

would be difficult to apply in the general case since the rate of change of altitude is required to an accuracy of two decimal places and this would be difficult to attain graphically.

REFERENCES

- ¹ Ranta, M. (1991). Position fixing in a fast moving ship by culmination of a celestial body. *This Journal*, 43, 276.
- ² Forsythe, G. E. (1957). Generation and use of orthogonal polynomials for data-fitting with a digital computer. *Journal of Industrial Applied Mathematics*, 5, 2.
- ³ Davis, P. L. H. (1918). *Altitude – Azimuth Tables*. HMSO.
- ⁴ Wilson, J. N. (1985). Position from observation of a single body. *Navigation* (Journal of the American Institute of Navigation), 32.

KEY WORDS

1. Astro. 2. Marine navigation.

On the Two-Body Running Fix

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If an observer can determine the altitudes of two bodies simultaneously, he can place himself at one of two positions on the Earth's surface. This fact has been the basis for several proposed methods for obtaining a fix without reference to a DR or assumed position, the latest of which is due to Chiesa and Chiesa.¹ Chiesa and Chiesa's procedure is straightforward and elegant if the sights are simultaneous, or if the sights are not simultaneous but the observer is stationary, but there seems to be a need for a simple method of extending it to cover a running fix. Chiesa and Chiesa themselves proposed transferring the geographical position of the first body before calculating the fix. Williams² questioned the accuracy of transferring the geographical position and, instead, solved the problem by transferring the first position circle pointwise; his method is sufficiently complex to require a computer, as he himself notes. Metcalf³ presented exact equations for transferring the geographical position of a body and its associated circles of altitude; however, application of Metcalf's equations assumes prior knowledge of the observer's position. Brown's approach⁴ was to move the observer's DR on to the first position circle before using it to advance the geographical position of the first body. Based on his experience with Chiesa and Chiesa's method, Pepperday⁵ argued that knowledge of a good DR position can be put to better use as part of a conventional running fix using position lines deduced by the Marcq St Hilaire or the modified Sumner method. Apparently, all users and advocates of Chiesa and Chiesa's method consider that the calculations involved are complex enough to require a computer.¹⁻⁶

Here, I present a simple solution for the two-body running fix without DR, which is accurate whenever the usual simplifying assumptions of the Marcq St. Hilaire method apply: namely, that the position lines are locally straight (equivalently, the azimuths are effectively independent of position) and the surface of the Earth is locally flat. The solution lends itself to an easy graphical construction, which will be described first. Subsequently, I show how to obtain the same solution with a hand calculator, and indicate how to find the position using Ageton's method of sight reduction and a time-sight table or formula.

Each of the two non-simultaneous sights gives rise to a position circle; the observer must be on the first position circle at the time of the first sight and on the second circle at the time of the second sight. If the observer were stationary he would be at one of the two points of intersection of the two circles. The correct choice can be made by reference to a DR position, even a very bad one, or by other means;⁵ the chosen point of intersection is a putative 'stationary fix'. The problem then becomes one of finding a point on the first position circle and another point on the second position circle, both of which are near the stationary fix, such that the two points can be connected by a segment of a rhumbline of length equal to the distance made good between sights and direction equal to the course made good. The exact mathematical solution to this problem was presented by Williams,² in the form of two simultaneous nonlinear equations whose solution is the latitude and longitude of the running fix. Williams' equations, which need to be solved iteratively with a computer, must be used when the simplifying assumptions referred to above are not valid (for instance, a large observed altitude or a very long run between sights). There is then no need to calculate the stationary fix, except possibly as a starting point for the iterative solution of the equations. However, when the simplifying assumptions are applicable, the problem becomes one involving straight lines in a plane. The graphical solution of this simplified problem is as follows.

(1) From the declinations, GHAS, and observed altitudes of the bodies at the actual times of the observations, calculate the latitude and longitude of the stationary fix. This is the point S in Figure 1.

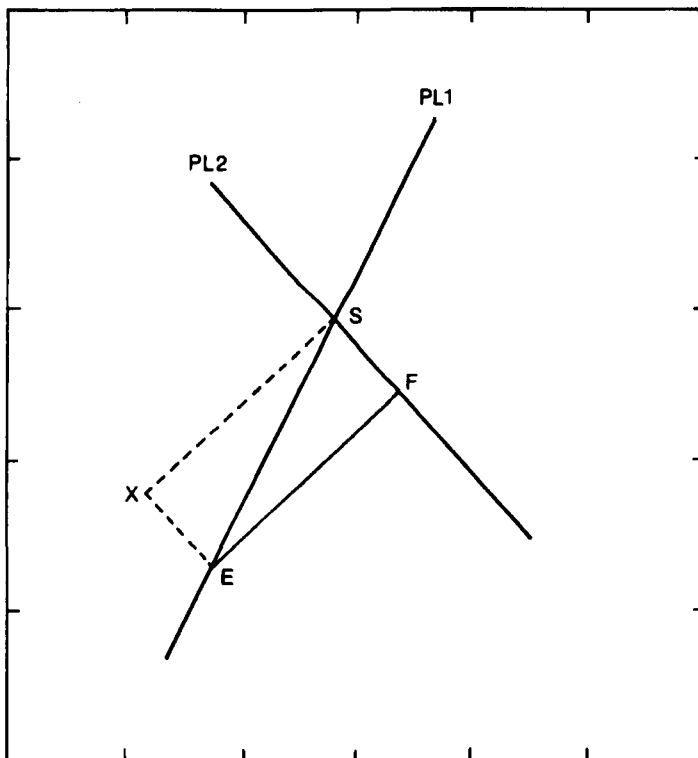


Fig. 1. Plot of the running fix

(2) From the declinations, G_{HAS} and observed altitudes of the bodies, and the latitude and longitude of the stationary fix, determine the azimuths of the bodies; this can be done by any standard method if the stationary fix is treated as a DR position.

(3) Plot position lines through the stationary fix. These are labelled PL₁ and PL₂ in Figure 1.

(4) From S, draw a line in the direction opposite to the course made good between sights, and of a length equal to the distance made good, ending at the point X.

(5) From X, draw a line parallel to the second position line to meet the first position line at the point E.

(6) Complete the parallelogram by drawing a line from E parallel to XS, to meet the second position line at the point F.

The observer was on the first position line at the time of the first sight and on the second position line at the time of the second sight. In between, he moved along a course parallel to EF through a distance equal to the length of EF. He therefore took the first sight at E and the second sight at F, which is the running fix.

The amount of plotting demanded by this procedure is about the same as is required for a running fix drawn from DR or assumed positions. However, only one distance needs to be laid off. Errors in the final position are therefore almost entirely due to errors in plotting course and position lines, and lines parallel to them.

The stationary fix can be found with a hand calculator by applying the cosine formula five times in succession⁴ (this formula is preferred because all intermediate quantities lie between 0° and 180°, and the arc cosine function on a calculator delivers an angle in this range). The azimuths can be found in this way also. To obtain the running fix with a hand calculator, first calculate the stationary fix and the azimuths.

Let ϕ and λ be the latitude and longitude of the stationary fix, let Z_1 and Z_2 be the azimuths of the first and second bodies, and let C and A be the course and distance made good between sights. The following calculations locate the running fix.

(7) Calculate

$$d = A \cos(C - Z_1) / \sin(Z_2 - Z_1)$$

disregarding the sign of the answer.

(8) Choose A and A^* to be the directions of the first position line pointing away from the stationary fix, such that both directions lie between 0° and 360° with $A^* = A + 180^\circ$. This can be achieved through one of the following choices:

(i) $A = Z_1 + 90^\circ$, $A^* = Z_1 + 270^\circ$ if $0^\circ \leq Z_1 < 90^\circ$;

(ii) $A = Z_1 - 90^\circ$, $A^* = Z_1 + 90^\circ$ if $90^\circ \leq Z_1 < 270^\circ$;

(iii) $A = Z_1 - 270^\circ$, $A^* = Z_1 - 90^\circ$ if $270^\circ \leq Z_1 < 360^\circ$.

(9) Set B equal to $Z_2 - 270^\circ$, $Z_2 - 90^\circ$, $Z_2 + 90^\circ$, or $Z_2 + 270^\circ$, chosen so that $A < B < A^*$.

(10) If $A \leq C \leq A^*$, set

$$\begin{aligned}\phi' &= \phi + d \cos B, \\ \lambda' &= \lambda - d \sin B / \cos(\phi + \frac{1}{2}d \cos B).\end{aligned}$$

If $0^\circ \leq C < A$ or $A^* < C < 360^\circ$, set

$$\begin{aligned}\phi' &= \phi - d \cos B, \\ \lambda' &= \lambda + d \sin B / \cos(\phi - \frac{1}{2}d \cos B),\end{aligned}$$

ϕ' and λ' are the latitude and longitude of the running fix. Step (7) calculates the length of the segment SF in Figure 1 by applying the sine formula for plane triangles to the triangle SEF. The formulae in step (10) are the familiar corrections for D.Lat. and D.Long. by traverse table.

If no computer or calculator is available, the stationary fix can be found with two applications of Ageton's method and one application of a time-sight table or formula. 'Navigational triangles' must be solved twice for 'altitude' and 'azimuth' and once for 'meridian angle'. Figure 2 illustrates two of the possible spatial arrangements between the elevated pole (P), the observer's position (O), and the geographical positions of the

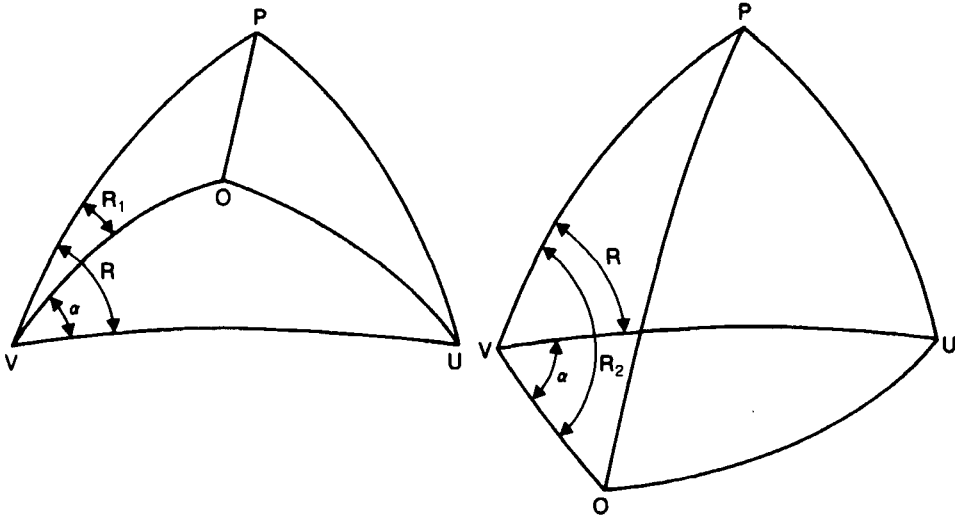


Fig. 2. Geometry of the stationary fix, showing two of the possible spatial arrangements

TABLE I

Sight no.	GMT	Altitude	Declination	GHA
1	15-06-00	62° 07.5'	22° 21.7' N	46° 58.4'
2	18-01-27	68° 19.7'	22° 22.6' N	90° 49.9'

first and second bodies (U and V). At each step, three of these four points are labelled 'pole', 'star' and 'DR', and the appropriate 'navigational triangle' is solved. First, in the triangle PUV label P 'pole', U 'star', and V 'DR'. To solve by Ageton's method, set 'latitude' equal to the declination of the second body, 'declination' equal to the declination of the first body, and 'meridian angle' equal to the difference between the GHAs of the two bodies. 'Altitude' is then equal to $90^\circ - D$, where D is the orthodromic distance between the two bodies (the notation is that of Chiesa and Chiesa¹). 'Azimuth' is equal to the angle PVU, or R in Chiesa and Chiesa's notation. Next, in the triangle OUV label V 'pole', U 'star', O 'DR'. To solve by time-sight, set 'latitude' equal to the observed altitude of the second body, 'declination' equal to $90^\circ - D$, and 'altitude' equal to the observed altitude of the first body. 'Meridian angle' is then equal to the angle OVU, or α in the notation of Chiesa and Chiesa. An observation of the approximate azimuth of the second body⁵ will allow a choice between $R_1 = R \sim \alpha$ and $R_2 = R + \alpha$; the chosen angle is needed in the final application of Ageton's method to the triangle OPV. For this step, label V 'pole', O 'star', and P 'DR'. Set 'latitude' equal to the declination of the second body, 'declination' equal to the observed altitude of the second body, and 'meridian angle' equal to the chosen angle R_1 or R_2 . 'Altitude' is then the latitude of the stationary fix, and 'azimuth' is the meridian angle of the second body

measured from the stationary fix. The longitude of the stationary fix can be found from this meridian angle and the GHA of the second body.

Once the stationary fix is known, the running fix can be found graphically. Obtaining a two-body stationary or running fix in this way requires somewhat more arithmetic than is needed for a running fix using DR positions. The difference is the second step, involving the solution of a time-sight navigational triangle. Tables for solving time-sights are hard to find nowadays, and it would usually be necessary to grind through an old-fashioned calculation with five-figure logarithms. The purpose of the last paragraph is to point out that it can be done if the need arises.

Example. On June 3, 1989, a small-boat navigator took two sets of five sights of the Sun, averaging each set in order to obtain a running fix. Course and distance made good between the two sets of sights were 049° , 17.5 miles. After averaging, data for sight reduction were as shown in Table 1.

Chiesa and Chiesa's procedure gives, using their notation: $D = 40^\circ 24.4'$, $R = 81^\circ 18.6'$, $\alpha = 42^\circ 33.3'$, $R_1 = 38^\circ 45.3'$, $R_2 = 123^\circ 51.9'$. The two possible stationary fixes are (1) $38^\circ 19.3' \text{ N}$, $73^\circ 41.7' \text{ W}$; (2) $9^\circ 24.6' \text{ N}$, $72^\circ 43.3' \text{ W}$. Comparison with the DR position ($38^\circ 30' \text{ N}$, $73^\circ 43' \text{ W}$) establishes the first stationary fix as the correct one. The azimuths calculated for this position are $Z_1 = 117.2^\circ$, $Z_2 = 227.5^\circ$. From Step (7), $d = 6.9'$. From step (8), $A = 27.2^\circ$ and $A^* = 207.2^\circ$; Step (9) then leads to $B = 137.5^\circ$. Finally, since $A \leq C \leq A^*$, Step (10) gives $\phi' = 38^\circ 14.2' \text{ N}$, $\lambda' = 73^\circ 35.7' \text{ W}$. It may be verified that this position satisfies equations (4) and (5) of Williams² with residual errors whose magnitude is $0.1'$ or less.

Figure 1 is a plot of this running fix. For comparison, the 1801 position computed from the original ten sights by the nonlinear least-squares algorithm of Severance⁷ was $38^\circ 13.8' \text{ N}$, $73^\circ 35.5' \text{ W}$. Also for comparison, calculation of the stationary fix using Ageton's method and the time-sight formula of Rust⁸ gave $90^\circ - D = 49^\circ 35.5'$, $R = 81^\circ 19.0'$, $\alpha = 42^\circ 33.5'$, $\phi = 38^\circ 19.0' \text{ N}$, $t = 17^\circ 08.5'$, and $\lambda = 73^\circ 41.5' \text{ W}$.

REFERENCES

- ¹ Chiesa, A. and Chiesa, R. (1990). A mathematical method of obtaining an astronomical vessel position. *This Journal*, **43**, 125.
- ² Williams, R. (1990). Computation of an astronomical running fix. *This Journal*, **43**, 444.
- ³ Metcalf, T. R. (1991). Advancing celestial circles of position. *Navigation*, **38**, 285.
- ⁴ Brown, C. (1991). Astro without azimuth. *This Journal*, **44**, 428.
- ⁵ Pepperday, M. (1992). The 'two-body fix' at sea. *This Journal*, **45**, 138.
- ⁶ Spencer, B. (1990). Astronomical fixes without an assumed position. *This Journal*, **43**, 449.
- ⁷ Severance, R. W. (1989). Overdetermined celestial fix by iteration. *Navigation*, **36**, 373.
- ⁸ Bowditch, N. (1977). *American Practical Navigator* (chapter XXI). US Defence Mapping Agency.

KEY WORD

1. Astro.

Shortest Spheroidal Distance

Tim Zukas

What is the shortest distance on the terrestrial spheroid between two widely separated points, given their latitudes and longitudes (and the dimensions of the chosen spheroid)?