\alpha\tau}\prod_{j=0}^{n} f_j(z_{t_j})\chi_{\{t_n<\tau<\infty\}}(w)\Big]$$
$$=\lim_{\varepsilon\downarrow 0}\int_{t_{n-1}}^{\infty} e^{-\beta t_n}\, dt_n M_a\Big[e^{-\alpha\tau}\prod_{j=0}^{n} f_j(x_{\tau-t_j-\varepsilon})\chi_{\{t_n+\varepsilon<\tau<\infty\}}(w)\Big]$$
$$=\lim_{\varepsilon\downarrow 0} M_a\Big[\int_{t_{n-1}}^{\tau-\varepsilon} e^{-\beta t_n}\, dt_n\, e^{-\alpha\tau}\prod_{j=0}^{n-1} f_j(x_{\tau-t_j-\varepsilon}) f_n(x_{\tau-t_n-\varepsilon})\,\chi_{\{\tau<\infty\}}\Big]$$
$$=\lim_{\varepsilon\downarrow 0} M_a\Big[\int_0^{\tau-t_{n-1}-\varepsilon} e^{-\beta(\tau-s-\varepsilon)-\alpha\tau}\, ds f_n(x_s)\prod_{j=0}^{n-1} f_j(x_{\tau-t_j-\varepsilon})\,\chi_{\{\tau<\infty\}}\Big]$$
$$=\lim_{\varepsilon\downarrow 0} M_a\Big[\int_0^{\infty} e^{-\beta(\tau-s-\varepsilon)-\alpha\tau}\, ds f_n(x_s)\prod_{j=0}^{n-1} f_j(x_{\tau-t_j-\varepsilon})\chi_{\{s<\tau-t_{n-1}-\varepsilon<\infty\}}\Big].$$

further we have, by making use of (L_2') and Lemma 2.2,

$$=\lim_{\varepsilon\downarrow 0} M_a\Big[\int_0^{\infty} e^{-\beta(\tau(w_s)-\varepsilon)-\alpha(\tau(w_s)+s)}\, ds f_n(x_s)\prod_{j=0}^{n-1} f_j(x_{\tau(w_s)-t_j-\varepsilon}(w_s))\chi_{\{t_{n-1}+\varepsilon<\tau(w_s)<\infty\}}\Big]$$
$$=\lim_{\varepsilon\downarrow 0} M_a\Big[\int_0^{\infty} e^{-\alpha s+\beta\varepsilon} ds\, f_n(x_s) M_{x_s}\Big[e^{-(\alpha+\beta)\tau}\prod_{j=0}^{n-1} f_j(x_{\tau-t_j-\varepsilon})\,\chi_{\{t_{n-1}+\varepsilon<\tau<\infty\}}\Big]$$

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$$= M_a\Big[\int_0^\infty e^{-\alpha s} ds\, f_n(x_s)\, M_{x_s}\Big[e^{-(\alpha+\beta)\tau}\prod_{j=0}^{n-1} f_j(z_{t_j})\,;\ t_{n-1}<\tau<\infty\Big]\Big]$$

$= \textit{the right member of}$ (3.3),

where $\alpha>0$, (this is verified for $\alpha=0$, by making use of σ-finiteness of $G_0(a,.)$ and $f_n \in C_0(E)$).

LEMMA 3.3. *Let τ be an almost L-time, and $L_\alpha^{f_0}(a) = M_a[f_0(z_0)e^{-\alpha\tau}]$. For any $n \geqq 1$, $0<\alpha_1, \alpha_2, \ldots, \alpha_n$, and $f_0, f_1, \ldots, f_n \in B(E)$ ($f_0 \equiv 1$, when z_0 does not exist), we have, for $\alpha>0$,*

$$(3.5)\qquad \int_0^\infty e^{-\alpha_1 t_1} dt_1 \int_{t_1}^\infty e^{-\alpha_2 t_2} dt_2 \int_{t_2}^\infty \cdots \int_{t_{n-1}}^\infty e^{-\alpha_n t_n} dt_n M_a\Big[e^{-\alpha\tau}\prod_{j=0}^{n} f_j(z_{t_j})\,;\ t_n<\tau<\infty\Big]$$

$$= \int\cdots\int G_\alpha(a, da_n) f_n(a_n) G_{\alpha_n+\alpha}(a_n, da_{n-1}) f_{n-1}(a_{n-1}) \cdots$$

$$\cdots G_{\alpha_2+\alpha_3+\ldots+\alpha_n+\alpha}(a_2, da_1) L^{f_0}_{\alpha_1+\alpha_2+\ldots+\alpha_n+\alpha}(a_1).$$

If $G_0(a,.)$ is σ-finite and $f_n \in B_0(E)$, then (3.5) *is valid for $\alpha=0$.*

Proof. We shall prove (3.5) by induction. When $n=1$, (3.5) is reduced to (3.3) with $n=1$. Because of Lemma 3.2 and the assumption of induction, we have

the left member of (3.5)

$$= \int G_\alpha(a, db) f_n(b) \int_0^\infty e^{-\alpha_1 t_1} dt_1 \cdots \int_{t_{n-2}}^\infty e^{-\alpha_{n-1} t_{n-1}} dt_{n-1}$$

$$M_b\Big[e^{-(\alpha+\alpha_n)\tau}\prod_{j=0}^{n-1} f_j(z_{t_j})\,;\ t_{n-1}<\tau<\infty\Big]$$

$$= \int\cdots\int G_\alpha(a, da_n) f_n(a_n) G_{\alpha_n+\alpha}(a_n, da_{n-1}) \cdots L^{f_0}_{\alpha_1+\alpha_2+\ldots+\alpha_n+\alpha}(a_1).$$

This concludes the proof.

Now we introduce several conditions under which we are able to derive (3.2) from the above lemmas.

(*Case 1*)

A.3.1. We assume that $G_0(a,.)$ is σ-finite and for a σ-finite measure ν, if we put

$$(3.6)\qquad \eta(A) = \int \nu(da) G_0(a, A), \text{ for } A \in \mathscr{B},$$

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then, there exists a transition probabability $\hat{p}(t, a, A)$[8] such as

$$\int T_t f(a) g(a) \eta(da) = \int f(a) \hat{T}_t g(a) \eta(da), \tag{3.7}$$

for each $f, g \in B_0(E)$. Here

$$T_t f(a) = M_a[f(x_t)], \tag{3.8}$$

and

$$\hat{T}_t f(a) = \int \hat{p}(t, a, db) f(b). \tag{3.9}$$

A.3.2. ν is a σ-finite measure on $(E, \mathscr{B})$ satisfying

$$P_\nu[z_t \in K] < \infty, \quad (t > 0), \tag{i}$$

and

$$\int_0^\infty e^{-\alpha t} dt P_\nu[z_t \in K] < \infty, \quad (\alpha > 0), \tag{ii}$$

for each compact K. Here z_t is defined by (3.1) for a given τ.

In the following we fix an almost L-time τ, and consider the reversed process z_t defined in (3.1).

THEOREM 3.4. *Let ν be a measure satisfying A.3.1 and A.3.2. Then, for any $n \geqq 2$, $0 < t_1 < t_2 < \cdots < t_{n-2}$, and $f_1, f_2, \ldots, f_n \in C_0(E)$, we have (put $t_{n-2} = 0$, if $n = 2$)*

$$\begin{aligned} &\int_{t_{n-2}}^\infty e^{-\alpha_{n-1} t_{n-1}} dt_{n-1} \int_{t_{n-1}}^\infty e^{-\alpha_n t_n} dt_n M_\nu\Big[\prod_{j=1}^n f_j(z_{t_j});\ t_n < \tau < \infty\Big] \\ &= \int_{t_{n-2}}^\infty e^{-\alpha_{n-1} t_{n-1}} dt_{n-1} \int_{t_{n-1}}^\infty e^{-\alpha_n t_n} dt_n M_\nu\Big[\prod_{j=1}^{n-1} f_j(z_{t_j}) \hat{T}_{t_n - t_{n-1}} f_n(z_{t_{n-1}});\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t_{n-1} < \tau < \infty\Big], \end{aligned} \tag{3.10}$$

where α_n and α_{n-1} are positive.

Proof. It suffices to prove (3.10) for $f_j \geqq 0$ and $f_j \in C_0(E)$, $(j = 1, 2, \ldots, n)$. Putting

$$\hat{G}_\alpha f(a) = \int_0^\infty e^{-\alpha t} dt \hat{T}_t f(a), \quad (\alpha > 0), \tag{3.11}$$

---

[8] $0 \leqq \hat{P}(t, a, A) \leqq 1$, and for fixed t, a, $\hat{P}(t, a, A)$ is a measure on $(E, \mathscr{B})$, and for fixed A, it is measurable in (t, a). Further it satisfies

$$\hat{P}(t+s, a, A) = \int \hat{P}(t, \alpha, db) \hat{P}(s, a, A), \quad (t, s \geqq 0).$$

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we have, by (3.7),

$$\int G_\alpha f(a)\, g(a)\, \eta(da) = \int f(a) \hat{G}_\alpha g(a) \eta(da). \tag{3.12}$$

Making use of Lemma 3.2 three times and (3.12), we have

the left member of (3.10)

$$= \int_{t_{n-2}}^{\infty} e^{-\alpha_{n-1} t_{n-1}} dt_{n-1} \iint \nu(da)\, G_0(a,\ db)\, f_n(b) \boldsymbol{M}_b\Big[e^{-\alpha_n \tau} \prod_{j=1}^{n-1} f_j(z_{t_j})\,;\ t_{n-1} < \tau < \infty \Big]$$

$$= \iint \eta(db) f_n(b) G_{\alpha_n}(b,\ dc) f_{n-1}(c) \boldsymbol{M}_c\Big[e^{-(\alpha_n + \alpha_{n-1})\tau} \prod_{j=1}^{n-2} f_j(z_{t_j})\,;\ t_{n-2} < \tau < \infty \Big]$$

$$= \int \eta(db)\, f_{n-1}(b)\, \hat{G}_{\alpha_n} f_n(b) \boldsymbol{M}_b\Big[e^{-(\alpha_n + \alpha_{n-1})\tau} \prod_{j=1}^{n-2} f_j(z_{t_t})\,;\ t_{n-2} < \tau < \infty \Big]$$

$$= \int_{t_{n-2}}^{\infty} e^{-(\alpha_n + \alpha_{n-1}) t_{n-1}} dt_{n-1} \boldsymbol{M}_\nu\Big[\prod_{j=1}^{n-1} f_j(z_{t_j}) \hat{G}_{\alpha_n} f_n(z_{t_{n-1}})\,;\ t_{n-1} < \tau < \infty \Big]$$

$=$ *the right member of* (3.10),

completing the proof.

A.3.3. For any $f \in C_0(E)$,

(i) $\hat{T}_t f(a)$ is right continuous in t, and

(ii) $\hat{G}_\alpha f(z_t)$ is right continuous in t, $\boldsymbol{P}_a - a.e.$ $(\alpha > 0)$.

Remark. $(A.3.1)$ and $(A.3.3)$ follow from the condition (F) of Hunt [3].

THEOREM 3.5. *If a Markov process X and a measure ν satisfy* $A.1.1$, $A.3.2$, *and* $A.3.3$, *then the reversed process* $(z_t,\ \boldsymbol{P}_\nu)$, $(t > 0)$, *of* $(x_t,\ \boldsymbol{P}_\nu)$ *from an almost L-time τ has temporally homogeneous Markov property and its transition probability is* $\hat{P}(t,\ a,\ A)$, i.e.

$$\begin{aligned} \boldsymbol{P}_\nu[z_t \in A \mid z_r,\ 0 < r < s] &= \boldsymbol{P}_\nu[z_t \in A \mid z_s] \\ &= \hat{P}(t-s,\ z_s,\ A),\ \boldsymbol{P}_\nu - a.e.\ on\ \{s < \tau < \infty\},\ (0 < s < t).^{9)} \end{aligned} \tag{3.13}$$

It must be noticed that the transition probability of the reversed process does depend only on the transition probability and the initial measure ν of the process x_t, and does not depend on the almost L-time τ.

---

[9] It was remarked by K. Sato that if the process $(x_t,\ \boldsymbol{P}_\nu)$ satisfies; (i) $\boldsymbol{P}_\nu[0 < \tau < \infty$ *and* $x_{\tau-0}$ *does not exist*$]=0$, (ii) $\boldsymbol{P}_\nu[z_0 \in K] < \infty$ for every compact set K, and (iii) $\hat{T}_t f(a) \in C(E)$ for each $f \in C_0(E)$, then in Theorem 3.5, we can replace $(z_t,\ \boldsymbol{P}_\nu)$, $(t > 0)$ by $(z_t,\ \boldsymbol{P}_\nu)$, $(t \geqq 0)$, and $(0 < s < t)$ in (3.13) by $(0 \leqq s < t)$.

We prepare a lemma.

LEMMA 3.6. *If $f_0, f_1, \ldots, f_n \in C_0(E)$, then*

$$u(t_{n-1}) = \int_{t_{n-1}}^{\infty} e^{-\alpha_n t_n} dt_n M_\nu\Big[\prod_{j=0}^{n} f_j(z_{t_j})\,;\ t_n<\tau<\infty\Big],\ (n\geqq 2), \tag{3.14}$$

and

$$v(t_n) = M_\nu\Big[\prod_{j=0}^{n} f_j(z_{t_j})\,;\ t_n<\tau<\infty\Big],\ (n\geqq 1), \tag{3.15}$$

are right continuous in t_{n-1} and t_n, respectively.

Proof. For any $\varepsilon>0$, we have

$$\begin{aligned} &|u(t_{n-1}+\varepsilon) - u(t_{n-1})| \\ &\leqq \int_{t_{n-1}+\varepsilon}^{\infty} e^{-\alpha_n t_n} dt_n M_\nu\Big[\prod_{j\neq n-1} |f_j(z_{t_j})|\cdot|f_{n-1}(z_{t_{n-1}+\varepsilon}) - f_{n-1}(z_{t_{n-1}})|\Big] \\ &\quad + \int_{t_{n-1}}^{t_{n-1}+\varepsilon} e^{-\alpha_n t_n} dt_n M_\nu\Big[\prod_{j=0}^{n} |f_j(z_{t_j})|\Big]. \end{aligned} \tag{3.16}$$

$$\begin{aligned} &|v(t_n+\varepsilon) - v(t_n)| \\ &\leqq M_\nu\Big[\prod_{j=0}^{n-1} |f_j(z_{t_j})|\cdot|f_n(z_{t_n+\varepsilon}) - f_n(z_{t_n})|\,;\ t_n+\varepsilon<\tau<\infty\Big] \\ &\quad + M_\nu\Big[\prod_{j=0}^{n} |f_j(z_{t_j})|\,;\ t_n<\tau\leqq t_n+\varepsilon\Big]. \end{aligned} \tag{3.17}$$

Both terms in (3.16) and (3.17) converges to zero with ε by *A.3.2* and the *bounded convergence theorem.*

Proof of Theorm 3.5. Making use of the previous lemma and *A.3.3*, we can strip off the integrations by t_{n-1} and t_n in (3.10), because of one to one property of the Laplace transform. Thus we have (3.2), from which the statements of the theorem immediately follow.

(*Case 2*)

B.3.1. (i) There exist a σ-finite measure m on $(E, \mathscr{B})$ and a function $p(t, a, b)$ $(t>0,\ a, b\in E)$ satisfying;

(3.18) $p(t, a, b)$ is non-negative, $\mathscr{B}\times\mathscr{B}$-measurable as a function of (a, b),

(3.19) $T_t f(a) = \int p(t, a, b) f(b)\, m(db)$.

Moreover, either

(ii) $p(t, a, b)$ satisfies

(3.20) $p(t, a, b)$ is right continuous function of t in $(0, \infty)$ and bounded in (a, b) for any fixed $t>0$;

or

(ii′) T_t maps $B_0(E)$ into $C(E)$.

***B**.3.2.* There exists a σ-finite measure μ such that

$$M_a[e^{-\alpha\tau};\ 0<\tau<\infty]=\int g_\alpha(a, b)\mu(db),\ (\alpha\geqq 0,\ a\in E), \tag{3.21}$$

where

$$g_\alpha(a, b)=\int_0^\infty p(t, a, b)e^{-\alpha t}dt,\ (\alpha\geqq 0). \tag{3.22}$$

***B**.3.2′.* (i) $P_a[x_{\tau-0}$ *does not exist in* $E]=0$, and (ii) there exists a σ-finite measure μ, satisfying

$$M_a[f(x_{\tau-0})e^{-\alpha\tau};\ 0<\tau<\infty]=\int g_\alpha(a, b)f(b)\mu(db), \tag{3.21′}$$

for $\alpha\geqq 0$, $a\in E$, and $f\in B(E)$.

Remark 1. **B**.3.2 is weaker than **B**.3.2′ (ii).

Remark 2. Let X be an A-diffusion, i.e. the diffusion process with the generator which is a closed extension of a second order elliptic differential operator A satisfying some regularity conditions (cf. [4], [12]). Then X satisfies the conditions **B**.3.1 and **B**.3.2 (or **B**.3.2′).

THEOREM 3.7. *Let X be a Markov process satisfying* **B**.3.1, *and τ be an almost L-time satisfying* **B**.3.2 *or* **B**.3.2′. *Then, for any* $\alpha\geqq 0$, $n\geqq 1$, $0=t_0<t_1<\cdots<t_n$, *and* $f_0, f_1, \dots, f_n\in C_0(E)$ (*when* **B**.3.2 *is assumed, we put* $f_0\equiv 1$), *we have*

$$\begin{aligned} M_a\Big[\prod_{j=0}^{n} f_j(z_{t_j})e^{-\alpha\tau};\ t_n<\tau<\infty\Big] &= e^{-\alpha t_n}\int\cdots\int g_\alpha(a, a_n)m(da_n)f_n(a_n)P(t_n-t_{n-1}, a_n, da_{n-1})\cdot \\ &\cdot f_{n-1}(a_{n-1})P(t_{n-1}-t_{n-2}, a_{n-1}, da_{n-2})f_{n-2}(a_{n-2})\cdots \\ &\cdot P(t_2-t_1, a_2, da_1)f_1(a_1)p(t_1, a_1, a_0)f_0(a_0)\mu(da_0). \end{aligned} \tag{3.23}$$

Proof. It suffices to prove (3.23) for non-negative $f_0, f_1, \dots, f_n\in C_0(E)$ and $\alpha>0$. We shall prove it by induction.

We assume that (3.23) is valid for $n-1$ ($n \geqq 2$). Then we shall at first prove for $\beta > 0$ that

$$(3.24)\qquad \int_{t_{n-1}}^{\infty} e^{-\beta t_n} dt_n \boldsymbol{M}_a\Big[\prod_{j=0}^{n} f_j(z_{t_j}) e^{-\alpha\tau};\ t_n < \tau < \infty\Big]$$

$$= \int_{t_{n-1}}^{\infty} e^{-(\alpha+\beta)t_n} dt_n \int \cdots \int G_\alpha(a,\ da_n) f_n(a_n) P(t_n - t_{n-1},\ a_n,\ da_{n-1})$$

$$\cdots P(t_2 - t_1,\ a_2,\ da_1) f_1(a_1) p(t_1,\ a_1,\ a_0) f_0(a_0) \mu(da_0).$$

On account of Lemma 3.2 and making use of (3.23) for $n-1$, we have

the left member of (3.24)

$$= \int G_\alpha(a,\ da_n) f_n(a_n) \boldsymbol{M}_{a_n}\Big[e^{-(\alpha+\beta)\tau} \prod_{j=0}^{n-1} f_j(z_{t_j});\ t_{n-1} < \tau < \infty\Big]$$

$$= \int G_\alpha(a,\ da_n) f_n(a_n) e^{-(\alpha+\beta)t_{n-1}} \int \cdots \int G_{\alpha+\beta}(a_n,\ da_{n-1}) f_{n-1}(a_{n-1})$$

$$\cdots P(t_2 - t_1,\ a_2,\ da_1) f_1(a_1) p(t_1,\ a_1,\ a_0) f_0(a_0) \mu(da_0)$$

$$= e^{-(\alpha+\beta)t_{n-1}} \int_0^\infty e^{-(\alpha+\beta)s} ds \int \cdots \int G_\alpha(a,\ da_n) f_n(a_n) P(s,\ a_n,\ da_{n-1})$$

$$\cdots p(t_1,\ a_1,\ a_0) f_0(a_0) \mu(da_0)$$

$=$ *the right member of* (3.24).

The right member of (3.23) is right continuous in t_n according to $\boldsymbol{B}.3.1$ (ii) or (ii′), since μ and m are finite on compact sets. On the other hand, the left member of (3.23) is also right continuous in t_n by Lemma 3.6 ($\boldsymbol{M}_\nu$ must be replaced by $\boldsymbol{M}_a$ in (3.15)). Consequently, (3.23) for $n \geqq 2$ is obtained from (3.24) by the one-to-one property of the Laplace transform. Applying the above discussions for $n=1$ and using (3.21) or (3.21′), we have (3.23) for $n=1$. This completes the proof.

Definition 3.8. *For a σ-finite measure ν, we put*

$$(3.25)\qquad \eta(b) = \int \nu(da) g_0(a,\ b),$$

and

$$(3.26)\qquad E_\eta = \{a;\ a \in E,\ 0 < \eta(a) < \infty\},\ \textit{and}\ E_0 = E \setminus E_\eta.$$

$\boldsymbol{B}.3.3.$ A σ-finite measure ν satisfies

$$(3.27)\qquad \boldsymbol{P}_\nu[z_t \in K] < \infty,\ (t \geqq 0),$$

for any compact set K (when z_0 does not exist, $t=0$ is omitted).

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LEMMA 3.9. *Under the condition* $\boldsymbol{B}.3.1$, $\boldsymbol{B}.3.2'$ (*or* $\boldsymbol{B}.3.2$), *and* $\boldsymbol{B}.3.3$, *we have* $\boldsymbol{P}_\nu[z_t = E_0] = 0$ $(t > 0)$, (*when* $\boldsymbol{B}.3.2'$ *is assumed,* $t=0$ *may be added*).

Proof. Choose a sequence of compact sets K_n such as $K_n \uparrow E$. Then, for $t>0$,

$$\boldsymbol{P}_\nu[z_t \in E_0 \cap K_n] = \iint p(t, a_1, a_0)\chi_{F_0 \cap K_n}(a_1)\eta(a_1)m(da_1)\mu(da_0), \tag{3.28}$$

and, if $\boldsymbol{B}.3.2'$ is assumed,

$$\boldsymbol{P}_\nu[z_0 \in E_0 \cap K_n] = \int \chi_{F_0 \cap k_n}(a_0)\eta(a_0)\mu(da_0), \tag{3.29}$$

by Theorem 3.7 and (3.21′). Since the right members of (3.28) and (3.29) are equal to zero or infinity, while the left members are finite, they must be zero. Letting $n \uparrow \infty$, we complete the proof.

THEOREM 3.10. *Let* X, τ, *and* ν *be a Markov process, an almost L-time and a* σ*-finite measure satisfying* $\boldsymbol{B}.3.1$, $\boldsymbol{B}.3.2$ *or* $\boldsymbol{B}.3.2'$, *and* $\boldsymbol{B}.3.3$, *respectively. Then the reversed process* $(z_t, \boldsymbol{P}_\nu)$ *from the almost L-time* τ *has temporally homogeneous Markov property and its transition probability is*

$$\boldsymbol{P}_\nu[z_t \in db \mid z_s = a] = p(t-s, b, a)\frac{\eta(b)}{\eta(a)}\chi_{F_\eta}(b)m(db), \ (0<s<t, a \in E_\eta). \tag{3.30}$$

Moreover, if $\boldsymbol{B}.3.2'$ *is assumed,* $s=0$ *is added in* (3.30) *and the initial measure of the reversed process is given by*

$$\boldsymbol{P}_\nu[z_0 \in db] = \eta(b)\mu(db), \tag{3.31}$$

where η *and* E_η *are defined in* (3.25) *and* (3.26).

Proof. (3.31) is obvious by (3.21′). For the proof of Markov property of $(z_t, \boldsymbol{P}_\nu)$, and (3.30), it suffices to verify an analogue of (3.2),

$$\begin{aligned} &\boldsymbol{M}_\nu\Big[\prod_{j=0}^{n} f_j(z_j) \ ; \ t_n < \tau < \infty\Big] \\ &= \boldsymbol{M}_\nu\Big[\prod_{j=0}^{n-1} f_j(z_{t_j})\chi_{E_\eta}(z_{t_{n-1}})\int p(t_n - t_{n-1}, a, z_{t_{n-1}})\frac{\eta(a)}{\eta(z_{t_{n-1}})}\chi_{F_\eta}(a)f_n(a)m(da) \\ &\qquad ; \ t_{n-1} < \tau < \infty\Big], \end{aligned} \tag{3.2'}$$

(when $\boldsymbol{B}.3.2'$ is not assumed, put $f_0 \equiv 1$).

On account of Theorem 3.7 and Lemma 3.9,

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the left member of (3.2′)

$$= M_\nu\Big[\prod_{j=0}^{n} f_j(z_{t_j})\chi_{E_\eta}(z_{n-1})\chi_{E_\eta}(z_{t_n})\,;\ t_n<\tau<\infty\Big]$$

$$= \int\cdots\int \eta(a)\,m(da)\,f_n(a)\chi_{E_\eta}(a)\,p(t_n-t_{n-1},\ a,\ a_{n-1})\frac{1}{\eta(a_{n-1})}\chi_{E_\eta}(a_{n-1})$$

$$\eta(a_{n-1})m(da_{n-1})\cdots p(t_2-t_1,\ a_2,\ a_1)m(da_1)f_1(a_1)p(t_1,\ a_1,\ a_0)f_0(a_0)\mu(da_0)$$

$=$ *the right member of* (3.2′),

completing the proof.

We shall give remarks about the conditions $\boldsymbol{B}.3.2$ and $\boldsymbol{B}.3.2'$.

Remark 1. Put, for $g_\alpha(a, b)$ in (3.22) and for $f\in B(E)$,

$$G_\alpha^* f(b) = \int m(da) f(a) g_\alpha(a,\ b),\ (\alpha<0). \tag{3.32}$$

$\boldsymbol{B}.3.4.$ X is standard, G_α^* maps $C_\infty(E)$ into $C_\infty(E)$, and $G_\alpha^*[C_\infty(E)]$ is dense in $C_\infty(E)$;

If X satisfies $\boldsymbol{B}.3.4$, then $\boldsymbol{B}.3.2$ is valid for every almost L-times (cf. Proposition 5.1, pp. 115-117 in [4]).

Remark 2.

$\boldsymbol{B}.3.5.$ $\boldsymbol{B}.3.2$ is satisfied and (i) $g_\alpha(a, b)$ given in (3.22) is α-excessive and α-harmonic in $E\setminus b$ as a function of a, and (ii) $M_a[e^{-\alpha\tau}\,;\ 0<\tau<\infty]$ is regularly α-excessive.

If X is standard and satisfyies $\boldsymbol{B}.3.5$, then $\boldsymbol{B}.3.2'$ (ii) is valid. (cf. Theorem 4.1 in [14]).

Remark 3. Let $p(t, a, b)$ satisfy $\boldsymbol{B}.3.1$ (i), and

(i) $\int p(t,\ a,\ b)m(db)p(s,\ b,\ c) = p(t+s,\ a,\ c),\ (t,\ s>0)$,

(ii) $G_\alpha^* f\in C(E)$, for any $f\in C_0(E)$,

(iii) for any $f\in C(E)$,

$$\lim_{t\downarrow 0}\int m(da) f(a) p(t,\ a,\ b) = f(b),\ \ (b\in E).$$

Then $g_\alpha(a, b)$ satisfies $\boldsymbol{B}.3.5$ (i).

Proof. We shall prove, for any open set U containing b,

$$g_\alpha(a,\ b) = M_a[e^{-\alpha\sigma_U} g_\alpha(x_{\sigma_U},\ b)] \equiv H_U^\alpha g_\alpha(a,\ b),\ (a\in E,\ b\in U) \tag{3.33}$$

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where σ_U is the first passage time to U.

We have easily (3.33) for fixed $a \in E$ and m-a.e. $b \in U$. Take $f \in C(E)$ such that $f(b)=1$, $f(a)=0$ for $a \notin U$, and $0 \leqq f \leqq 1$, and take any $h \in C_0(E)$, then we have

$$\int h(a)\, m(da)\, g_\alpha(a, b)$$

$$= \lim_{t \downarrow 0} \iint h(a)\, m(da)\, g_\alpha(a, c)\, m(dc)\, f(c) e^{-\alpha t} p(t, c, b)$$

$$= \lim_{t \downarrow 0} \iint h(a)\, m(da)\, H_U^\alpha g_\alpha(a, c)\, m(dc)\, f(c)\, e^{-\alpha t} p(t, c, b)^{10)}$$

$$= \lim_{t \downarrow 0} \iint h(a)\, m(da)\, H_U^\alpha(a, dc) \int_t^\infty e^{-\alpha s} p(s, c, b) ds$$

$$- \lim_{t \downarrow 0} \iint h(a)\, m(da)\, H_U^\alpha g_\alpha(a, c)(1 - f(c))\, m(dc)\, e^{-\alpha t} p(t, c, b).$$

The first term

$$= \int h(a)\, m(da)\, H_U^\alpha g_\alpha(a, b), \text{ and}$$

$|$*the second term*$|$

$$\leqq K \lim_{t \downarrow 0} \int (1 - f(c))\, m(dc)\, p(t, c, b) = 0,$$

where $K \equiv K(h, \alpha)$ such as

$$\left| \int h(a)\, m(da)\, H_U^\alpha g_\alpha(a, c) \right| \leqq \int |h(a)|\, m(da) g_\alpha(a, c) \leqq K < \infty.$$

Consequently, we have (3.33) for m-a.e. $a \in E$ and for any $b \in E$, but since the both sides of (3.33) are α-excessive, therefore fine continuous, as functions of $a \in E$, (3.33) is valid for any $a \in E$, completing the proof.

We state here a corollary of Theorem 3.10.

COROLLARY. *Let X be a standard process satisfying $\boldsymbol{B}.3.1$ and $\boldsymbol{B}.3.5$ (i) and $\boldsymbol{B}.3.2$ (or $\boldsymbol{B}.3.4$) for the last exit time ξ_D where D is an open set with regular boundary such as*

$$\boldsymbol{P}_a[\text{there exists } x_{\xi_D} \in E \mid 0 < \xi_D < \infty] = 1 \quad (a \in E).$$

Let ν be a σ-finite measure satisfying $\boldsymbol{B}.3.3$. Then the reversed process $(z_t, \boldsymbol{P}_\nu)$

10) $H_U^\alpha(a, A) = \boldsymbol{M}_a[e^{-\alpha\sigma_U}\chi_A(x_{\sigma_U})]$ and $H_U^\alpha f(a) = \int H_U^\alpha(a, db) f(b)$.

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of $(x_t, \boldsymbol{P}_\nu)$ *from the last exit time* ξ_D *has temporal homogeneity and the transition probability*

$$\boldsymbol{P}_\nu[z_t \in db \mid z_s = a] = p(t-s, b, a)\frac{\eta(b)}{\eta(a)}\chi_{E_\eta}(b)\, m(db), \quad (0 \leqq s < t),$$

and the initial measure

$$\boldsymbol{P}_\nu[z_0 \in db] = \eta(b)\,\mu(db),$$

where $\eta(b)$ *is defined in* (3.25).

Proof is immediately obtained by making use of Proposition 2.16.

(*Case 3*)

C.3.1. There exist a σ-finite measure m and $\mathscr{B}\times\mathscr{B}$-measurable function $g_\alpha(a, b) \geqq 0$, $(\alpha \geqq 0)$ with the properties that

(i) $G_\alpha f(a) = \int g_\alpha(a, b) f(b) m(db)$,

and

(ii) $\hat{G}_\alpha f(b) = \int m(da) f(a) g_\alpha(a, b)$ gives the resolvent operator of some Markov process $\hat{X}$, in the sense of § 1.

C.3.2. ν is a σ-finite measure satisfying

$$\int_0^\infty e^{-\alpha t} dt\, \boldsymbol{P}_\nu[z_t \in K] < \infty, \quad (\alpha > 0), \tag{3.34}$$

and

$$\boldsymbol{P}_\nu[z_0 \in K] < \infty, \tag{3.35}$$

for any compact set K, where z_t is the reversed process of x_t from an almost L-time τ satisfying $\boldsymbol{B}$.3.2′.

Given an excessive function $e(a)$, the super-harmonic transform X^e of a Markov process X by e is the Markov process with the transition probability

$$P^e(t, a, A) = \frac{1}{e(a)} \boldsymbol{M}_a[\chi_A(x_t)\, e(x_t)], \quad (a \in E_e), \tag{3.36}$$

where $E_e = \{a;\ 0 < e(a) < \infty\}$ is the state space of X^e.

Kunita and Watanabe [8] proved that the process X^e preserves main properties of X. For example, if X has right continuous paths (this is always asuumed in this paper), then X^e does, and moreover if X is standard, X^e is also standard. This fact is essential in the proof of the following

THEOREM 3.11. *Under the assumptions* C.3.1 *and* C.3.2, *the reversed process* $(z_t, \boldsymbol{P}_\nu)$ *of* $(x_t, \boldsymbol{P}_\nu)$ *is a version of* $(\hat{x}_t^\eta, \hat{\boldsymbol{P}}_{\eta\mu}^\eta)$, *which is the super-harmonic transform* $\hat{X}^\eta$ *of* $\hat{X}$ *by* η *with an initial measure* $\eta\mu$. *Here* $\eta(a)$ *is defined in* (3.25).

We omit the proof, and give a brief sketch in foot note.[11]

Remark. If X and $\hat{X}$ are standard and satisfy C.3.1, and if $g_\alpha(a, b)$ is α-excessive for X in $a \in E$ and for $\hat{X}$ in $b \in E$, then the last exit time ξ_D satisfies (3.21′) for open set D with regular boundary.

§ 4. Time reversion of approximate Markov processes

Following Hunt [2] we first mention some notations and definitions on approximate Markov processes.

Let $(\Omega, \mathscr{M})$ be a measurable space, and $\boldsymbol{P}$ be a measure on $(\Omega, \mathscr{M})$ (possibly with infinite total mass), and $\alpha(\omega)$ and $\beta(\omega)$ be $\mathscr{M}$-measurable functions such that $-\infty \leqq \alpha(\omega) < \infty$, $-\infty < \beta(\omega) \leqq \infty$, and $\alpha(\omega) \leqq \beta(\omega)$.

Let $y_t(\omega)$ be a function on $(\alpha(\omega), \beta(\omega)) \times \Omega$ taking values in a locally compact Hausdorff space E with countable base. A system $(y_t, \alpha, \beta, \boldsymbol{P})$ is said to be a random process if (i) $\{y_t \in A, \alpha < t < \beta\} \in \mathscr{M}$, $(t \in (-\infty, \infty)$, $A \in \mathscr{B})$ and (ii) $\boldsymbol{P}[y_t \in A, \alpha < t < \beta] < \infty$, for every compact set A.

Put $\Omega_t = \{\alpha < t < \beta\}$, and let $\mathscr{M}_t$ be the σ-field of Ω_t generated by $\{y_s \in A, \alpha < t < \beta\}$ for every $s \leqq t$ and $A \in \mathscr{B}$. Then $(y_t, \alpha, \beta, \boldsymbol{P})$ is said to have *Markov property* if for every $\mathscr{M}_t$-measurable and locally $\boldsymbol{P}$-integrable function f,[12]

$$\boldsymbol{M}[f(\omega) \mid \mathscr{M}_t] = \boldsymbol{M}[f(\omega) \mid y_t], \ \boldsymbol{P}\text{-}a.e. \text{ on } \Omega_t, \tag{4.1}$$

---

11) A proof was given by H. Kunita in the case of $\tau = \zeta$. For the proof it suffices to verify an analogue of (3.2),

$$\boldsymbol{M}_\nu\Big[\prod_{j=0}^{n} f_j(z_{t_j});\ t_n < \tau < \infty\Big] = \hat{\boldsymbol{M}}_{\eta\mu}^\eta\Big[\prod_{j=0}^{n} f_j(\hat{x}_{t_j}^\eta)\Big], \tag{3.2″}$$

which can be obtained from its multiple Laplace transforms by making use of Lemma 3.3, a lemma similar to Lemma 3.9, and the right continuity of $\hat{X}^\eta$.

12) $\boldsymbol{M}[f(\omega); B] = \int_B f(\omega)\boldsymbol{P}[d\omega]$.
f is said to be locally $\boldsymbol{P}$-integrable if $\boldsymbol{M}[\,|f(\omega)|\,; B] < \infty$, where $B = \{\alpha < t < \beta\} \cap \{y_t \in A\}$ for every compact set A and $t \in (-\infty, \infty)$.

and said to have *temporally homogeneous Markov property with a transition probability* $\boldsymbol{P}(t, a, A)$, if it has Markov property and

$$\boldsymbol{P}[y_t \in A \mid y_s] = P(t-s, y_s, A), \ \boldsymbol{P}\text{-}a.e. \text{ on } \Omega_s, \tag{4.2}$$

for every $s \leqq t$ and every compact A.

$\sigma(\omega)$ on Ω is said to be *reducing time* for $(y_t, \alpha, \beta, \boldsymbol{P})$ if (i) σ is $\mathscr{M}$-measurable and $\alpha \leqq \sigma \leqq \beta$, and (ii) $(y'_t, 0, \gamma, \boldsymbol{P})$ on $(\Omega', \mathscr{M}|_{\Omega'})$ has temporally homogeneous Markov property, where $\Omega' = \{-\infty < \sigma < \infty\}$, $\gamma(\omega) = \beta(\omega) - \sigma(\omega)$, and $y'_t(\omega) = y_{\sigma(\omega)+t}(\omega)$ for $\omega \in \Omega'$, $t \geqq 0$. We shall call $(y'_t, 0, \gamma, \boldsymbol{P})$ the *reduced process* by σ.

A random process $(y_t, \alpha, \beta, \boldsymbol{P})$ is said to be an *approximate* (temporally homogeneous) *Markov process*, if there exists a sequence of reducing times $\{\alpha_i\}$ such that (i) $-\infty < \alpha_i(\omega) < +\infty$ and $\alpha_i \downarrow \alpha (i \to \infty) \boldsymbol{P}\text{-}a.e.$, and (ii) each reduced process by α_i has temporally homogeneous Markov property with a transition probability $P(t, a, A)$ independent of i.

In order to apply the results in the previous sections, we add some assumptions.

A.4.1. (e_1) The sample paths of y_t are right continuous. (e_2) there exists $y^i_0 \in E$, $\boldsymbol{P}\text{-}a.e.$[13] and each reduced process by α_i is equivalent to a standard process, i.e. the reduced process $(y'_t, 0, \gamma^i, \boldsymbol{P})$ and $(x_t, \boldsymbol{P}_{\nu_i})$ are equivalent for any $i \geqq 0$, where $X = (x_t, \zeta, \mathscr{N}_t, \boldsymbol{P}_a)$ is a standard process and $\nu_i(A) = \boldsymbol{P}[y^i_0 \in A]$, and (e_3) $\eta(A) = \boldsymbol{M}\left[\int_\alpha^\beta \chi_A(y_t)dt\right] < \infty$, for any compact set A.

Define the last exit times from $D \subset E$ of y_t and y^i_t by

$$\xi_D(\omega) = \sup\{t;\ y_t(\omega) \in D\}, \ (\sup \phi = -\infty), \tag{4.3}$$

and

$$\xi^i_D(\omega) = \sup\{t \geqq 0;\ y^i_t(\omega) \in D\}, \ (\sup \phi = -\infty), \tag{4.4}$$

respectively.

Then we have, by the right continuity of paths,

LEMMA 4.1. *If D is open, ξ_D and ξ^i_D are $\mathscr{M}$-measurable.*

LEMMA 4.2. *There exists $i_0(\omega) < \infty$ such that*

13) $y^i_0(\omega) = \lim_{t \downarrow 0} y^i_t(\omega)$.

(4.5) $$\alpha_i(\omega)+\xi_D^i(\omega)=\xi_D(\omega),\ \textit{for any } i\geqq i_0(\omega).$$

Proof. Put $\Omega_1=\{\omega;$ there exist $i_0(\omega)<\infty$ and $0\leqq t<\infty$, such as $y_t^{i_0}(\omega)\in D\}$ and $\Omega_2=\{\omega;$ for any $i\geqq 0$ and $0\leqq t<\infty$, $y_t^i(\omega)\notin D\}$. If $\omega\in\Omega_2$, $\xi_D(\omega)=-\infty$ and $\xi_D^i(\omega)=-\infty$ for any $i\geqq 0$. If $\omega\in\Omega_1$, $\xi_D(\omega)\geqq\alpha_i(\omega)$ for every $i>i_0(\omega)$. Therefore $\xi_D(\omega)=\sup\{t+\alpha_i;\ y_{\alpha_i+t}\in D\}=\alpha_i(\omega)+\xi_D^i(\omega)$, completing the proof.

Put

(4.6) $$\eta(A)=\boldsymbol{M}\Big[\int_\alpha^\beta\chi_A(y_t)\,dt\Big],$$

and

(4.7) $$\eta^i(A)=\boldsymbol{M}\Big[\int_0^{\gamma i}\chi_A(y_t^i)\,dt\Big].$$

Then, we have

LEMMA 4.3. *η and η^i for any i are excessive relative to $P(t,a,A)$, and $\eta^i(A)\uparrow\eta(A)$, $(i\to\infty)$ for each $A\in\mathscr{B}$.*

Proof. For any $A\in\mathscr{B}$, we have

$$\eta^i(A)=\boldsymbol{M}\Big[\int_0^{\beta-\alpha_i}\chi_A(y_{\alpha_i+t})dt\Big]=\boldsymbol{M}\Big[\int_{\alpha_i}^{\beta}\chi_A(y_t)dt\Big]\uparrow\eta(A),\ (i\to\infty),$$

and for compact set A

$$\int\eta^i(da)P(t,a,A)=\boldsymbol{M}\Big[\int_0^{\gamma i}P(t,y_s^i,A)\,ds\Big]=\boldsymbol{M}\Big[\int_0^{\gamma i}\boldsymbol{M}[\chi_A(y_{s+t}^i)\,|\,y_s^i]ds\Big]$$
$$=\boldsymbol{M}\Big[\int_t^{\gamma i}\chi_A(y_s^i)ds\Big]\leqq\eta^i(A)\leqq\eta(A),$$

and, therefore,

$$\int\eta(da)P(t,a,A)=\lim_{i\to\infty}\int\eta^i(da)P(t,a,A)\leqq\eta(A),\ (A\in\mathscr{B}),$$

completing the proof.

We define now the *reversed process* $(\hat{y}_t,\hat{\alpha},\hat{\beta},\boldsymbol{P})$ of an approximate Markov process $(y_t,\alpha,\beta,\boldsymbol{P})$ by

(4.8) $$\hat{\alpha}(\omega)=-\beta(\omega),\ \hat{\beta}(\omega)=-\alpha(\omega),$$

and

(4.9) $$\hat{y}_t(\omega)=y_{-t-0}(\omega).$$

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In the following we shall prove that the reversed process is also an approximate Markov process.

A.4.2. The standard process in $A.4.1$ (e_2) satisfies; (r_1) There exist σ-finite measure m and a transition density $p(t, a, b)$ such as

$$P(t, a, A) = \int_A p(t, a, b)m(db);$$

(r_2) $p(t, a, b)$ satisfies $\boldsymbol{B}.3.1$ and $g_\alpha(a, b)$ satisfies $\boldsymbol{B}.3.4$ and $\boldsymbol{B}.3.5$ (i), where $g_\alpha(a, b)$ is defined in (3.22), or

(r_2') $g_\alpha(a, b)$ satisfies $\boldsymbol{C}.3.1$ and $\boldsymbol{B}.3.5$ (i);

(r_3) There exists a sequence of open sets $\{D_k\}$ with regular boundary and with compact closure such as $\overline{D}_k \subset D_{k+1}$ and $D_k \uparrow E$;

(r_4) For any open set D with compact closure,

$$\text{there exists } y_{\xi_D - 0}, \ \boldsymbol{P}\text{-a.e. on } \{0 \leqq \xi_D < \infty\} \tag{4.10}$$

A.4.3. For any k, $t \geqq 0$, and compact set K,

$$\boldsymbol{P}[y(\xi_{D_k} - t - 0) \in K, \ \xi_{D_k} - t > \alpha] < \infty, \tag{4.11}$$

where $y(t) = y_t(\omega)$.

THEOREM 4.4. *Let $(y_t, \alpha, \beta, \boldsymbol{P})$ be an approximate Markov process satisfying A.4.1 and A.4.2, then for any $0 = t_0 < t_1 < \cdots < t_n$, and $A_0, A_1, \ldots, A_n \in \mathscr{B}$, we have*

$$\begin{aligned}
&\boldsymbol{P}[y^i(\xi^i_{D_k} - t_j - 0) \in A_j, \ 0 \leqq j \leqq n; \ t_n < \xi^i_{D_k} < \infty] \\
&= \int \cdots \int \mu_{D_k}(da_0)\chi_{A_0}(a_0)p(t_1, a_1, a_0)\chi_{A_1}(a_1)m(a_1) \cdots \\
&\qquad\qquad p(t_n - t_{n-1}, a_n, a_{n-1})\chi_{A_n}(a_n)\eta^i(da_n),
\end{aligned} \tag{4.12}$$

where μ_{D_k} is a mearure, given in $\boldsymbol{B}$.3.2 for ξ_{D_k}, independent of i.

Proof. We have

$$\begin{aligned}
&\boldsymbol{P}[y^i(\xi^i_{D_k} - t_j - 0) \in A_j, \ 0 \leqq j \leqq n; \ t_n < \xi^i_{D_k} < \infty] \\
&= \boldsymbol{P}_{\nu_i}[x(\xi_{D_k} - t_j - 0) \in A_j, \ 0 \leqq j \leqq n; \ t_n < \xi_{D_k} < \infty]
\end{aligned}$$

where $x(t) = x_t$.

Therrfore, if (r_2) is assumed, we have (4.12), applying Theorem 3.7 and Remark 1 after Theorem 3.10. If (r_2') is assumed, we have

$$(4.13)\qquad \begin{aligned} &\boldsymbol{P}_{\nu_i}[x(\xi_{D_k}-t_j-0)\in A_j,\ 0\leqq j\leqq n;\ t_n<\xi_{D_k}<\infty] \\ &=\hat{\boldsymbol{M}}^v_{v\mu}\Big[\prod_{j=0}^{n}\chi_{A_j}(\hat{x}^v_{t_j})\Big],\quad (v(a)\equiv\eta^i(a),\ \mu\equiv\mu_{D_k}),\end{aligned}$$

by means of Theorem 3.11. Hence we have (4.12), because of

$$(4.14)\qquad P^v(t,\ a,\ A)=\int_A p(t,\ b,\ a)\frac{v(b)}{v(a)}m(db),$$

completing the proof.

THEOREM 4.5. *Put*

$$(4.15)\qquad \begin{aligned}\hat{P}(t,\ a,\ A)&=\int_A p(t,\ b,\ a)\frac{\eta(b)}{\eta(a)}\chi_{E_\eta}(b)m(db),\ \textit{for}\ a\in E_\eta,\\ &=0,\ \textit{for}\ a\in E\setminus E_\eta,\end{aligned}$$

where

$$(4.16)\qquad \eta(b)=\lim_{i\to\infty}\int\nu_i(da)g_0(a,\ b),^{14)}\ \textit{and}\ E_\eta=\{a;\ 0<\eta(a)<\infty\}.$$

Let $(y_t,\ \alpha,\ \beta,\ \boldsymbol{P})$ *be an approximate Markov process satisfying* $\boldsymbol{A}$.4.1, $\boldsymbol{A}$.4.2, *and* $\boldsymbol{A}$.4.3. *Then the reversed process* $(\hat{y},\ \hat{\alpha},\ \hat{\beta},\ \boldsymbol{P})$ *is an approximate Markov process with transition probability* $\hat{P}(t,\ a,\ A)$, *and its sequence of reducing times is* $\{-\xi_{D_k}\}$.[15]

---

[14] Since $\int\nu_i(da)g_0(a,\ b)$ is non-decreasing m-a.e. $b\in E$, there exists a e. limit $\eta(b)$ and $\eta(db)=\eta(b)m(db)$.

[15] Time reversion of an approximate Markov chain is an essential tool in Hunt's treatment of Martin boundaries of Markov chains [2]. This theorem, therefore, will give a basis for theory of Martin boundaries analoguous to Hunt's in the case of more general Markov processes.

For this purpose we must construct an approximate Markov process from a given (standard) process X and a given excessive measure η such as the reduced processes have the same transition probability as X and satisfy

$$\boldsymbol{M}\Big[\int_\alpha^\beta\chi_A(y_t)\,dt\Big]=\eta(A).$$

But the author is able to prove weak facts that: Let X be a standard process satisfying that $G_0f\in C_\infty(E)$ for each $f\in C_0(E)$ and $G_0[C_0(E)]$ is dense in $C_\infty(E)$. Then there exists a system $(y_t,\ \alpha,\ \beta,\ \boldsymbol{P})$ on $(\Omega,\ \mathfrak{M})$ consisting of : (i) $\mathfrak{M}$ is a field (not necessarily σ-field) on Ω, and $\boldsymbol{P}$ is a finitely additive non-negative function on $\mathfrak{M}$; (ii) $\alpha(\omega)$ and $\beta(\omega)$ are functions on Ω such as $-\infty<\alpha(\omega)<\beta(\omega)\leqq+\infty$; (iii) $y_t(\omega)$ is a function defined on $[\alpha,\beta)\times\Omega$ taking values in a locally compact space E, and right continuous in t; (iv) there exists a sequence of functions $\{\alpha_i\}$ on Ω such as $\alpha_i\leqq\beta$ and $\alpha_i\downarrow\alpha$ $(i\to\infty)$. If we put $\gamma^i=\beta-\alpha_i$ and $y^i_t(\omega)=y_{\alpha_i+t}(\omega)$ $(0\leqq t\leqq\gamma^i)$, and let $\mathfrak{M}^i$ be the σ-field on Ω generated by $\{y^i_t;\ t\geqq0\}$, then $(y^i_t,\ 0,\ \gamma^i,\ \boldsymbol{P})$ is a temporally homogeneous Markov process with the same transition probability as X; and (v)

$$\lim_{i\to\infty}\boldsymbol{M}\Big[\int_{\alpha_i}^{\beta}\chi_A(y_t)\,dt\Big]=\eta(A),\ (A\in\mathscr{B},\ \bar{A}\text{ is compact}).$$

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Proof. Putting

$$B=\{y(\xi_{D_k}-t_j-0)\in A_j,\ 0\leqq j\leqq n,\ \textit{and}\ \xi_{D_k}-t_n>\alpha\},$$

and

$$B_i=\{y^i(\xi^i_{D_k}-t_j-0)\in A_j,\ 0\leqq j\leqq n,\ \textit{and}\ \xi^i_{D_k}>t_n\},$$

for $0=t_0<t_1<\cdots<t_n$, and $A_0, A_1, \ldots, A_n\in\mathscr{B}$ with compact closure, we have, making use of Theorem 4.4 and Lemma 4.2,

$$(4.17)\quad \boldsymbol{P}[B_i]=\int\cdots\int\mu_{D_k}(da_0)\chi_{A_0}(a_0)\cdots p(t_n-t_{n-1},\ a_n,\ a_{n-1})\chi_{A_n}(a_n)\eta^i(da_n).$$

Because $B_i\uparrow B$, we have, letting $i\to\infty$,

$$(4.18)\quad \boldsymbol{P}[B]=\int\cdots\int\mu_{D_k}(da_0)\chi_{A_0}(a_0)\cdots p(t_n-t_{n-1},\ a_n,\ a_{n-1})\chi_{A_n}(a_n)\eta(da_n)$$
$$=\int\cdots\int\mu_{D_k}(da_0)\cdots\chi_{A_{n-1}}(a_{n-1})\chi_{E_\eta}(da_{n-1})\eta(da_{n-1})\hat{P}(t_n-t_{n-1},\ a_{n-1},\ A_n)$$
$$=\boldsymbol{M}[\hat{P}(t_n-t_{n-1},\ y(\xi_{D_k}-t_{n-1}-0),\ A_n);\ y(\xi_{D_k}-t_j-0)\in A_j,\ 0\leqq j\leqq n-1].$$

Noticing that $y(\xi_{D_k}-t-0)=\hat{y}(-\xi_{D_k}+t)$ and putting $\hat{y}^k(t)=\hat{y}(-\xi_{D_k}+t)$, (4.18) is written as

$$(4.19)\quad \begin{aligned}&\boldsymbol{P}[\hat{y}^k(t_j)\in A_j,\ 0\leqq j\leqq n]\\ &=\boldsymbol{M}[\hat{P}(t_n-t_{n-1},\ \hat{y}^k(t_{n-1}),\ A_n);\ \hat{y}^k(t_j)\in A_j,\ 0\leqq j\leqq n-1],\end{aligned}$$

from which (4.1) is deduced.

Thus, we have shown that $-\xi_{D_k}$ reduces $(\hat{y}_t,\ \hat{\alpha},\ \hat{\beta},\ \boldsymbol{P})$ to a process $(\hat{y}^k_t,\ 0,\ \hat{\gamma}^k,\ \boldsymbol{P})$ with temporally homogenous Markov property and transition probability $\hat{P}(t,\ a.\ A)$. On the other hand, $-\xi_{D_k}$ decreases to $-\beta=\hat{\alpha}$ and $-\infty<-\xi_{D_k}<+\infty$, completing the proof.

References

[1] Dynkin, E. B., Markov processes. Moskow (1963), (Russian).

[2] Hunt, G. A., Markoff chains and Martin boundaries. Ill. J. Math. **4** (1960), 316–340.

[3] Hunt, G. A., Markoff processes and potentials, III. Ill. J. Math. **2** (1958), 151–213.

[4] Ikeda, N., K. Sato, H. Tanaka and T. Ueno, Boundary problems in multi-dimensisnal diffusion process, I, II. Seminar on Prob. **5** (1960), **6** (1961), (Japanese).

[5] Ikeda, N., M. Nagasawa and K. Sato, A time reversion of Markov processes with killing. Kōdai Math. Sem. Rep. **16** (1964), 88–97.

[6] Kolmogoroff, A., Zur Theorie der Markoffschen Ketten. Math. Ann. **112** (1936), 155–160.

[7] Kolmogoroff, A., Zur Unkehrbarkeit der statistischen Naturgesetze, Math. Ann. **113** (1937), 766–772.

[8] Kunita, H. and T. Watanabe, Note on transformations of Markov processes connected with multiplicative functionals. Mem. Fac. Sci. Kyushu Univ. Series A. Math. **17** (1963), 181–191.

[9] McKean, H. P. Jr. and H. Tanaka, Additive functionals of the Brownian path. Mem. Coll. Sci. Univ. Kyoto, Ser. A **33** (1961), 476–506.

[10] Meyer, P. A., Fonctionnelles multiplicatives et additives de Markov. Ann. Inst. Fourier, **12** (1962), 125–230.

[11] Motoo, M., Additive functionals of Markov processes. Seminar on Prob. **15** (1963), 1–123, (Japanese).

[12] Nagasawa, M., The adjoint process of a diffusion with reflecting barrier, Kōdai Math. Sem. Rep. **13** (1961), 235–248.

[13] Nagasawa, M, and K. Sato, Remarks to "The adjoint process of a diffusion with reflecting barrier" Kōdai Math. Sem. Rep. **14** (1962), 119–122.

[14] Nagasawa, M. and K. Sato, Some theorems on time change and killing of Markov processes. Kōdai Math. Sem. Rep. **15** (1963), 195–219.

[15] Nelson, E., The adjoint Markov process. Duke Math. J. **25** (1958), 671–690.

[16] Sur, M. G., Continuous additive functionals of Markov processes and excessive functions, Dokl. Acad. Nauk. SSSR **137** (1961), 800–803, (Russian).

[17] Volkonskii, V. A., Additive functionals of Markov processes, Trudy Mosk. Mat. Ob. **9** (1960), 143–189, (Russian).

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